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STUDY ON LORENTZIAN PARA-KENMOSTU MANIFOLDS ADMITTING GENERALIZED TANAKA-WEBSTER CONNECTION

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Abstract: In this study, we analyze the curvature properties and Ricci solitons in Lorentzian para-Kenmotsu manifolds using the generalized Tanaka-Webster connection. Here, using a generalized Tanaka-Webster connection, we examine the recurring conditions of the Lorentzian para-Kenmotsu manifold, projective curvature tensor, and conharmonic curvature tensor. Furthermore, using a generalized Tanaka-Webster connection, we investigate Ricci solitons on Lorentzian para-Kenmotsu manifolds.

Keywords and Phrases: Lorentzian para-Kenmotsu Manifold, Generalized Tanaka - Webster connection, ϕ-recurrent, Ricci-solitons.

2020 Mathematics Subject Classification: 53C05, 53C07, 53C50.

1. Introduction, Notations and Definitions

On a non-degenerate pseudo-Hermitian CR-manifold, the Tanaka-Webster connection is a canonical affine connection [18, 21]. For contact metric manifolds, Tanno [19] defined the generalized Tanaka-Webster connection via the canonical connection, which is equivalent to the Tanaka-Webster connection provided that the corresponding CR-structure is integrable. Numerous writers have recently examined the generalized Tanaka-Webster link in Kenmotsu manifolds [5, 13, 15].

Hamilton [6] first proposed the idea of Ricci solitons in 1982. They are Einstein metrics in their most natural form.

$$(L_X g)(U,V) + 2S(U,V) + 2\lambda g(U,V) = 0, \tag{1.1}$$

is the definition of a Ricci soliton (g, V, λ) on a Riemannian manifold (M, g), where L_X is the Lie-derivative of Riemannian metric g along a vector field X, λ is a constant, and U and V are arbitrary vector fields.
An n-dimensional differentiable manifold M equipped with a structure (ϕ, ξ, η, g) is classified as a Lorentzian almost paracontact metric manifold if it possesses a (1, 1)-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g that satisfy the following conditions [1]

$$\eta(\xi) = -1, \tag{1.2}$$

$$\phi^2 U = U + \eta(U)\xi, \tag{1.3}$$

$$\phi\xi = 0, \eta(\phi U) = 0, \tag{1.4}$$

$$g(\phi U, \phi V) = g(U,V) + \eta(U)\eta(V), \tag{1.5}$$

$$g(U,\xi) = \eta(U), \tag{1.6}$$

$$\Phi(U,V) = \Phi(V,U) = g(U, \phi V). \tag{1.7}$$

A Lorentzian almost Paracontact manifold M is said to be a Lorentzian para-Sasakian manifold if

$$(\nabla_U \phi)V = g(U,V)\xi + \eta(V)U + 2\eta(U)\eta(V)\xi, \tag{1.8}$$

for any vector fields U, V on M. Now, we define Lorentzian para-Kenmotsu manifold: **(i)** A Lorentzian almost paracontact manifold M is referred as a Lorentzian para-Kenmotsu manifold if the following condition holds for any vector fields U and V on M:

$$(\nabla_U \phi)V = -g(\phi U, V)\xi - \eta(V)\phi U, \tag{1.9}$$

for any vector fields U, V on M.

In a Lorentzian para-Kenmotsu manifold, we have

$$\nabla_U \xi = -U - \eta(U)\xi, \tag{1.10}$$

$$(\nabla_U \eta)V = -g(U,V) - \eta(U)\eta(V), \tag{1.11}$$

where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g. Further, on a Lorentzian para-Kenmotsu manifold M, the following results hold:

$$g(R(U,V)W,\xi) = g(V,W)\eta(U) - g(U,W)\eta(V), \tag{1.12}$$

$$R(\xi,U)V = -R(U,\xi)V = g(U,V)\xi - \eta(V)U, \tag{1.13}$$

$$R(U,V)\xi = \eta(V)U - \eta(U)V, \tag{1.14}$$

$$R(\xi,U)\xi = U + \eta(U)\xi, \tag{1.15}$$

$$S(U,\xi) = (n-1)\eta(U), S(\xi,\xi) = -(n-1), \tag{1.16}$$

$$Q\xi = (n-1)\xi, \tag{1.17}$$

$$S(\phi U,\phi V) = S(U,V) + (n-1)\eta(U)\eta(V), \tag{1.18}$$

for any vector fields $U, V, W \in \chi(M)$.

