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SOME RESULTS ON SUBMANIFOLDS OF A α-COSYMPLECTIC MANIFOLD WITH TORQUED VECTOR FIELD

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Abstract: In this paper, we examine a submanifold N of an α-cosymplectic manifold equipped with a torqued vector field τ. We also investigate submanifolds that admit a $*$-η-Ricci soliton within the framework of α-cosymplectic manifolds with torqued vector field τ. We establish the necessary conditions for such a submanifold to reduce to a simpler form and demonstrate that the tangential component of τ acts as a torse-forming vector field on N. Finally, we present an example of a 3-dimensional submanifold of a 5-dimensional α-cosymplectic manifold which verifies our results.

Keywords and Phrases: α-cosymplectic manifold, $*$-η-Ricci soliton, Torqued vector field.

2020 Mathematics Subject Classification: 53C15, 53C25, 53C17, 53D15, 53D10.

1. Introduction

The study of manifolds is highly regarded by geometers and physicists for its broad applications in geometry, physics, and relativity. By examining manifolds, geometers have utilized two essential tools-the Riemannian curvature tensor and the Ricci tensor-to understand their differential geometric properties. Over time, these tools have enabled the introduction of several new concepts to describe complex structures. One such concept is the $*$-Ricci tensor S^*, initially introduced by

Tachibana for almost Hermitian manifolds [18] and later explored by Hamada [11] on real hypersurfaces in non-flat complex space forms. The $*$-Ricci tensor on a Riemannian manifold $\tilde{M}$ is defined by [18]

$$S^*(X_1, X_2) = \frac{1}{2}(trace\{\phi \circ R(X_1, \phi X_2)\}), \tag{1.1}$$

for any vector fields $X_1, X_2 \in \Gamma(T\tilde{M})$. Here, $\tilde{R}$ represents the Riemannian curvature tensor, S^* denotes the $*$-Ricci tensor of type $(0, 2)$, ϕ is a tensor field of type $(1, 1)$ and $\Gamma(T\tilde{M})$ refers to the set of all smooth vector fields of $\tilde{M}$.

On the other hand, Hamilton [12] introduced the concept of a Ricci soliton in 1988 as a generalization of Einstein manifolds. Since then, various classes of Ricci solitons have been developed. An important example is the $*$-Ricci soliton, which was defined by Kaimakamis et al. in 2014. They explored this concept in the setting of real hypersurfaces within complex space forms [14]. A Riemannian metric g on a smooth manifold $\tilde{M}$ is termed a $*$-Ricci soliton, if there exists a smooth vector field V such that [14]

$$(L_V g)(X_1, X_2) + 2S^*(X_1, X_2) + 2\lambda g(X_1, X_2) = 0, \quad \lambda \in \mathbb{R},$$

for any vector fields $X_1, X_2 \in \tilde{M}$. Here, L_V denotes the Lie-derivative operator along the vector field V.

Dey and Roy introduced the concept of $*$-η-Ricci soliton as a generalization of $*$-Ricci soliton, defined as follows [6]:

$$(L_V g)(X_1, X_2)+2S^*(X_1, X_2)+2\lambda g(X_1, X_2)+2\mu\eta(X_1)\eta(X_2) = 0, \quad \lambda, \mu \in \mathbb{R}, \tag{1.2}$$

for any vector fields $X_1, X_2 \in \tilde{M}$. If $L_V g = \lambda g$, then the potential vector field V is conformal Killing, where λ is a function. If λ vanishes identically, then V is said to be a Killing vector field.

The $*$-η-Ricci solitons have been explored by several geometers, such as Dey and Roy [6], Dey et al. [7], Dey and Turki [8] and, others.

An important class of almost contact manifolds is the cosymplectic manifolds, which were introduced by Goldberg and Yano [10] in 1969. The simplest examples of almost cosymplectic manifolds include the products of almost Kählerian manifolds with the real line $\mathbb{R}$ or the circle S^1 [17]. Many mathematicians have studied almost cosymplectic manifolds, as seen in the literature ([17], [9], and [13]).

