South East Asian J. of Mathematics and Mathematical Sciences
Vol. 21, No. 1 (2025), pp. 77-96

DOI: 10.56827/SEAJMMS.2025.2101.7 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

ITERATIVE ALGORITHM FOR GENERALIZED VARIATIONAL INEQUALITY PROBLEM AND APPLICATIONS

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(**Received:** *Jan. 13, 2024* **Accepted:** *Apr. 06, 2025* **Published:** *Apr. 30, 2025)*

Abstract: The purpose of this study is to look at a generalized variational inequality problem. We start a iterative method [18] and evaluate its convergence. We estimate the common solution of a generalized variational inequality problem and the fixed points of a nonexpansive mapping using iterative method [18]. A numerical example is provided to demonstrate our existence result. Furthermore, we show that the considered iterative technique converges quicker than the earlier iterative scheme. We also use our suggested iterative approach to estimate the solution to a convex minimization problem and a split feasibility problem.

Keywords and Phrases: Variational inequalities problem, Iterative methods, nonexpansive mappings, Convex minimization and split feasibility problem.

2020 Mathematics Subject Classification: 47J40, 58E35, 47H10.

1. Introduction

G. Stampacchia, an Italian mathematician, defined variational inequalities at the end of the 1960's and the beginning of the 1970's [26]. In recent years, the concepts and methods of variational inequalities have been employed in a wide range of pure and practical disciplines and have proven to be fruitful and inventive. It has been demonstrated that this theory gives the most natural, straightforward, simple, unified, and efficient framework for a broad class of linear and nonlinear problems. Using new and novel methodologies, variational inequalities have been generalized and expanded in numerous directions. Noor [19] developed and investigated general variational inequality, a novel class of variational inequalities involving two operators, in 1988. We point out that Noor variational inequalities are another name for general variational inequalities.

Finding the fixed points of the nonexpansive mappings is an issue that is related to variational inequalities and is currently being researched through functional analysis. It appears sense to think about combining these two distinct issues. Noor and Huang [20], who were motivated and inspired by the study in this area, took into consideration the issue of identifying the common component of the set of variational inequalities solutions and the set of nonexpansive mappings fixed points. We use and analyse new iterative techniques for discovering common solutions to general variational inequalities and nonexpansive mappings in this study. We also look at the convergence of the suggested three-step iterative process under some mild conditions. Our results represent a major and original advancement above the already established results.

2. Preliminaries

This section contains some well-known concepts and results that will be referenced throughout the paper.

All through this study, we presume that $\mathbb{H}$ is a real Hilbert space equipped with norm $||.||$ induced by inner product $\langle \cdot, \cdot \rangle$. Let Ω be a nonempty closed convex subset of $\mathbb{H}$ and $f, g \colon \Omega \to \mathbb{H}$ be nonlinear mappings. The generalized nonlinear variational inequality is to locate a point $a \in \mathbb{H}$ such that

$$\langle f(a), g(a) - g(b) \rangle \geq 0, \quad \forall b \in \Omega, g(a), g(b) \in \mathbb{H}, \tag{2.1}$$

which was introduced by Noor [21]. We denote the set of solutions of (2.1) by $Sol(\Omega, f, g)$.

If $g = I$, then generalized nonlinear variational inequality (2.1) reduces to the classical variational inequality studied by Stampacchia [26], which is to allocate a point $a \in \mathbb{H}$, such that

$$\langle f(a), b - a \rangle \geq 0, \quad \forall b \in \mathbb{H}. \tag{2.2}$$

If $\Omega^* = \{a \in \mathbb{H} \colon \langle a, b \rangle \geq 0, \forall b \in \Omega\}$ is a dual cone of a convex cone Ω, then generalized nonlinear variational inequality (2.1) coincides to generalized nonlinear complementary problem which is to locate a point $a \in \mathbb{H}$ such that

$$\langle f(a), g(a)\rangle = 0, \quad g(a) \in \Omega, f(a) \in \Omega^*. \tag{2.3}$$

It is important to note that variational inequalities, which are an unusual and impressive extension of variational principles, offer a well-organized, unified framework for solving a variety of nonlinear issues that arise in operations research, control theory, economics, physics, and a host of other fields. For instance, [2, 8, 15, 23, 24, 27, 29, 36] and the references cited therein.

Recall that a mapping $f\colon \Omega \subset \mathbb{H} \to \mathbb{H}$ is called:
(i) λ-Lipschitzian if for all $a^*, b^* \in \Omega$, there exists a constant $\lambda > 0$ such that

$$||f(a^*) - f(b^*)|| \leq \lambda ||a^* - b^*||, \tag{2.4}$$

(ii) nonexpansive if for all $a^*, b^* \in \Omega$, we have

$$||f(a^*) - f(b^*)|| \leq ||a^* - b^*||, \tag{2.5}$$

(iii) α-inverse strongly monotone if for all $a^*, b^* \in \Omega$, there exists a constant $\alpha > 0$, such that

$$\langle f(a^*) - f(b^*), a^* - b^*\rangle \geq \alpha ||f(a^*) - f(b^*)||^2. \tag{2.6}$$

Note that α-inverse strongly monotone mapping is $\frac{1}{\alpha}$-Lipschitz continuous.

It is usual to point out that similar optimization issues such as variational inequalities, variational inclusions, and fixed-point problems might be presented. By using fixed point iterative techniques, this novel framework dominates the study of variational inequalities and nonlinear problems.