First Bianchi identity with respect to Levi-Civita connection is given by

$$R(U,V)W + R(V,W)U + R(W,U)V = 0. \tag{1.19}$$

(ii) If Ricci tensor of Lorentzian para-Kenmotsu manifold satisfies the condition

$$S(U,V) = ag(U,V) + b\eta(U)\eta(V) \tag{1.20}$$

where a and b are scalar functions on M, then manifold is said to be η-Einstein manifold. If the Ricci tensor of the Lorentzian para-Kenmotsu manifold satisfies the condition

$$S(U,V) = ag(U,V) + b\eta(U)\eta(V) + c\varphi(U,V), \tag{1.21}$$

where a,b and c are scalar functions on M and $\varphi(U,V) = g(\phi U,V)$, then manifold is said to be generalized η-Einstein manifold [23]. If $c = 0$, then the manifold reduces to an $\eta-$Einstein manifold.

2. Curvature properties of Lorentzian para-Kenmotsu manifolds admitting generalized Tanaka-Webster connection

In this paper, we use the symbol $*$ to represent the quantities associated with the generalized Tanaka-Webster connection. The generalized Tanaka-Webster connection ∇^* in terms of Levi-Civita connection ∇ is given by [5, 20]

$$\nabla^*_U V = \nabla_U V - \eta(V)\nabla_U \xi + (\nabla_U \eta)(V)\xi - \eta(U)\phi V, \tag{2.1}$$

for any vector fields U, V on M. Using (1.10) and (1.11), the above equation reduces

$$\nabla^*_U V = \nabla_U V - g(U,V)\xi + \eta(V)U - \eta(U)\phi V. \tag{2.2}$$

By taking $V = \xi$ in (2.2) and using (1.10), we get

$$\nabla^*_U \xi = -2U - 2\eta(U)\xi. \tag{2.3}$$

We now find Riemann curvature tensor R^*, using (2.2), we get

$$\begin{aligned} R^*(U,V)W &= R(U,V)W + 3g(V,W)U - 3g(U,W)V \\ &\quad + 2\eta(U)g(V,W)\xi - 2\eta(V)g(U,W)\xi - 2\eta(U)\eta(W)V \\ &\quad + 2\eta(V)\eta(W)U - 2\eta(U)\eta(W)\phi V + 2\eta(V)\eta(W)\phi U. \end{aligned} \tag{2.4}$$

Taking $W = \xi$ in (2.4), and using (1.4), we get

$$R^*(U,V)\xi = 2\eta(V)U - 2\eta(U)V + 2\eta(U)\phi V - 2\eta(V)\phi U. \tag{2.5}$$

On contracting (2.4), we obtain the Ricci tensor S^* of a Lorentzian para-Kenmotsu manifold with respect to the generalized Tanaka-Webster connection ∇^* as

$$S^*(U,V) = S(U,V) + (3n-5)g(U,V) + (2n + 2\psi - 4)\eta(U)\eta(V), \tag{2.6}$$

where $\psi = trace(\phi)$.

This gives

$$Q^*V = QV + (3n-5)V + (2n + 2\psi - 4)\eta(V)\xi. \tag{2.7}$$

Contracting with respect to U and V in (2.6), we get

$$r^* = r + (3n-4)(n-1) - 2\psi, \tag{2.8}$$

where, r^* is the scalar curvatures with respect to the generalized Tanaka-Webster connection ∇^* and r is the scalar curvature with respect to the Levi-Civita connection ∇.