On the other hand, Riemannian manifolds with torqued vector fields, which are defined as a combination of concurrent and recurrent vector fields, were first introduced by Chen in [2]. A vector field τ on a Riemannian manifold $\tilde{M}$ is called a torqued vector field, if it satisfies the following two conditions:

$$\tilde{\nabla}_{X_1}\tau = fX_1 + \delta(X_1)\tau, \quad \delta(\tau) = 0, \tag{1.3}$$

where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$, and for any $X_1 \in \Gamma(T\tilde{M})$. The function f is referred to as the torqued function, and the 1-form δ is called the torqued form of τ. Chen characterized and studied rectifying submanifolds in Riemannian manifolds equipped with a torqued vector field [2]. In [3], Chen classified all torqued vector fields on Riemannian manifolds and investigated Ricci solitons with torqued potential fields.

Inspired by the aforementioned studies, this paper investigates submanifold N of an α-cosymplectic manifold $\tilde{M}$ equipped with a torqued vector field τ. We show that if the characteristic vector field ξ on N is both a torqued and recurrent vector field, then N must be a cosymplectic manifold. Additionally, we explore the characteristics of the tangential component of the vector field τ on the submanifold N of $\tilde{M}$. We demonstrate that if N admits a $*$-η-Ricci soliton within α-cosymplectic manifold $\tilde{M}$ with torqued vector field τ, then N becomes η-Einstein. Finally, we provide an example of a 3-dimensional submanifold of a 5-dimensional α-cosymplectic manifold to illustrate and verify some of our results.

2. Preliminaries

Let $\tilde{M}$ be an m-dimensional differential manifold equipped with the structure tensors (ϕ, ξ, η, g), where ϕ is a (1, 1)-tensor field, ξ is a characteristic vector field, η is a 1-form and g is a Riemannian metric. These tensors satisfy the following conditions [1]

$$\phi^2 X_1 = -X_1 + \eta(X_1)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi X_1) = 0, \tag{2.2}$$

$$g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2), \quad \eta(X_1) = g(X_1, \xi) \tag{2.3}$$

for all vector fields X_1 and X_2 on $\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-civita connection associated with the Riemannian metric g. Then $\tilde{M}$ is said to admit almost contact structure (ϕ, ξ, η, g) [1].
The fundamental 2-form Φ on $\tilde{M}$ is defined as follows:

$$\Phi(X_1, X_2) = g(X_1, X_2),$$

for all $X_1, X_2 \in \Gamma(T\tilde{M})$. An almost contact metric manifold $(\tilde{M}, \phi, \xi, \eta, g)$ is said to be almost cosymplectic [10] if $d\eta = 0$ and $d\Phi = 0$, where d denotes the exterior operator. An almost contact manifold $(\tilde{M}, \phi, \xi, \eta, g)$ is considered normal if the Nijenhuis torsion

$$N_\phi(X_1, X_2) = [\phi X_1, \phi X_2] - \phi[\phi X_1, X_2] - [X_1, \phi X_2] + \phi^2[X_1, X_2] + 2d\eta(X_1, X_2)\xi$$

vanishes for any vector fields X_1 and X_2. A normal almost cosymplectic manifold is called a cosymplectic manifold.
An almost contact metric manifold $\tilde{M}$ is said to be almost α-Kenmotsu if it satisfies the conditions $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, where α is a non-zero real constant.

Kim and Pak [15] introduced a new class of manifolds called almost α-cosymplectic manifolds by combining the concepts of almost α-Kenmotsu and almost cosymplectic manifolds, where α is a scalar. An almost α-cosymplectic manifold is defined by the following conditions:

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi,$$

for any real number α. A normal almost α-cosymplectic manifold is referred to as an α-cosymplectic manifold. Specifically, an α-cosymplectic manifold is:

- cosymplectic when $\alpha = 0$, or
- α-Kenmotsu when $\alpha \neq 0$, with $\alpha \in \mathbb{R}$.

In a α-cosymplectic manifold, we have [13]:

$$(\tilde{\nabla}_{X_1}\phi)X_2 = \alpha(g(\phi X_1, X_2)\xi - \eta(X_2)\phi X_1), \tag{2.4}$$

$$\tilde{\nabla}_{X_1}\xi = -\alpha\phi^2 X_1, \tag{2.5}$$

$$\tilde{R}(X_1, X_2)\xi = \alpha^2(\eta(X_1)X_2 - \eta(X_2)X_1), \tag{2.6}$$

$$S(X_1, \xi) = -\alpha^2(m-1)\eta(X_1), \tag{2.7}$$

for all $X_1, X_2 \in \Gamma(\tilde{M})$ and $\alpha \in \mathbb{R}$, where $\tilde{R}$ is the Riemannian curvature tensor and S is the Ricci curvature tensor of $\tilde{M}$, respectively.