Lemma 2.1. *Let $\mathbb{P}_\Omega\colon \mathbb{H} \to \Omega$ be a projection mapping of $\mathbb{H}$ onto Ω. For a given $v \in \mathbb{H}$, $u \in \Omega$ satisfies the inequality*

$$\langle u - v, w - u\rangle \geq 0, \ \forall w \in \Omega \iff u = \mathbb{P}_\Omega(v). \tag{2.7}$$

It should be noted that the projection mapping $\mathbb{P}_\Omega$ is nonexpansive [17]. [12] provides more information on projection mapping $\mathbb{P}_\Omega$. The generalized nonlinear variational inequality (2.1) can be designed as a fixed-point problem using Lemma 2.1 as follows:

Lemma 2.2. (see [19]). *Let $\mathbb{P}_\Omega\colon \mathbb{H} \to \Omega$ be a projection mapping. For any $\mu > 0$, $a \in \mathbb{H}$,$g(a) \in \Omega$ solves the generalized nonlinear variational inequality (2.1) if and only if*

$$g(a) = \mathbb{P}_\Omega[g(a) - \mu f(a)]. \tag{2.8}$$

Relationship (2.8) can be rewritten as

$$a = a - g(a) + \mathbb{P}_\Omega[g - \mu f](a). \tag{2.9}$$

Let $\Im$ be a nonexpansive mapping and $F(\Im)$ denotes the set of fixed points of $\Im$. If $a \in F(\Im) \cap Sol(\mathbb{H}, f, g)$,

$$\begin{aligned} a = \Im(a) &= a - g(a) + \mathbb{P}_\Omega[g - \mu f](a) \\ &= \Im\{a - g(a) + \mathbb{P}_\Omega[g - \mu f](a)\}, \quad \mu > 0. \end{aligned} \tag{2.10}$$

It is significant to achieve better rate of convergence if two or more iterative algorithms converge to the same point for a given problem. We recall the following concepts which are versatile tools to find finer convergence rate for different iterative methods.

Definition 2.3. (see [3]). *Let $\{u_n\}$ and $\{v_n\}$ be two real sequences converging to u and v, respectively. Suppose that $\lim_{n\to\infty} ||u_n - u||/||v_n - v|| = k$ exists. Then,*
(i) $\{u_n\}$ converges faster then $\{v_n\}$ if $k = 0$,
(ii) $\{u_n\}$ and $\{v_n\}$ converges with identical rates if $0 < k < \infty$.

Definition 2.4. (see [3]). *Let $\{u_n\}$ and $\{v_n\}$ be two real sequences converging to the same fixed point ρ. If $\{p_n\}$ and $\{q_n\}$ are two sequences of positive real numbers converging to 0 such that $||u_n - \rho|| \leq p_n$ and $||v_n - \rho|| \leq q_n$ for all $n \in \mathbb{N}$. Then, $\{u_n\}$ converges to ρ faster than $\{v_n\}$ if $\{p_n\}$ converges faster ρ then $\{q_n\}$.*

Lemma 2.5. (see [4]). *Let $\{\eta_n\}$ and $\{\xi_n\}$ be non-negative sequences of real numbers satisfying*

$$\eta_{n+1} \leq \kappa \eta_n + \xi_n, \qquad \forall n \in \mathbb{N}, \tag{2.11}$$

where $\kappa \in (0, 1)$ and $\lim_{n\to\infty} \xi_n = 0$. Then $\lim_{n\to\infty} \eta_n = 0$.

Lemma 2.6. (see [34]). *Let $\{\zeta_n\}$, $\{\eta_n\}$ and $\{\xi_n\}$ be non-negative sequences of real numbers satisfying*

$$\zeta_{n+1} \leq (1 - \eta_n)\zeta_n + \xi_n, \qquad \forall n \in \mathbb{N}, \tag{2.12}$$

where $\eta_n \in (0, 1)$, $\sum_{n=1}^{\infty} \eta_n = \infty$ and $\xi_n = o(\eta)$ Then $\lim_{n\to\infty} \zeta_n = 0$.

Mann, Ishikawa, and Halpern iterative approaches are key tools for solving nonexpansive mapping fixed-point problems. A number of fixed point iterative algorithms have recently been developed and implemented to solve various classes of nonlinear mappings [2, 9, 10, 22, 25, 28, 37]. Agarwal and colleagues [1] developed the S-iteration approach, which converges quicker than well-known iterative algorithms like Mann, Ishikawa, and Picard for contraction and nonexpansive mappings. A number of researchers were drawn to study fixed-point problems, minimization difficulties, variational inclusions, variational inequalities, and alternate points problems in various contexts because of the super convergence rate. In [21], Noor utilized formulation (2.10) to propose following iterative algorithm:

$$\begin{cases} x_0 \in \Omega, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n \Im\{x_n - g(x_n) + \mathbb{P}_\Omega[g(x_n) - \mu f(x_n)]\}, \end{cases} \tag{2.13}$$

where $\{\alpha_n\}$ is a sequence in (0,1). The author demonstrated the proposed iterative algorithm's strong convergence. Furthermore, the typical S-iterative technique is expected to converge faster than the Mann and Picard iterative algorithms. Gursoy and colleagues [14] investigated the following standard S-iterative method to analyse (2.1) because of its simplicity and faster convergence rate:

$$\begin{cases} u_0 \in \Omega, \\ u_{n+1} = \Im\{v_n - g(v_n) + \mathbb{P}_\Omega[g(v_n) - \mu f(v_n)]\}, \\ v_n = (1-\psi_n)u_n + \psi_n \Im\{u_n - g(u_n) + \mathbb{P}_\Omega[g(u_n) - \mu f(u_n)]\}, \end{cases} \tag{2.14}$$