The projective curvature tensor P^* [22] associated with the generalized Tanaka-Webster connection ∇^* is defined as follows:

$$P^*(U,V)W = R^*(U,V)W - \frac{1}{(n-1)}\{S^*(V,W)U - S^*(U,W)V\}. \tag{2.9}$$

If the projective curvature tensor P^* associated with the generalized Tanaka-Webster connection ∇^* is zero, then from (2.9), it follows that

$$R^*(U,V)W = \frac{1}{n-1}\{S^*(V,W)U - S^*(U,W)V\}. \tag{2.10}$$

Now, in view of (2.4) and (2.6), (2.10) takes the form

$$\begin{aligned} &g(R(U,V)W,Y) + 3g(V,W)g(U,Y) - 3g(U,W)g(V,Y) + 2\eta(U)g(V,W)\eta(Y) \\ &- 2\eta(V)\eta(Y)g(U,W) - 2\eta(U)\eta(W)g(V,Y) + 2\eta(V)\eta(W)g(U,Y) \\ &- 2\eta(U)\eta(W)g(\phi V,Y) + 2\eta(V)\eta(W)g(\phi U,Y) = \frac{1}{n-1}\{S(V,W)g(U,Y) \\ &+ (3n-5)g(V,W)g(U,Y) + (2n+2\psi-4)\eta(V)\eta(W)g(U,Y) \\ &- S(U,W)g(V,Y) - (3n-5)g(U,W)g(V,Y) - (2n+2\psi-4)\eta(U)\eta(W)g(V,Y)\}. \end{aligned} \tag{2.11}$$

Now, taking $Y = \xi$ and $U = \xi$ in (2.11), we obtain

$$S(V,W) = (-n+3)g(V,W) + (-2n+4)\eta(V)\eta(W). \tag{2.12}$$

Contracting the above equation (2.12), we get

$$r = -(n-1)(n-4). \tag{2.13}$$

This leads to the following:

Theorem 2.1. *If in a Lorentzian para-Kenmotsu manifold, projective curvature tensor with respect to generalized Tanaka-Webster connection vanished then manifolds reduces to η-Einstein Manifold.*

Now, interchanging U and V in (2.9), we get

$$P^*(V,U)W = R^*(V,U)W - \frac{1}{n-1}\{S^*(U,W)V - S^*(V,W)U\}. \tag{2.14}$$

After summing up (2.9) and (2.14) and applying the formula $R(U,V)W = -R(V,U)W$, we obtain

$$P^*(U,V)W + P^*(V,U)W = 0. \tag{2.15}$$

From (2.4), (2.9) and (1.19) we obtain

$$P^*(U,V)W + P^*(V,W)U + P^*(W,U)V = 0. \tag{2.16}$$

The projective curvature tensor with respect to the generalized Tanaka-Webster connection in a Lorentzian para-Kenmotsu manifold is therefore skew-symmetric and cyclic, as we may infer from (2.15) and (2.16).

(i) A Lorentzian para-Kenmotsu manifold is called ϕ-projectively semi-symmetric with respect to the generalized Tanaka-Webster connection ∇^* if

$$P^*(U,V)\cdot\phi = 0, \tag{2.17}$$

for all vector fields U,V on M.

Now, from (2.17), we have

$$(P^*(U,V)\cdot\phi)W = P^*(U,V)\phi W - \phi P^*(U,V)W = 0. \tag{2.18}$$

Using (2.9), (2.4) and (2.6) in (2.18), we get

$$\begin{aligned}
&R(U,V)\phi W + 3g(V,\phi W)U - 3g(U,\phi W)V + 2\eta(U)g(V,\phi W)\xi \\
&\quad - 2\eta(V)g(U,\phi W)\xi - \phi(R(U,V)W) - 3g(V,W)\phi U + 3g(U,W)\phi V \\
&\quad + 2\eta(U)\eta(W)\phi V - 2\eta(V)\eta(W)\phi U + 2\eta(U)\eta(W)V - 2\eta(V)\eta(W)U \\
&\quad + \frac{1}{n-1}\{S(V,W)\phi U + (3n-5)g(V,W)\phi U + (2n+2\psi-4)\eta(V)\eta(W)\phi U \\
&\quad - S(U,W)\phi V - (3n-5)g(U,W)\phi V - (2n+2\psi-4)\eta(U)\eta(W)\phi V - S(V,\phi W)U \\
&\quad - (3n-5)g(V,\phi W)U + S(U,\phi W)V + (3n-5)g(U,\phi W)V\} = 0.
\end{aligned} \tag{2.19}$$