Let N be an n-dimensional $(n \leq m)$ submanifold of an α-cosymplectic manifold $\tilde{M}$ with induced metric g. Let $\Gamma(TN)$ and $\Gamma(T^\perp N)$ denote the tangent and normal subspaces of N in $\tilde{M}$, respectively.
Then the Gauss and Weingarten formulas are given by [4]:

$$\tilde{\nabla}_{X_1}X_2 = \nabla_{X_1}X_2 + \sigma(X_1, X_2) \tag{2.8}$$

and

$$\tilde{\nabla}_{X1}X_5 = -A_{X_5}X_1 + \nabla^\perp_{X_1}X_5, \tag{2.9}$$

for all X_1, $X_2 \in \Gamma(TN)$ and $X_5 \in \Gamma(T^{\perp}N)$, where ∇ and $\nabla^{\perp}$ denote the induced connections on the tangent bundle TN and $T^{\perp}N$ of N, respectively.
The second fundamental form σ and shape operator A are related by the following equation:

$$g(A_{X_5}X_1, X_2) = g(\sigma(X_1, X_2), X_5), \tag{2.10}$$

for any X_1, $X_2 \in \Gamma(TN)$ and $X_5 \in \Gamma(T^{\perp}N)$.
The first covariant derivative of the second fundamental form σ is defined by [4]

$$(\tilde{\nabla}_{X_1}\sigma)(X_2, X_3) = \nabla^{\perp}_{X_1}\sigma(X_2, X_3) - \sigma(\nabla_{X_1}X_2, X_3) - \sigma(X_2, \nabla_{X_1}X_3), \tag{2.11}$$

for any $X_1, X_2, X_3 \in \Gamma(TN)$.

The mean curvature vector H of N is given by

$$H = \frac{1}{n}tr(\sigma) = \frac{1}{n}\sum_{i=1}^{n}\sigma(e_i, e_i), \tag{2.12}$$

where n is the dimension of N and $\{e_1, e_2, ...e_n\}$ is the local orthonormal frames of N.
A submanifold N is said to be totally umbilical if

$$\sigma(X_1, X_2) = g(X_1, X_2)H, \tag{2.13}$$

for any X_1, $X_2 \in \Gamma(TN)$. A submanifold N is said to be totally geodesic if $\sigma(X_1, X_2) = 0$ and N is said to be minimal if $H = 0$.
The equation of Gauss for any submanifold N of a Riemannian manifold $\tilde{M}$ is given by [4]

$$\begin{aligned}\tilde{R}(X_1, X_2)X_3 =& R(X_1, X_2)X_3 + A_{\sigma(X_1,X_3)}X_2 - A_{\sigma(X_2,X_3)}X_1 + (\tilde{\nabla}_{X_1}\sigma)(X_2, X_3)\\ &- (\tilde{\nabla}_{X_2}\sigma)(X_1, X_3),\end{aligned} \tag{2.14}$$

where R is the Rimannian curvature tensor of N.
The tangential component of the equation (2.14) is given by

$$\begin{aligned}g(R(X_1, X_2)X_3, X_5) = g(\tilde{R}(X_1, X_2)X_3, X_5) + g(\sigma(X_1, X_5), \sigma(X_2, X_3))\\ - g(\sigma(X_1, X_3), \sigma(X_2, X_5)\end{aligned} \tag{2.15}$$

for any X_1, X_2,$X_3 \in \Gamma(TN)$.
Using equations (2.5) and (2.8), we obtain:

$$\nabla_{X_1}\xi = \alpha(X_1 - \eta(X_1)\xi), \tag{2.16}$$

$$\sigma(X_1, \xi) = 0, \tag{2.17}$$

for all X_1, $X_2 \in \Gamma(TN)$.
By substituting equations (2.11), (2.16) and (2.17) in equation (2.14), we find