where $\{\psi_n\}$ is a sequence in (0,1). In 2018, Ullah and Arshad [30] introduced a more efficient iterative algorithm called the M-iterative method for Suzuki's generalized nonexpansive mappings as follows:

$$\begin{cases} x_1 \in \Omega, \\ z_n = (1-\alpha_n)x_n + \alpha_n \Im x_n, \\ y_n = \Im z_n, \\ x_{n+1} = \Im y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{2.15}$$

where $\{\alpha_n\}$ is a sequence in (0,1). They investigated convergence and discovered that their iterative approach is faster than the Picard S [13] and S-iteration processes [1]. In 2020, Garodia and Uddin [11] developed a new iterative algorithm for Suzuki's generalized nonexpansive mappings as follows:

$$\begin{cases} x_1 \in \Omega, \\ z_n = \Im x_n, \\ y_n = \Im((1-\alpha_n)z_n + \alpha_n \Im z_n), \\ x_{n+1} = \Im y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{2.16}$$

where $\{\alpha_n\}$ is a sequence in (0,1). The authors approximated fixed-points and inspected the convergence. Also, they proved that the posed iterative method converges with faster rate than that of the M-iterative method.

Question 1. Is it possible to develop an iteration process which rate of convergence for nonexpansive maps is faster than the iteration process (2.16) and the other iteration processes?

As an answer, Manoj et al. [18] developed a new iterative algorithm for Suzuki's generalized nonexpansive mappings as follows:

$$\begin{cases} x_1 = x \in \Omega, \\ z_n = \Im((1-\alpha_n)x_n + \alpha_n \Im x_n), \\ y_n = \Im z_n, \\ x_{n+1} = \Im y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{2.17}$$

where $\{\alpha_n\}$ is a sequence in (0,1). Now, we present a example which show that our iteration process in the case (2.17) converges at a rate faster than the other iteration processes.

Example 1. Let $\Omega = [2,5]$ be equipped with the usual metric space and $\Omega\colon \Im \to \Im$ be a mapping defined by $\Im(x) = \sqrt{2x+3}$, for any $x \in \Omega$. Choose $\alpha_n = 0.7$. Also the initial value $x_1 = 4$. It is obvious that 3 is a fixed point of $\Im$.

No. of Iteration	S-iteration	M-iteration	Garodia iteration	New iteration
0	4	4	4	4
1	3.2209	3.0558	3.0183	3.0003
2	3.0507	3.0032	3.0003	3
3	3.0117	3.0001	3	3
4	3.0027	3	3	3
5	3.0006	3	3	3
6	3	3	3	3

In this study, we investigate method (2.17) to estimate the common solution of fixed points of a nonexpansive mapping $\Im$ and the generalized nonlinear variational inequality (2.1). This investigation is motivated by the work stated in the aforementioned references:

$$\begin{cases} x_1 \in \Omega, \\ w_n = (1-\alpha_n)x_n + \alpha_n \Im[x_n - g(x_n) + \mathbb{P}_\Omega\{g(x_n) - \mu f(x_n)\}], \\ z_n = \Im[w_n - g(w_n) + \mathbb{P}_\Omega\{g(w_n) - \mu f(w_n)\}], \\ y_n = \Im[z_n - g(z_n) + \mathbb{P}_\Omega\{g(z_n) - \mu f(z_n)\}], \\ x_{n+1} = \Im[y_n - g(y_n) + \mathbb{P}_\Omega\{g(y_n) - \mu f(y_n)\}], \quad \forall n \in \mathbb{N}, \end{cases} \tag{2.18}$$

where $\{\alpha_n\}$ is a sequence in (0,1) satisfying certain assumptions. Under certain simple assumptions, we investigate the strong convergence of our suggested iterative algorithm (2.18). In addition, we propose a modified version of our iterative algorithm (2.18) for investigating convex optimization and split feasibility problems. An instructive numerical illustration validates theoretical findings. Our existence and convergence results might be viewed as generalizations and manifestations of previously known results.

3. Main result

Theorem 3.1. *Let $f, g\colon \Omega \to \mathbb{H}$ be α_1, α_2-inverse strongly monotone mappings, respectively, and $\Im\colon \mathbb{H} \to \Omega$ be a nonexpansive mapping such that $F(\Im) \cap Sol(\Omega, f, g) \neq \emptyset$. Suppose that the assumption*

$$|\mu - \alpha_1| < \alpha_1(1 - \Upsilon), \tag{3.1}$$

holds, where $\Upsilon = 2|\alpha_2 - 1/\alpha_2|$. Then, the iterative sequence $\{x_n\}$ defined by (2.18) converges strongly to $w^* \in F(\Im) \cap Sol(\Omega, f, g)$ with the following

$$||x_{n+1} - w^*|| \leq \varphi^{3(n+1)} ||x_0 - w^*|| \prod_{k=0}^{n} [1 - \alpha_k(1 - \varphi)], \quad \forall n \in \mathbb{N}, \tag{3.2}$$

where

$$\varphi = 2\left|\frac{\alpha_2 - 1}{\alpha_2}\right| + \left|\frac{\alpha_1 - \mu}{\alpha_1}\right|. \tag{3.3}$$

Proof. Note that

$$w^* = \Im[w^* - g(w^*) + \mathbb{P}_\Omega\{g(w^*) - \mu f(w^*)\}]. \tag{3.4}$$