Taking $V=\xi$ in (2.19), using (1.4) and (1.6), we get

$$S(U,\phi W)\xi = (-n+3)g(U,\phi W)\xi + 2\psi\eta(W)\phi U - 2(n-1)\eta(W)U - 2(n-1)\eta(U)\eta(W)\xi. \tag{2.20}$$

Taking inner product of (2.20) with ξ and replacing U by ϕU and using (1.5) and (1.8), we get

$$S(U,W) = (-n+3)g(U,W) + (-2n+4)\eta(U)\eta(W), \tag{2.21}$$

and

$$r = (-n+4)(n-1). \tag{2.22}$$

Again, by substituting (2.21) in (2.9), we obtain

$$\begin{aligned}
P^*(U,V)W = {}& R(U,V)W + g(V,W)U - g(U,W)V + 2\eta(U)g(V,W)\xi \\
& - 2\eta(V)g(U,W)\xi + (\frac{2\psi}{n-1} - 2)\eta(U)\eta(W)V + (2 - \frac{2\psi}{n-1})\eta(V)\eta(W)U \\
& - 2\eta(U)\eta(W)\phi V + 2\eta(V)\eta(W)\phi U.
\end{aligned} \tag{2.23}$$

Thus we state the following:

Theorem 2.2. *If a Lorentzian para-Kenmotsu manifold with respect to the generalized Tanaka-Webster connection ∇^* is ϕ- projectively semi-symmetric then it is η-Einstein manifold.*

If in a Lorentzian para-Kenmotsu manifold M, $(P^*(U,V)\cdot S^*)(W,Y) = 0$. Then, we have

$$S^*(P^*(U,V)W,Y) + S^*(W,P^*(U,V)Y) = 0. \tag{2.24}$$

Using the equations (2.9), (2.4), (2.6) and (2.7) into the above equation, we have

$$\begin{aligned} &2(2n-2\psi-2)\eta(Y)g(V,W)-2\eta(W)S(V,Y)-2(3n-5)\eta(W)g(V,Y)\\ &-2(2n+2\psi-4)\eta(V)\eta(W)\eta(Y)+2\eta(W)S(\phi V,Y)+2(3n-5)\eta(W)g(\phi V,Y)\\ &+2(2n-2\psi-2)\eta(W)g(V,Y)-2\eta(Y)S(V,W)-2(3n-5)\eta(Y)g(V,W)\\ &-2(2n+2\psi-4)\eta(V)\eta(W)\eta(Y)+2\eta(Y)S(\phi V,W)+2(3n-5)\eta(Y)g(\phi V,W)=0. \end{aligned} \tag{2.25}$$

Using $U=Y=\xi$ into above equation, we get

$$\begin{aligned} &(n+2\psi-4)g(V,W)+(2n+2\psi-4)\eta(V)\eta(W)\\ &+S(V,W)-S(\phi V,W)-(3n-5)g(\phi V,W)=0. \end{aligned} \tag{2.26}$$

Thus, we can state that

Theorem 2.3. *If a Lorentzian para-Kenmotsu manifold M with respect to the generalized Tanaka-Webster connection ∇^* satisfies $P^*\cdot S^*=0$, then $(n+2\psi-4)g(V,W)+(2n+2\psi-4)\eta(V)\eta(W)+S(V,W)-S(\phi V,W)-(3n-5)g(\phi V,W)=0$.*

With respect to the generalized Tanaka-Webster connection ∇^* the conharmonic curvature tensor [4], K^* is defined by

$$\begin{aligned} K^*(U,V)W = R^*(U,V)W - \frac{1}{(n-2)}\{&S^*(V,W)U - S^*(U,W)V\\ &+g(V,W)Q^*U - g(U,W)Q^*V\}. \end{aligned} \tag{2.27}$$