$$R(X_1, X_2)\xi = \alpha^2(\eta(X_1)X_2 - \eta(X_2)X_1). \tag{2.18}$$

Contraction of previous equation gives us

$$S(X_1, \xi) = -\alpha^2(n-1)\eta(X_1). \tag{2.19}$$

Lemma 2.1. [13] *In an m-dimensional α-cosymplectic manifold $\tilde{M}$, the $*$-Ricci tensor is given by*

$$S^*(X_1, X_2) = S(X_1, X_2) + \alpha^2(m-2)g(X_1, X_2) + \alpha^2\eta(X_1)\eta(X_2), \tag{2.20}$$

for any $X_1, X_2 \in \Gamma(T\tilde{M})$, where S and S^* are the Ricci tensor and $*$-Ricci tensor of type $(0, 2)$, respectively.

Definition 2.1. *A submanifold N of an α-cosymplectic manifold $\tilde{M}$ is said to be η-Einstein if its Ricci tensor S satisfies the following expression:*

$$S(X_1, X_2) = ag(X_1, X_2) + b\eta(X_1)\eta(X_2),$$

where a and b are smooth functions on N.

A vector field v on a Riemannian manifold $(\tilde{M}, g)$ is called torse-forming if it satisfies the condition

$$\tilde{\nabla}_{X_1}v = fX_1 + \delta(X_1)v, \tag{2.21}$$

for any $X_1 \in \Gamma(T\tilde{M})$, where f is a function and δ is a 1-form. The 1-form δ is referred to as the generating form, and the function f is called the conformal scalar of v.
If the 1-form δ in (2.21) vanishes identically, then the vector field v is called concircular. If $f = 1$ and $\delta = 0$, the vector field v is called concurrent. The vector field v is referred to as recurrent if it satisfies (2.21) with $f = 0$. Additionally, if both $f = 0$ and $\delta = 0$, the vector field v is called parallel.

Let $u : N \to \tilde{M}$ be an isometric immersion of a submanifold N into the Riemannian manifold $\tilde{M}$. For each point $p \in N$, we denote T_pN and $T_p^{\perp}N$ as the tangent and the normal spaces at p, respectively. There is a natural orthogonal decomposition given by [2]

$$T_p\tilde{M} = T_pN + T_p^{\perp}N.$$

Let N be a submanifold of a α-cosymplectic manifold $\tilde{M}$ endowed with a torqued vector field τ, and let $\psi : N \to \tilde{M}$ is an isometric immersion. Then, we have [2, 19]

$$\tau = \tau^T + \tau^\perp, \tag{2.22}$$

where τ^T and $\tau^\perp$ represent the tangential and normal components of τ on $\tilde{M}$, respectively.

3. Submanifold of α-cosymplectic manifold admitting $*$-η-Ricci soliton and with torqued vector field

This section examines the study of a submanifold N of an α-cosymplectic manifold $\tilde{M}$ endowed with a torqued vector field.

Theorem 3.1. *Let N be a submanifold of an α-cosymplectic manifold $\tilde{M}$ equipped with a torqued vector field τ, and let the characteristic vector field ξ be a torse-forming vector field on N. Then, N is cosymplectic submanifold provided ξ is torqued vector field.*

Proof. Let τ be a torqued vector field on $\tilde{M}$. Then, from equation (1.3) we have

$$\tilde{\nabla}_{X_1}\tau = fX_1 + \delta(X_1)\tau, \quad \delta(\tau) = 0, \tag{3.1}$$

for any $X_1 \in \Gamma(T\tilde{M})$. Assuming ξ is a torse-forming vector field on $\tilde{M}$, replacing τ with ξ in equation (3.1) gives us

$$\tilde{\nabla}_{X_1}\xi = fX_1 + \delta(X_1)\xi. \tag{3.2}$$

Substituting equations (2.8) and (2.17) in (3.2), we obtain

$$\nabla_{X_1}\xi = fX_1 + \delta(X_1)\xi, \tag{3.3}$$

for any $X_1 \in \Gamma(TN)$.
Taking the inner product of (3.3) with ξ, we get

$$\delta(X_1) = -f\eta(X_1). \tag{3.4}$$

By using (2.16) and (3.4) in (3.3), we find

$$\alpha(X_1 - \eta(X_1)\xi) = f(X_1 - \eta(X_1)\xi). \tag{3.5}$$

Taking the inner product of (3.5) with an arbitrary vector field X_2, we obtain

$$(f - \alpha)g(\phi X_1, \phi X_2) = 0. \tag{3.6}$$

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the tangent space T_pN, $\forall p \in N$. By setting $X_1 = X_2 = e_i$ in (3.6) and summing over $i = 1, 2, ..., n$, we obtain

$$(f - \alpha)n = 0. \tag{3.7}$$

Since $n \neq 0$, which implies that $f = \alpha$.