Since f being α_1-inverse strongly monotone is $1/\alpha_1$-Lipschitz continuous mapping, $\Im$ and $\mathbb{P}_\Omega$ are the nonexpansive mappings. Then, from (2.18 and 3.4), we obtain

$$\begin{aligned} ||z_n - w^*|| &= ||\Im[w - g(w) + \mathbb{P}_\Omega\{g(w) - \mu f(w)\}] \\ &\quad - \Im[w^* - g(w^*) + \mathbb{P}_\Omega\{g(w^*) - \mu f(w^*)\}]|| \\ &\leq 2||w_n - w^* - \{g(w_n) - g(w^*)\}|| \\ &\quad + ||w_n - w^* - \mu\{f(w_n) - f(w^*)\}||. \end{aligned} \tag{3.5}$$

Since f is α_1-inverse strongly monotone mapping, then we have

$$\begin{aligned}||w_n - w^* - \mu\{f(w_n) - f(w^*)\}||^2 &= ||w_n - w^*||^2 + \mu^2||\{f(w_n) - f(w^*)\}||^2 \\ &\quad - 2\mu \langle w_n - w^*, f(w_n) - f(w^*)\rangle \\ &\leq ||w_n - w^*||^2 + \frac{\mu^2}{\alpha_1^2}||w_n - w^*||^2 \\ &\quad - 2\mu\alpha_1||f(w_n) - f(w^*)||^2 \\ &\leq \left(\frac{\alpha_1 - \mu}{\alpha_1}\right)^2 ||w_n - w^*||^2.\end{aligned} \tag{3.6}$$

Also, g is α_2-inverse strongly monotone mapping, then we have

$$\begin{aligned}||w_n - w^* - \{g(w_n) - g(w^*)\}||^2 &= ||w_n - w^*||^2 + ||\{g(w_n) - g(w^*)\}||^2 \\ &\quad - 2 \langle w_n - w^*, g(w_n) - g(w^*)\rangle \\ &\leq ||w_n - w^*||^2 + \frac{1}{\alpha_2^2}||w_n - w^*||^2 \\ &\quad - 2\alpha_2||g(w_n) - g(w^*)||^2 \\ &\leq \left(\frac{\alpha_2 - 1}{\alpha_2}\right)^2 ||w_n - w^*||^2.\end{aligned} \tag{3.7}$$

Thus, from (3.5), (3.6) and (3.7), we have

$$||z_n - w^*|| \leq \left(2\left|\frac{\alpha_2 - 1}{\alpha_2}\right| + \left|\frac{\alpha_1 - \mu}{\alpha_1}\right|\right) ||w_n - w^*|| = \varphi||w_n - w^*||, \tag{3.8}$$

where φ is defined by (3.3), and from (3.1), we have $\varphi < 1$. Again, following the same steps (3.5)-(3.8) and from (2.17).

Next, we estimate

$$\begin{aligned}||w_n - w^*|| &= ||(1 - \alpha_n)x_n + \alpha_n\Im[x_n - g(x_n) \\ &\quad + \mathbb{P}_\Omega\{g(x_n) - \mu f(x_n)\}] - w^*|| \\ &\leq (1 - \alpha_n)||x_n - w^*|| + \alpha_n\varphi||x_n - w^*|| \\ &\leq 1 - \alpha_n(1 - \varphi)||x_n - w^*||.\end{aligned} \tag{3.9}$$

Also, we have

$$\begin{aligned}||y_n - w^*|| &\leq \varphi||z_n - w^*|| \\ &\leq \varphi^2||w_n - w^*|| \\ &\leq \varphi^2[1 - \alpha_n(1 - \varphi)||x_n - w^*||].\end{aligned} \tag{3.10}$$

So, we have

$$
\begin{aligned}
||x_{n+1} - w^*|| = ||&\Im[y_n - g(y_n) + \mathbb{P}_\Omega\{g(y_n) - \mu f(y_n)\}] \\
&- \Im[w^* - g(w^*) + \mathbb{P}_\Omega\{g(w^*) - \mu f(w^*)\}]|| \\
&\leq \varphi||y_n - w^*|| \leq \varphi^3[1 - \alpha_n(1-\varphi)||x_n - w^*||].
\end{aligned} \tag{3.11}
$$

Since, $1-\alpha_n(1-\varphi) < 1$. Therefore, we get $||x_{n+1}-w^*|| \leq \varphi^3||x_n-w^*||$, $\forall n \in \mathbb{N}$. By repeating the process in this fashion, we obtain

$$
||x_{n+1} - w^*|| \leq \varphi^{3(n+1)}||x_0 - w^*||, \quad \forall n \in \mathbb{N}, \tag{3.12}
$$

which gives that $\lim_{n\to\infty} ||x_n - w^*|| = 0$.

Theorem 3.2. *Let $\mathbb{H}$ be a real Hilbert space and Ω be a nonempty closed convex subset of $\mathbb{H}$. Let $f, g, \Im$, and φ be same as defined in Theorem 3.1. Let $\{u_n\}$ and $\{x_n\}$ be the sequences defined by (2.14) and (2.18), respectively. Suppose that (3.1) holds and $F(\Im) \cap (\Omega, f, g) \neq \emptyset$. Then, the following statements hold:*

(i) If $\{(1+\varphi^3)/\psi\}$ is bounded and $\sum_{n=1}^{\infty} a_n = \infty$, then the sequence $\{x_n - u_n\}$ converges strongly to 0 with following error estimates:

$$
\begin{aligned}
||x_{n+1} - u_{n+1}|| \leq{}& [1 - \psi(1-\varphi)]||x_n - u_n|| \\
&+ (1-\varphi^3)||x_n - w^*||, \quad \forall n \in \mathbb{N}.
\end{aligned} \tag{3.13}
$$

(ii) If $\{u_n\}$ converges strongly to $w^* \in F(\Im) \cap Sol(\Omega, f, g)$, then $\{x_n - u_n\}$ converges strongly to 0 with following error estimates:

$$
||u_{n+1} - x_{n+1}|| \leq \varphi^3||u_n - x_n|| + (1-\varphi^3)||u_n - w^*||, \quad \forall n \in \mathbb{N}. \tag{3.14}
$$