If the conharmonic curvature tensor K^* associated with the generalized Tanaka-Webster connection ∇^* is zero, then from equation (2.27), we obtain

$$R^*(U,V)W = \frac{1}{(n-2)}\{S^*(V,W)U - S^*(U,W)V + g(V,W)Q^*U - g(U,W)Q^*V\}. \tag{2.28}$$

By using the equations (2.4), (2.6) and (2.7) into equation (2.28), we get

$$\begin{aligned} &(n-2)g(R(U,V)W,Y)-(3n-4)g(V,W)g(U,Y)+(3n-4)g(U,W)g(V,Y)\\ &-2\psi\eta(U)\eta(Y)g(V,W)+2\psi\eta(V)\eta(Y)g(U,W)+2\psi\eta(U)\eta(W)g(V,Y)-2\psi\eta(V)\\ &\eta(W)g(U,Y)-2(n-2)\eta(U)\eta(W)g(\phi V,Y)+2(n-2)\eta(V)\eta(W)g(\phi U,Y)\\ &=S(V,W)g(U,Y)-S(U,W)g(V,Y)+g(V,W)g(QU,Y)-g(U,W)g(QV,Y). \end{aligned} \tag{2.29}$$

Taking $U=Y=\xi$ into the above equation, we get

$$S(V,W)=(-3n+2\psi+3)g(V,W)+(-4n+2\psi+4)\eta(V)\eta(W). \tag{2.30}$$

So, we state the following:

Theorem 2.4. *If a Lorentzian para-Kenmotsu manifold is conharmonically flat with respect to the generalized Tanaka-Webster connection, then it reduces to an η-Einstein manifold.*

A Lorentzian para-Kenmotsu manifold associated with the generalized Tanaka-Webster connection ∇^* is termed recurrent if its curvature tensor R^* meets the following condition:

$$(\nabla^*_Y R^*)(U,V)W = A(Y)R^*(U,V)W, \tag{2.31}$$

where R^* is the curvature tensor corresponding to the connection ∇^*, and A is a 1-form. using (2.31), we have

$$\begin{aligned}&\nabla^*_Y(R^*(U,V)W) - R^*(\nabla^*_Y U,V)W - R^*(U,\nabla^*_Y V)W\\&- R^*(U,V)\nabla^*_Y W = A(Y)R^*(U,V)W.\end{aligned} \tag{2.32}$$

Making use of (2.2), (2.4) and (2.6) in (2.32), we get

$$\begin{aligned}&- g(Y,R(U,V)W)\xi + \eta(R(U,V)W)Y - \eta(Y)\phi(R(U,V)W) - 3g(V,W)g(Y,U)\xi\\&- 4\eta(U)g(V,W)Y + 3g(U,W)g(Y,V)\xi + 4\eta(V)g(U,W)Y - 8\eta(U)\eta(Y)g(V,W)\xi\\&+ 8\eta(Y)\eta(V)g(U,W)\xi + 8\eta(U)\eta(Y)\eta(W)V + 4\eta(U)g(W,Y)V - 8\eta(V)\eta(W)\eta(Y)U\\&- 4\eta(V)g(W,Y)U - 8\eta(V)\eta(W)\eta(Y)\phi U + 4\eta(U)g(W,Y)\phi V + \eta(U)\eta(W)g(Y,\phi V)\xi\\&- 4\eta(W)g(V,Y)\phi U - \eta(V)\eta(W)g(Y,\phi U)\xi - \eta(U)R(Y,V)W + 8\eta(U)\eta(W)\eta(Y)\phi V\\&+ \eta(Y)R(\phi U,V)W - 3\eta(Y)g(\phi U,W)V - 4\eta(Y)\eta(V)g(\phi U,W)\xi - \eta(V)R(U,Y)W\\&+ \eta(Y)R(U,\phi V)W + 6\eta(Y)g(\phi V,W)U + 4\eta(Y)\eta(U)g(\phi V,W)\xi\\&+ g(Y,W)R(U,V)\xi - \eta(W)R(U,V)Y + \eta(Y)R(U,V)\phi W - 3\eta(W)g(V,Y)U\\&+ 3\eta(W)g(U,Y)V - 3\eta(Y)g(U,\phi W)V - 2\eta(U)\eta(W)g(V,Y)\xi + 2\eta(V)\eta(W)g(U,Y)\xi\\&- 4\eta(V)g(Y,W)\phi U + 4\eta(W)g(U,Y)\phi V = A(Y)\{3g(V,W)U - 3g(U,W)V\\&+ 2\eta(U)g(V,W)\xi - 2\eta(V)g(U,W)\xi - 2\eta(U)\eta(W)V + 2\eta(V)\eta(W)U\\&- 2\eta(U)\eta(W)\phi V + 2\eta(V)\eta(W)\phi U\}.\end{aligned} \tag{2.33}$$