Now, we consider the following cases:

Case I: If ξ is a torqued vector field on N, then we get $\delta(\xi) = 0$. From (3.4), it follows that $f = 0$. In this case, N is a cosymplectic submanifold and thus ξ is a Killing vector field.

Case II: If ξ is a recurrent vector field on N, then we have that $f = 0$. Consequently, N is a cosymplectic submanifold, and ξ is a Killing vector field.

Theorem 3.2. *Let N be a submanifold of an α-cosymplectic manifold $\tilde{M}$ equipped with a torse-forming vector field τ. The submanifold N is totally geodesic if and only if the tangential component τ^T of τ is a torse-forming vector field on N whose conformal scalar is the restriction of the torqued function, and whose generating form is the restriction of the torqued function of τ on N.*

Proof. Since τ is a torqued vector field on the ambient space $\tilde{M}$, it follows from equations (1.3), (2.22), (2.8) and (2.9) that

$$\nabla_{X_1}\tau^T + h(X_1, \tau^T) + \nabla^{\perp}_{X_1}\tau^{\perp} - A_{\tau^{\perp}}X_1 = fX_1 + \delta(X_1)\tau^T + \delta(X_1)\tau^{\perp}, \tag{3.8}$$

for any $X_1 \in \Gamma(TN)$.

By comparing the tangential and normal components of equation (3.8), we arrive at

$$\nabla_{X_1}\tau^T - A_{\tau^{\perp}}X_1 = fX_1 + \delta(X_1)\tau^T, \tag{3.9}$$
$$h(X_1, \tau^T) + \nabla^{\perp}_{X_1}\tau^{\perp} = \delta(X_1)\tau^{\perp}.$$

If N is a totally geodesic submanifold of $\tilde{M}$, then (3.9) simplifies to

$$\nabla_{X_1}\tau^T = fX_1 + \delta(X_1)\tau^T, \tag{3.10}$$

which implies that τ^T is torse-forming on N. The converse is straightforward.

From now on, we assume that the submanifold N admits a $*$-η-Ricci soliton in Theorem (3.2). Considering equation (3.9), we have the following cases:

Case I: If we take $\tau^T \in \Gamma(D)$, then from equations (2.10), (2.16), (2.17) and (3.9), we get

$$g(\nabla_{X_1}\tau^T, \xi) = g(fX_1, \xi), \tag{3.11}$$

where $TN = D \oplus span\xi$, for any $X_1 \in \Gamma(TN)$. Since g is non-degenerate, from equation (3.11) we have

$$\nabla_{X_1}\tau^T = fX_1, \tag{3.12}$$

which shows that the vector field τ^T is concircular on N.

On the other hand, from the definition of the Lie derivative and (3.12), we obtain

$$\begin{aligned}(L_{\tau^T}g)(X_1, X_2) &= g(\nabla_{X_1}\tau^T, X_2) + g(X_1, \nabla_{X_2}\tau^T) \\ &= 2fg(X_1, X_2),\end{aligned} \tag{3.13}$$

for any $X_1, X_2 \in \Gamma(TN)$, which means that the vector field τ^T is conformal Killing. Also, from (1.2), (2.20) and (3.13), we obtain

$$S(X_1, X_2) = -(\lambda + f + \alpha^2(n-2))g(X_1, X_2) - (\mu + \alpha^2)\eta(X_1)\eta(X_2),$$

where S is the Ricci tensor of N. Hence, N is η-Einstein.
Case II: If we use ξ instead of τ^T in (3.10), we have

$$\nabla_{X_1}\xi = fX_1 + \delta(X_1)\xi. \tag{3.14}$$

Taking inner product of (3.14) with ξ, we get

$$g(\nabla_{X_1}\xi, \xi) = f\eta(X_1) + \delta(X_1).$$

Utilizing (2.16) in previous equation, we get

$$\delta(X_1) = -f\eta(X_1).$$

It is straightforward to see that $\delta(\xi) \neq 0$. Therefore, ξ is a torse-forming vector field on N.