Proof. (i) It follows from Theorem 3.1 that $\lim_{n\to\infty} ||x_n - w^*|| = 0$. Next we prove that $\lim_{n\to\infty} ||u_n - w^*|| = 0$. Following (2.14) and (2.18) and steps as in (3.5)-(3.8), we obtain

$$
\begin{aligned}
||x_{n+1} - u_{n+1}|| = ||&\Im[y_n - g(y_n) + \mathbb{P}_\Omega\{g(y_n) - \mu f(y_n)\}] \\
&- \Im\{v_n - g(v_n) + \mathbb{P}_\Omega[g(v_n) - \mu f(v_n)]\}|| \\
&\leq \varphi||y_n - v_n||,
\end{aligned} \tag{3.15}
$$

where φ is same as in (3.3). Again, utilizing (2.14), (2.18) and (3.15), we have

$$\begin{aligned}
||x_{n+1}-u_{n+1}|| &\leq \varphi||y_n-(1-\psi_n)u_n-\psi_n\Im\{u_n-g(u_n)+\mathbb{P}_\Omega[g(u_n)-\mu f(u_n)]\}|| \\
&\leq \varphi||y_n-w^*||+(1-\psi_n)\varphi||u_n-w^*||+\psi_n\varphi||\Im\{u_n-g(u_n) \\
&\quad+\mathbb{P}_\Omega[g(u_n)-\mu f(u_n)]\}-\Im\{w^*-g(w^*)+\mathbb{P}_\Omega[g(w^*)-\mu f(w^*)]\}|| \\
&\leq \varphi||y_n-w^*||+(1-\psi_n)\varphi||u_n-w^*||+\psi_n\varphi||u_n-w^*|| \\
&= \varphi[||y_n-w^*||+1-\psi_n(1-\varphi)||u_n-w^*||] \\
&\leq \varphi[\varphi^2(1-\alpha_n(1-\varphi))||x_n-w^*||+1-\psi_n(1-\varphi)||u_n-w^*||] \\
&= \varphi[\varphi^2(1-\alpha_n(1-\varphi))||x_n-w^*||+1-\psi_n(1-\varphi)||x_n-w^*|| \\
&\quad+1-\psi_n(1-\varphi)||x_n-u_n||] \\
&\leq \varphi[1-\psi_n(1-\varphi)]||x_n-u_n||+(1+\varphi^3)max\{1-\alpha_n(1-\varphi), \\
&\quad 1-\psi_n(1-\varphi)\}||x_n-w^*|| \\
&\leq [1-\psi_n(1-\varphi)]||x_n-u_n||+(1+\varphi^3)||x_n-w^*||.
\end{aligned} \tag{3.16}$$

Let $\sigma_n=||x_n-u_n||$, $\hbar_n=\psi_n(1-\varphi)$, $\psi_n=(1+\varphi^3)||x_n-w^*||$ and $\zeta_n=||x_n-w^*||$, $\forall n\in\mathbb{N}$. It follows from assumption of the theorem that $\{(1+\varphi^3)/\psi_n\}$ is bounded therefore, $\{(1+\varphi^3)/\psi_n(1-\varphi)\}$ is also bounded. Then, there exists a constant $M>0$, such that $|(1+\varphi^3)/\psi_n(1-\varphi)|<M, \forall n\in\mathbb{N}$. Since $\lim_{n\to\infty}\zeta_n=0$ and $\{(1+\varphi^3)/\psi_n(1-\varphi)\}$ is bounded therefore $\{(1+\varphi^3)/\psi_n(1-\varphi)\zeta\}\to 0$ as $n\to\infty$, i.e., $\lim_{n\to\infty}(\psi_n/\hbar_n)=0$, which amounts to say that $\psi_n=o(\hbar_n)$. Thus, all the assumption of Lemma 2.6 are fulfilled. Hence, $\lim_{n\to\infty}||x_n-u_n||=0$ and $||u_n-w^*||\leq||x_n-u_n||+||x_n-w^*||$. Thus, we have $\lim_{n\to\infty}||u_n-w^*||=0$.
(ii) Next, we estimate that $\{u_n-x_n\}\to 0$. Since $\{u_n\}$ converges to $w^*\in F(\Im)\cap Sol(\Omega,f,g)$, then following the same arguments as in (3.15) and (3.16), we obtain

$$\begin{aligned}
||u_{n+1}-x_{n+1}|| &\leq \varphi[1-\psi_n(1-\varphi)]||u_n-w^*||+\varphi[\varphi^2(1-\alpha_n(1-\varphi))]||x_n-w^*|| \\
&\leq \varphi^3[1-\alpha_n(1-\varphi)]||u_n-x_n||+\varphi^3[1-\alpha_n(1-\varphi)]||u_n-w^*|| \\
&\quad+\varphi[1-\psi_n(1-\varphi)]||u_n-w^*|| \\
&\leq \varphi^3[1-\alpha_n(1-\varphi)]||u_n-x_n||+\varphi^3||u_n-w^*||+||u_n-w^*|| \\
&\leq \varphi^3||u_n-x_n||+(1+\varphi^3)||u_n-w^*||.
\end{aligned} \tag{3.17}$$

Let $\sigma'_n=||u_n-x_n||$, $\psi'_n=(1+\varphi^3)||x_n-w^*||$, $\forall n\in\mathbb{N}$. By the assumption $\{u_n\}$ converges to w^* and utilizing the fact that $(1+\varphi^3)$ is bounded. we obtain that $\psi'_n\to 0$ as $n\to\infty$. Thus, all the assumptions of Lemma 2.5 are fulfilled. Hence, $\lim_{n\to\infty}||u_n-x_n||=0$. Also, we know that $||x_n-w^*||\leq||u_n-x_n||+||u_n-w^*||$, $\forall n\in\mathbb{N}$. Thus, $\lim_{n\to\infty}||x_n-w^*||=0$. Hence, $\{u_n-x_n\}\to 0$ as $n\to\infty$.