Replacing W by ξ and using (1.2), (1.3), (1.4), (1.13) and (1.14) into equation (2.33) we get

$$\begin{aligned}&- \eta(V)g(Y,U)\xi + \eta(U)g(Y,V)\xi - \eta(Y)\eta(V)\phi U + \eta(Y)\eta(U)\phi V\\&- 5\eta(V)g(U,Y)\xi + 5\eta(U)g(V,Y)\xi - 4\eta(U)\eta(Y)V + 4\eta(V)\eta(Y)U\\&+ 5\eta(V)\eta(Y)\phi U - 5\eta(U)\eta(Y)\phi V - \eta(U)g(Y,\phi V)\xi + 4g(V,Y)\phi U\\&+ \eta(V)g(Y,\phi U)\xi + R(U,V)Y + 3g(V,Y)U - 3g(U,Y)V\\&- 4g(U,Y)\phi V = A(Y)\{\eta(V)U - \eta(U)V + 2\eta(U)V - 2\eta(V)\phi U\}.\end{aligned} \tag{2.34}$$

Taking an inner product with Z in (2.34), we have

$$\begin{aligned}
&-6\eta(V)\eta(Z)g(Y,U)+6\eta(U)\eta(Z)g(V,Y)+4\eta(V)\eta(Y)g(\phi U,Z)\\
&-4\eta(U)\eta(Y)g(\phi V,Z)-4\eta(U)\eta(Y)g(V,Z)+4\eta(V)\eta(Y)g(U,Z)\\
&-\eta(U)\eta(Z)g(Y,\phi V)+4g(V,Y)g(\phi U,Z)+\eta(V)\eta(Z)g(Y,\phi U)\\
&+g(R(U,V)Y,Z)+3g(V,Y)g(U,Z)-3g(U,Y)g(V,Z)\\
&-4g(U,Y)g(\phi V,Z)=A(Y)\{\eta(V)g(U,Z)-\eta(U)g(V,Z)\\
&+2\eta(U)g(\phi V,Z)-2\eta(V)g(\phi U,Z)\}.
\end{aligned} \tag{2.35}$$

Let $\{e_1, e_2, e_3, ..., e_n = \xi\}$ be a local orthonormal basis of vector fields in M.Then by putting $U = Z = e_i$ in (2.35) and summing over i from 1 to n, we obtain

$$\begin{aligned}
&(4n+4\psi-10)\eta(V)\eta(Y)+(-9+3n+4\psi)g(V,Y)+S(V,Y)\\
&-3g(Y,\phi V)=(n-1-2\psi)A(Y)\eta(V).
\end{aligned} \tag{2.36}$$

Suppose the associated 1-form A is equal to the associated 1-form η, then from (2.36), we get

$$S(U,V)=(-3n-6\psi+9)\eta(U)\eta(V)+(9-3n-4\psi)g(U,V)+3g(\phi U,V). \tag{2.37}$$

Thus, we state the following:

Theorem 2.5. *If a Lorentzian para-Kenmotsu manifold with respect to the generalized Tanaka-Webster connection ∇^* is recurrent and the associated 1-form A is equal to the associated 1-form η, then the manifold is a generalized η-Einstein manifold.*