Theorem 3.3. *Let N be a submanifold of an α-cosymplectic manifold $\tilde{M}$ endowed with a torse-forming vector field τ. If the submanifold N is totally geodesic and the vector field τ^T is orthogonal to the characteristic vector field ξ, then τ^T is a recurrent vector field on N.*
Proof. Let τ be a torse-forming vector field on $\tilde{M}$. Then, from the definition of Lie derivative and from equation (1.2), we have

$$\begin{aligned}(L_\tau g)(X_1, X_2) &= L_\tau g(X_1, X_2) - g(L_\tau X_1, X_2) - g(X_1, L_\tau X_2) \\ &= g(\tilde{\nabla}_{X_1}\tau, X_2) + g(X_1, \tilde{\nabla}_{X_2}\tau).\end{aligned} \tag{3.15}$$

By substituting equation (2.22) in the above equation, we get

$$(L_\tau g)(X_1, X_2) = g(\tilde{\nabla}_{X_1}\tau^T + \tilde{\nabla}_{X_1}\tau^\perp, X_2) + g(X_1, \tilde{\nabla}_{X_2}\tau^T + \tilde{\nabla}_{X_2}\tau^\perp). \tag{3.16}$$

Now, by applying (2.8), (2.9) in (3.16) and if N is a totally geodesic submanifold of $\tilde{M}$, then we obtain

$$(L_\tau g)(X_1, X_2) = g(\nabla_{X_1}\tau^T, X_2) + g(X_1, \nabla_{X_2}\tau^T). \tag{3.17}$$

Applying (3.10) to (3.17), we arrive at

$$(L_\tau g)(X_1, X_2) = 2fg(X_1, X_2) + \delta(X_1)g(\tau^T, X_2) + \delta(X_2)g(X_1, \tau^T), \tag{3.18}$$

for any $X_1, X_2 \in \Gamma(TN)$.
Upon substituting $X_1 = X_2 = \xi$ in (3.18), we obtain

$$(L_\tau g)(\xi, \xi) = 2f + 2\delta(\xi)\eta(\tau^T). \tag{3.19}$$

However, by using (2.2), (2.8), (2.16), (2.17) and (2.22), we get

$$\begin{aligned}(L_\tau g)(\xi, \xi) &= -2g(L_\tau \xi, \xi)\\ &= -2g(\tilde{\nabla}_\tau \xi, \xi) + 2g(\tilde{\nabla}_\xi \tau, \xi)\\ &= -2g(\nabla_\tau \xi, \xi) - 2g(h(\tau, \xi), \xi) + 2g(\nabla_\xi \tau, \xi) + 2g(h(\tau, \xi), \xi)\\ &= 2g(\nabla_\xi(\tau^T + \tau^\perp), \xi)\\ &= 2g(\nabla_\xi \tau^T, \xi).\end{aligned} \tag{3.20}$$

Furthermore, it is straightforward to show that $\nabla_\xi(g(\tau^T, \xi)) = g(\nabla_\xi \tau^T, \xi)$. Therefore, from (3.19) and (3.20), we get

$$f + \delta(\xi)\eta(\tau^T) = \nabla_\xi(g(\tau^T, \xi)). \tag{3.21}$$

If the vector field τ^T is orthogonal to ξ, then from equation (3.21), we obtain $f = 0$. It follows from (3.10) that τ^T is a recurrent vector field on N.

Thus, the proof is complete.