Theorem 3.3. *Let $\mathbb{H}$ be a real Hilbert space and Ω be a nonempty closed convex subset of $\mathbb{H}$. Let $f, g, \Im$, and φ be same as defined in Theorem 3.1. Let $\{u_n\}$ and $\{x_n\}$ be the sequences defined by (2.14) and (2.18), respectively. Suppose that (3.1) holds and $F(\Im) \cap (\Omega, f, g) \neq \emptyset$. If $u_0 = x_0$, then $\{x_n\}$ converges faster then $\{u_n\}$ to w^*, such that $w^* \in F(\Im) \cap (\Omega, f, g)$.*

Proof. It follows from (3.11) that

$$||x_{n+1} - w^*|| \leq \varphi^3[1 - \alpha_n(1-\varphi)]||x_n - w^*||. \tag{3.18}$$

since $\{\alpha_n\}$ is sequence in (0,1), we can choose a constant $\alpha \in R$, such that $0 < \alpha \leq \alpha_n < 1,\ \forall n \in \mathbb{N}$. Then

$$||x_{n+1} - w^*|| \leq \varphi^3[1 - \alpha(1-\varphi)]||x_n - w^*||. \tag{3.19}$$

By repeating the process, we obtain

$$||x_{n+1} - w^*|| \leq \varphi^{3(n+1)}[1 - \alpha(1-\varphi)^{n+1}]||x_0 - w^*||,\ \forall n \in \mathbb{N}. \tag{3.20}$$

Also, it follows from (2.14) that

$$\begin{aligned} ||u_{n+1} - w^*|| &= ||\Im\{v_n - g(v_n) + \mathbb{P}_\Omega[g(v_n) - \mu f(v_n)]\} \\ &\quad - \Im\{w^* - g(w^*) + \mathbb{P}_\Omega[g(w^*) - \mu f(w^*)]\}|| \\ &\leq ||v_n - w^* - (g(v_n) - g(w^*))|| \\ &\quad + ||g(v_n) - g(w^*) - \mu(f(v_n) - f(w^*))|| \\ &\leq 2||v_n - w^* - (g(v_n) - g(w^*))|| \\ &\quad + ||v_n - w^* - \mu(f(v_n) - f(w^*))||. \end{aligned} \tag{3.21}$$

By following the arguments as discussed from (3.5) to (3.8), we have

$$||u_{n+1} - w^*|| \leq \varphi||v_n - w^*||. \tag{3.22}$$

Also,

$$\begin{aligned} ||v_n - w^*|| &= ||(1-\psi_n)u_n + \psi_n\Im\{u_n - g(u_n) + \mathbb{P}_\Omega[g(u_n) - \mu f(u_n)]\} - w^*|| \\ &\leq (1-\psi_n)||u_n - w^*|| + 2\psi_n||u_n - w^* - (g(u_n) - g(w^*))|| \\ &\quad + \psi_n||u_n - w^* - \mu(f(u_n) - f(w^*))|| \\ &\leq (1-\psi_n)||u_n - w^*|| + \psi_n\varphi||u_n - w^*|| \\ &= [1 - \psi_n(1-\varphi)]||u_n - w^*||. \end{aligned} \tag{3.23}$$

By combining (3.22) and (3.23), we get

$$||u_{n+1} - w^*|| \leq \varphi[1 - \psi_n(1 - \varphi)]||u_n - w^*||. \tag{3.24}$$

since $\{\psi_n\}$ is sequence in (0,1), we can choose a constant $\psi \in R$, such that $0 < \psi \leq \psi_n < 1,\ \forall n \in \mathbb{N}$. Then

$$||u_{n+1} - w^*|| \leq \varphi[1 - \psi(1 - \varphi)]||u_n - w^*||. \tag{3.25}$$

Thus, by repeating the process, we obtain

$$||u_{n+1} - w^*|| \leq \varphi^{n+1}[1 - \psi(1 - \varphi)]^{n+1}||u_0 - w^*||,\ \forall n \in \mathbb{N}. \tag{3.26}$$

Set $\alpha_n = \varphi^{3(n+1)}[1 - \alpha(1 - \varphi)]^{n+1}||x_0 - w^*||$, $\beta_n = \varphi^{n+1}[1 - \psi(1 - \varphi)]^{n+1}||u_0 - w^*||$; then,

$$\begin{aligned} \mathbb{A}_n = \frac{\alpha_n}{\beta_n} &= \frac{\varphi^{3(n+1)}[1 - \alpha(1 - \varphi)]^{n+1}||x_0 - w^*||}{\varphi^{n+1}[1 - \psi(1 - \varphi)]^{n+1}||u_0 - w^*||} \\ &\to 0 \ as\ n \to \infty. \end{aligned} \tag{3.27}$$

Hence, $\{x_n\}$ converges faster then $\{u_n\}$.