3. Ricci solitons in Lorentzian para-Kenmotsu manifold with generalized Tanaka Webster connection

Suppose the Lorentzian para-Kenmotsu manifold M supports a Ricci soliton with respect to the generalized Tanaka-Webster connection ∇^*. Then, we have:

$$(L^*_X g)(U,V)+2S^*(U,V)+2\lambda g(U,V)=0. \tag{3.1}$$

If the potential vector field X is pointwise collinear with the structure vector field ξ, meaning $X = b\xi$ where b is a function on M, then equation (3.1) leads to:

$$bg(\nabla^*_U\xi,V)+(Ub)\eta(V)+bg(U,\nabla^*_V\xi)+(Vb)\eta(U)+2S^*(U,V)+2\lambda g(U,V)=0. \tag{3.2}$$

Using (2.3) and (2.6) into (3.2), we get

$$\begin{aligned}
&(6n-4b+2\lambda-10)g(U,V)+(4n-4b+4\psi-8)\eta(U)\eta(V)\\
&+(Ub)\eta(V)+(Vb)\eta(U)+2S(U,V)=0.
\end{aligned} \tag{3.3}$$

By setting $V = \xi$ in (3.3), we get

$$Ub = \{\xi b + 4n - 4 + 2\lambda - 4\psi\}\eta(U). \tag{3.4}$$

Again, replacing U by ξ in (3.4), we get

$$(\xi b) = -2n + 2 - \lambda + 2\psi. \tag{3.5}$$

Substituting this into (3.4), we get

$$Ub = \{2n - 2\psi + \lambda - 2\}\eta(U). \tag{3.6}$$

By applying exterior derivative on (3.6), we get

$$(2n - 2\psi + \lambda - 2)d\eta = 0. \tag{3.7}$$

Since $d\eta \neq 0$ from (3.7), we get

$$\lambda = 2\psi + 2 - 2n. \tag{3.8}$$

Hence, the Ricci soliton is shrinking, steady and expanding according as $\psi < (n-1), \psi = (n-1)$ and $\psi > (n-1)$, respectively.

Theorem 3.1. *If a Lorentzian para-Kenmotsu manifold with respect to the generalized Tanaka-Webster connection admits a Ricci soliton (g, X, λ) where, X is pointwise collinear with ξ, then the Ricci soliton is classified as shrinking, steady, or expanding based on whether $\psi < (n-1)$, $\psi = (n-1)$, or $\psi > (n-1)$, respectively.*

4. Example of a Lorentzian para-Kenmotsu manifold admitting generalized Tanaka-Webster connection

We consider the 3-dimensional manifold

$$M^3 = \{(x, y, z) \in R^3 : z > 0\}, \tag{4.1}$$

where (x, y, z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M^3 defined by

$$e_1 = z\frac{\partial}{\partial x}, e_2 = z\frac{\partial}{\partial y}, e_3 = z\frac{\partial}{\partial z} = \xi. \tag{4.2}$$

Clearly, the above vectors are linearly independent at each point of M^3 and hence form a basis of T_pM^3. Let g be the Lorentzian metric defined by

$$g_{ij} = \begin{cases} 1, & \text{if } i = j \neq 3 \\ 0, & \text{if } i \neq j \\ -1, & \text{if } i = j = 3. \end{cases}$$

Let η be the 1-form on M^3 defined as $\eta(X) = g(X, e_3) = g(X, \xi)$ for all $X \in \chi(M)$ and let ϕ be the (1,1)-tensor field on M^3 defined as

$$\phi e_1 = -e_2, \phi e_2 = -e_1, \phi e_3 = 0. \tag{4.3}$$

Using linear property of ϕ and g, we have $\eta(\xi) = -1, \phi^2 X = X + \eta(X)\xi, \eta(\phi X) = 0$, $g(X, \xi) = \eta(X), g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$ for all $X, Y \in \chi(M)$. Let ∇ be the Levi-civita connection with respect to the Lorentzian metric g. Then, we have

$$[e_1, e_2] = 0, [e_2, e_1] = 0, [e_1, e_3] = -e_1, [e_3, e_1] = e_1, [e_2, e_3] = -e_2, [e_3, e_2] = e_2. \tag{4.4}$$