Theorem 3.4. *Let $\tilde{M}$ be an α-cosymplectic manifold endowed with a torqued vector field τ, and let N be a submanifold that admits a $*$-η-Ricci soliton of $\tilde{M}$. Then $(N, g, \xi, \lambda, \mu)$ is η-Einstein and the constants λ and μ satisfy the relation $\lambda = -\mu$.*
Proof. If we substitute ξ for τ^T in (3.9), we have

$$\nabla_{X_1}\xi - A_{\tau^\perp}X_1 = fX_1 + \delta(X_1)\xi. \tag{3.22}$$

Making use of (2.16) in (3.22), we get

$$A_{\tau^\perp} X_1 = (\alpha - f)X_1 - \alpha\eta(X_1)\xi - \delta(X_1)\xi. \tag{3.23}$$

Additionally, using the equations (2.3), (2.10) and taking the inner product with X_2 in the above equation, we obtain

$$g(\sigma(X_1, X_2), \tau^T) = (\alpha - f)g(X_1, X_2) - (\alpha\eta(X_1) + \delta(X_1))\eta(X_2). \tag{3.24}$$

By interchanging the roles of X_1 and X_2 in (3.24), we obtain

$$g(\sigma(X_2, X_1), \tau^T) = (\alpha - f)g(X_2, X_1) - (\alpha\eta(X_2) + \delta(X_2))\eta(X_1). \tag{3.25}$$

As σ and g exhibit symmetry, from (3.24) and (3.25) we have

$$2g(\sigma(X_1, X_2), \tau^T) = 2(\alpha - f)g(X_1, X_2) - 2\alpha\eta(X_1)\eta(X_2) - \delta(X_1)\eta(X_2) - \delta(X_2)\eta(X_1), \tag{3.26}$$

for any $X_1, X_2 \in \Gamma(TN)$.
On the other hand, using the definition of Lie derivative along with equations (2.3), (2.10), (3.22) and (3.26), we obtain

$$\begin{aligned}(L_\xi g)(X_1, X_2) &= g(\nabla_{X_1}\xi, X_2) + g(X_1, \nabla_{X_2}\xi)\\ &= g(fX_1 + \delta(X_1)\xi + A_{\tau^\perp}X_1, X_2)\\ &\quad + g(fX_2 + \delta(X_2)\xi + A_{\tau^\perp}X_2, X_1)\\ &= 2\alpha g(X_1, X_2) - 2\alpha\eta(X_1)\eta(X_2).\end{aligned} \tag{3.27}$$

Since N is a submanifold admitting a $*$-η-Ricci soliton, from equations (1.2) and (3.27), we have

$$S^*(X_1, X_2) = -(\alpha + \lambda)g(X_1, X_2) - (\mu - \alpha)\eta(X_1)\eta(X_2). \tag{3.28}$$

Utilizing (2.20) in (3.28), we find

$$S(X_1, X_2) = -(\lambda + \alpha + \alpha^2(n-2))g(X_1, X_2) - (\mu - \alpha + \alpha^2)\eta(X_1)\eta(X_2). \tag{3.29}$$

which implies that N is η-Einstein.
Taking $X_2 = \xi$ in (3.29), we get

$$S(X_1, \xi) = -(\lambda + \mu + \alpha^2(n-1))\eta(X_1). \tag{3.30}$$

From (2.19) and (3.30), we obtain

$$\lambda + \mu = 0 \Longrightarrow \lambda = -\mu. \tag{3.31}$$

Thus, the theorem is proved.

4. Example: 3-dimensional submanifold of an 5-dimensional α- cosymplectic manifold

In this section, we give an example of a 5-dimensional α-cosymplectic manifold. Subsequently, we utilize this example to construct a three-dimensional submanifold, thereby verifying the results we have obtained.

Let us consider a five-dimensional manifold $\tilde{M} = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, t \in \mathbb{R}^5\}$, here $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, t)$ represent the standard coordinates of $\mathbb{R}^5$. The linearly independent vector fields $\{\mathring{e}_1, \mathring{e}_2, \mathring{e}_3, \mathring{e}_4, \mathring{e}_5\}$ on $\tilde{M}$ are given by

$$\mathring{e}_1 = \alpha e^{\alpha t}\frac{\partial}{\partial \tilde{x}_1}, \quad \mathring{e}_2 = \alpha e^{\alpha t}\frac{\partial}{\partial \tilde{x}_2}, \quad \mathring{e}_3 = \alpha e^{\alpha t}\frac{\partial}{\partial \tilde{x}_3}, \quad \mathring{e}_4 = \alpha e^{\alpha t}\frac{\partial}{\partial \tilde{x}_4}, \quad \mathring{e}_5 = -\frac{\partial}{\partial t}.$$

Now, let us define Riemannian metric g on $\tilde{M}$ as

$$g(\mathring{e}_i, \mathring{e}_j) = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j, \quad for \quad 1 \leq i, j \leq 5. \end{cases}$$

The 1-form η corresponding to the metric g is defined as $\eta(X_1) = g(X_1, \xi)$ and setting $\mathring{e}_5 = \xi$ we observe that $\eta(\mathring{e}_5) = 1$, and $\eta(\mathring{e}_i) = 0$ for i=1, 2, 3, 4.