4. Numerical example

Example 4.1. Let $\mathbb{H} = \mathbb{R}$, $\Omega = [1, 2]$ be equipped with norm $||x|| = |x|$ and inner product $< x, y >= x.y$. Let $f, g, \Im \colon [1, 2] \to \mathbb{R}$ be defined by

$$f(x) = x^2,\ g(x) = \frac{x^3}{4} + \frac{3}{4},\ \Im(x) = \frac{x^2 + x^3}{16} + \frac{7}{8}. \tag{4.1}$$

Then, for all $x, y \in \Omega$, observe that

$$\begin{aligned} < f(x) - f(y), x - y >&= (x - y)^2(x + y) \geq 2|x - y|^2, \\ < g(x) - g(y), x - y >&= \frac{1}{16}(x - y)^2(x^2 + xy + y^2) \geq \frac{3}{4}|x - y|^2, \\ |\Im(x) - \Im(y)| &= \frac{1}{16}|x - y||x^2 + xy + y^2 + x + y| \leq |x - y|. \end{aligned} \tag{4.2}$$

Then, f and g are 2 and 3/4-inverse strongly monotone mapping, respectively, and $\Im$ is nonexpansive mapping. One can easily verify that $x^* = 1 \in \Omega$ is the unique fixed point of $\Im$. Also,

$$< f(x^*), g(y) - g(x^*) >= \frac{y^3 - 1}{4} \geq 0,\ \forall y \in \Omega \tag{4.3}$$

Thus, we have $x^* = 1 \in F(\Im) \cap Sol(\Omega, f, g)$.

5. Applications

5.1. Convex Minimization Problem:

Now, we solve convex minimization problem as an application of Theorem 3.1.

Let Ω be a closed convex subset of a real Hilbert space $\mathbb{H}$, $\mathbb{P}_\Omega \colon \mathbb{H} \to \Omega$ be a projection, and $F \colon \Omega \to \mathbb{R}$ be a convex, Fréchet differentiable mapping. We consider the following convex minimization problem:

$$\min_{x^* \in \Omega} F(x^*). \tag{5.1}$$

Clearly, $x^* \in \Omega$ is a solution of $\mathbb{P}_\Omega(I - \mu\nabla F)$ if and only if

$$\langle \nabla F(x^*), x - x^* \rangle \geq 0, \ \forall x \in \Omega. \tag{5.2}$$

More precisely, $x^* \in \Omega$ solves problem (5.1) if and only if x^* is a fixed point of the projection mapping $\mathbb{P}_\Omega(I - \mu\nabla F)$, i e.,

$$x^* = \mathbb{P}_\Omega[x^* - \mu\nabla F(x^*)], \tag{5.3}$$

where ∇F is the gradient of mapping F. This formulation is known as gradient projection, which plays a key role in solving problem (5.1). So far, several iterative methods have been employed to solve minimization problems [7, 29, 35]. By considering $f \colon = \nabla F$ and assuming $\Im = g = I$, the identity mapping, we propose the following modified gradient projection algorithm for solving $\mathbb{P}_\Omega(I - \mu\nabla F)$ as follows:

$$\begin{cases} x_n \in \Omega, \\ w_n = (1-\alpha_n)x_n + \alpha_n \mathbb{P}_\Omega[x_n - \mu\nabla f(x_n)], \\ z_n = \mathbb{P}_\Omega[w_n - \mu\nabla f(w_n)], \\ y_n = \mathbb{P}_\Omega[z_n - \mu\nabla f(z_n)], \\ x_{n+1} = \mathbb{P}_\Omega[y_n - \mu\nabla f(y_n)], \end{cases} \tag{5.4}$$

where $\{\alpha_n\}$ is a sequence in (0,1). Now, we approximate the proposed algorithm (5.4) to estimate the solution of (5.1).

Theorem 5.1. *Let Ω be a nonempty closed convex subset of real Hilbert space $\mathbb{H}$. Let $F \colon \Omega \to \mathbb{R}$ be a convex, Freschet differentiable mapping, and ∇F is a-inverse strongly monotone mapping. Suppose that the convex minimization problem (5.1) has a solution and condition (3.1) holds. Then, the sequence $\{x^*\}$ generated by (5.4) converges strongly to x^* which solves convex minimization problem (5.1) with the following error estimates:*

$$||x_{n+1} - w^*|| \leq \varphi^{3(n+1)}||x_0 - w^*|| \prod_{k=0}^{n}[1 - \alpha_k(1-\varphi)], \quad \forall n \in \mathbb{N}, \tag{5.5}$$

where

$$\varphi = \left| \frac{\alpha_1 - \mu}{\alpha_1} \right|. \tag{5.6}$$

Proof. The desired conclusion is accomplished by taking $f = \nabla F$ and $T, g = I$ in Theorem 3.1.

Example 5.2. Let $\mathbb{H} = L^2[0,1] = \{G\colon [0,1] \to \mathbb{R}\colon \int_0^1 G^2(x)dx < \infty\}$. Then, $(\mathbb{H}, ||.||_2)$ is a Hilbert space given by

$$||G(x)||_2{}^2 = \langle G(x), G(x) \rangle = \int_0^1 G^2(x)dx. \tag{5.8}$$

Consider a closed convex subset $\Omega = \{G \in L^2[0,1]\colon ||G(x)||_2{}^2 \leq 1\}$ of $\mathbb{H}$. Define $F\colon \Omega \to \mathbb{R}$ by $F(G) = ||G(x)||_2{}^2$. Then, $G(x) = 0$ is a unique minimum of a convex function f, and f is the Fréchet differentiable at G. The gradient $\nabla F\colon \Omega \to \mathbb{H}$ is evaluated as $\nabla F(G) = 2G$. Then, for all $G_1, G_2 \in \Omega$, we get