The Riemannian connection ∇ of the Lorentzian metric g is given by

$$\begin{aligned} 2g(\nabla_U V, W) &= Ug(V, W) + Vg(W, U) - Wg(U, V) \\ &\quad - g(U, [V, W]) + g(V, [W, U]) + g(W, [U, V]), \end{aligned} \tag{4.5}$$

which is known as *Koszul's formula.* we can easily calculate

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = -e_1, \nabla_{e_2} e_1 = 0,$$

$$\nabla_{e_2} e_2 = -e_3, \nabla_{e_2} e_3 = -e_2, \nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = 0. \tag{4.6}$$

Let

$$U = \sum_{i=1}^{3} U^i e_i = U^1 e_1 + U^2 e_2 + U^3 e_3,$$

$$V = \sum_{i=1}^{3} V^i e_i = V^1 e_1 + V^2 e_2 + V^3 e_3,$$

$$W = \sum_{i=1}^{3} W^i e_i = W^1 e_1 + W^2 e_2 + W^3 e_3,$$

for all $U, V, W \in \chi(M)$.
Using

$$U = \sum_{i=1}^{3} U^i e_i = U^1 e_1 + U^2 e_2 + U^3 e_3,$$

and the properties of connection we can easily verify that $\nabla_U \xi = -U - \eta(U)\xi$ and $(\nabla_U \phi)V = -g(\phi U, V)\xi - \eta(V)\phi U$.
which shows that the chosen manifold is a Lorentzian para-Kenmotsu manifold of

dimension 3.
We know that

$$R(U,V)W = \nabla_U\nabla_V W - \nabla_V\nabla_U W - \nabla_{[U,V]}W. \tag{4.7}$$

From the equations (4.6) and (4.7), it can be verified that

$$\begin{aligned} R(e_1,e_2)e_1 = -e_2, R(e_1,e_3)e_1 = -e_3, R(e_2,e_3)e_1 = 0,\\ R(e_1,e_2)e_2 = e_1, R(e_1,e_3)e_2 = 0, R(e_2,e_3)e_2 = -e_3,\\ R(e_1,e_2)e_3 = 0, R(e_1,e_3)e_3 = -e_1, R(e_2,e_3)e_3 = -e_2. \end{aligned} \tag{4.8}$$

Further, we find the following:

$$\nabla^*_{e_1}e_1 = -2e_3, \nabla^*_{e_1}e_2 = 0, \nabla^*_{e_1}e_3 = -2e_1, \nabla^*_{e_2}e_1 = 0, \tag{4.9}$$

$$\nabla^*_{e_2}e_2 = -2e_3, \nabla^*_{e_2}e_3 = -2e_2, \nabla^*_{e_3}e_1 = -e_2, \nabla^*_{e_3}e_2 = -e_1, \nabla^*_{e_3}e_3 = 0.$$

Now, we calculate the value of $R^*(U,V)W$

$$\begin{aligned} R^*(e_1,e_2)e_1 = -4e_2, R^*(e_1,e_3)e_1 = -2e_3, R^*(e_2,e_3)e_1 = 0,\\ R^*(e_1,e_2)e_2 = 4e_1, R^*(e_1,e_3)e_2 = 0, R^*(e_2,e_3)e_2 = -2e_3,\\ R^*(e_1,e_2)e_3 = 0, R^*(e_1,e_3)e_3 = -2e_1 - 2e_2, R^*(e_2,e_3)e_3 = -2e_1 - 2e_2. \end{aligned} \tag{4.10}$$

Using the above results we obtain the Ricci tensor as follows:

$$S(e_1,e_1) = g((R(e_1,e_1)e_1,e_1)) + g((R(e_2,e_1)e_1,e_2)) + g((R(e_3,e_1)e_1,e_3)) = 2. \tag{4.11}$$

Similarly, we have

$$S(e_2,e_2) = 2, S(e_3,e_3) = -2.$$

Now, by using the formula

$$S^*(U,V) = g(R^*(e_1,U)V,e_1) + g(R^*(e_2,U)V,e_2) + g(R^*(e_3,U)V,e_3),$$

We get the following results:

$$S^*(e_1,e_1) = 6, S^*(e_2,e_2) = 6, S^*(e_3,e_3) = -4.$$

Therefore, above manifold is an Lorentzian para-Kenmotsu manifold admitting generalized Tanaka-Webster connection.

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