$$\begin{aligned} \langle \nabla F(G_1) - \nabla F(G_2), G_1 - G_2 \rangle &= \int_0^1 (2G_1(x) - 2G_2(x))(G_1 - G_2)du \\ &= 2\int_0^1 (G_1(x) - G_2(x))^2 du \\ &\geq -\frac{1}{4}(2G_1(x) - 2G_2(x))^2 du \\ &= \frac{1}{4}||\nabla F(G_1) - \nabla F(G_2)||_2{}^2, \end{aligned} \tag{1}$$

i.e., ∇F is $1/4$ inverse strongly monotone. Also, $\varphi < 1$ for $\mu = 1/4$. Thus, all the assumptions of Theorem 5.1 are satisfied, and for $\alpha_n = 1/n+1$, the sequence $\{x_n\}$ generated by (5.4) is given as

$$\begin{cases} x_0 \in \Omega, \\ w_n = \left(1 - \frac{1}{n+1}\right) x_n + \frac{1}{n+1}\mathbb{P}_\Omega \left[\frac{1}{2}x_n\right], \\ z_n = \mathbb{P}_\Omega \left[\frac{1}{2}w_n\right], \\ y_n = \mathbb{P}_\Omega \left[\frac{1}{2}z_n\right], \\ x_{n+1} = \mathbb{P}_\Omega \left[\frac{1}{2}y_n\right], \end{cases} \tag{5.9}$$

where $\mathbb{P}_\Omega = \begin{cases} G, & G \in \Omega, \\ G/||G||, & G \notin \Omega. \end{cases}$ Then the sequence $\{x_n\}$ generated by (5.4) converges to 0 function.

5.2. Split Feasibility Problem:

This subsection is devoted to utilization of Theorem 3.1 to examine a split feasibility problem (SFP). Let Ω_1 and $\Omega 2$ be nonempty closed convex subsets of real Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Let $A\colon \mathbb{H}_1 \to \mathbb{H}_2$ be a bounded linear operator. The SFP is to locate a point x^*, such that

$$x^* \in \Omega_1 \colon Ax^* \in \Omega_2. \tag{5.10}$$

Let γ denotes the solution set of SFP (5.10); then,

$$\gamma =\colon \{x^* \in \Omega_1 \colon Ax^* \in \Omega_2\} = \Omega_1 \cap A^{-1}\Omega_2. \tag{5.11}$$

A class of inverse problems has been solved by using SFP, for example, [6]. In [35], Xu established the relationship between SFP (5.10) and the fixed point of problem $\mathbb{P}_{\Omega_1}[I - \mu A^*(I - \mathbb{P}_{\Omega_2})A]$. More precisely, for $\mu > 0$, $x^* \in \Omega_1$ solves SFP (5.10) if and only if $\mathbb{P}_{\Omega_1}[I - \mu A^*(I - \mathbb{P}_{\Omega_2})A](x^*) = x^*$. Byrne [5] posed the following iterative algorithm for solving SFP (5.10) as follows:

$$x_{n+1} = \mathbb{P}_{\Omega_1}[I - \mu A^*(I - \mathbb{P}_{\Omega_2})A](x^*), \ \forall n \geq 0. \tag{5.12}$$

where $0 < \mu < 2/||A||^2$, A^* is the adjoint of operator A, and $\mathbb{P}_{\Omega_1}$ and $\mathbb{P}_{\Omega_2}$ are the projections onto Ω_1 and Ω_2, respectively. Note that the operator $\mathbb{P}_{\Omega_1}[I - \mu A^*(I - \mathbb{P}_{\Omega_2})A]$ with $0 < \mu < 2/||A||^2$ is nonexpansive. Now, we propose following iterative algorithm to solve SFP (5.10):

$$\begin{cases} x_n \in \Omega_1, \\ w_n = (1-\alpha_n)x_n + \alpha_n \mathbb{P}_{\Omega_1}[I - \mu A^*(I - \mathbb{P}_{\Omega_2})A](x_n), \\ z_n = \mathbb{P}_{\Omega_1}[I - \mu A^*(I - \mathbb{P}_{\Omega_2})A](w_n), \\ y_n = \mathbb{P}_{\Omega_1}[I - \mu A^*(I - \mathbb{P}_{\Omega_2})A](z_n), \\ x_{n+1} = \mathbb{P}_{\Omega_1}[I - \mu A^*(I - \mathbb{P}_{\Omega_2})A](y_n), \end{cases} \tag{5.13}$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $0 < \mu < 2/||A||^2$.

Theorem 5.3. *Suppose that $\gamma \neq \emptyset$ and condition (3.1) holds. Then, the sequence $\{x_n\}$ initiated in (5.13) converges weakly to x^*, which solves SFP (5.10) with following error estimates:*

$$||x_{n+1} - w^*|| \leq \varphi^{3(n+1)}||x_0 - w^*|| \prod_{k=0}^{n}[1 - \alpha_k(1-\varphi)], \quad \forall n \in \mathbb{N}, \tag{5.14}$$

where

$$\varphi = \left| \frac{\alpha_1 - \mu}{\alpha_1} \right|. \tag{5.15}$$

Proof. The desired conclusion follows by taking $\nabla F = A^*[(I - \mathbb{P}_{\Omega_2})A]$ and $\Im, g = I$ in Theorem 3.1.

6. Conclusion

In this study, a new iterative algorithm (2.18) has been proposed and employed to explore convergence analysis. Using MH iterative procedure, a common solution of the generalized variational inequality problem and fixed points of nonexpansive mapping is investigated, and theoretical findings are verified by a numerical example. Furthermore, we show that the MH iterative technique converges faster than the previous iterative scheme. Finally, we used the MH iterative approach to analyze the convex optimization problem and the split feasibility problem.

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