Furthermore, let us define the (1, 1)-tensor field ϕ as follows:

$$\phi(\mathring{e}_1) = -\mathring{e}_4, \quad \phi(\mathring{e}_2) = \mathring{e}_1, \quad \phi(\mathring{e}_3) = -\mathring{e}_2, \quad \phi(\mathring{e}_4) = \mathring{e}_3, \quad \phi(\mathring{e}_5) = 0.$$

Considering the equations above, it is straightforward to verify that $\phi^2 X_1 = -X_1 + \eta(X_1)\xi$ and $g(\phi X_1, \phi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2)$, for any vector fields $X_1, X_2 \in T\tilde{M}^5$. This demonstrates that the structure $\tilde{M}(\phi, \xi, \eta, g)$ constitutes an almost contact metric manifold.

Suppose $\tilde{\nabla}$ represents the Levi-Civita connection corresponding to the metric g. This leads to:

$$[\mathring{e}_i, \mathring{e}_j] = \begin{cases} \alpha \mathring{e}_i, & \text{if} \quad i = 1, 2, 3, 4; j = 5 \\ 0, & \text{otherwise}, \end{cases} \tag{4.1}$$

here [., .] represents the Lie bracket.

By utilizing Koszul's formula and (4.1), we derive the following:

$$\begin{gathered}
\tilde{\nabla}_{\mathring{e}_1}\mathring{e}_1 = -\alpha\mathring{e}_5, \tilde{\nabla}_{\mathring{e}_1}\mathring{e}_2 = 0, \tilde{\nabla}_{\mathring{e}_1}\mathring{e}_3 = 0, \tilde{\nabla}_{\mathring{e}_1}\mathring{e}_4 = 0, \tilde{\nabla}_{\mathring{e}_1}\mathring{e}_5 = \alpha\mathring{e}_1, \\
\tilde{\nabla}_{\mathring{e}_2}\mathring{e}_1 = 0, \tilde{\nabla}_{\mathring{e}_2}\mathring{e}_2 = -\alpha\mathring{e}_5, \tilde{\nabla}_{\mathring{e}_2}\mathring{e}_3 = 0, \tilde{\nabla}_{\mathring{e}_2}\mathring{e}_4 = 0, \tilde{\nabla}_{\mathring{e}_2}\mathring{e}_5 = \alpha\mathring{e}_2, \\
\tilde{\nabla}_{\mathring{e}_3}\mathring{e}_1 = 0, \tilde{\nabla}_{\mathring{e}_3}\mathring{e}_2 = 0, \tilde{\nabla}_{\mathring{e}_3}\mathring{e}_3 = -\alpha\mathring{e}_5, \tilde{\nabla}_{\mathring{e}_3}\mathring{e}_4 = 0, \tilde{\nabla}_{\mathring{e}_3}\mathring{e}_5 = \alpha\mathring{e}_3, \\
\tilde{\nabla}_{\mathring{e}_4}\mathring{e}_1 = 0, \tilde{\nabla}_{\mathring{e}_4}\mathring{e}_2 = 0, \tilde{\nabla}_{\mathring{e}_4}\mathring{e}_3 = 0, \tilde{\nabla}_{\mathring{e}_4}\mathring{e}_4 = -\alpha\mathring{e}_5, \tilde{\nabla}_{\mathring{e}_4}\mathring{e}_5 = \alpha\mathring{e}_4, \\
\tilde{\nabla}_{\mathring{e}_5}\mathring{e}_1 = 0, \tilde{\nabla}_{\mathring{e}_5}\mathring{e}_2 = 0, \tilde{\nabla}_{\mathring{e}_5}\mathring{e}_3 = 0, \tilde{\nabla}_{\mathring{e}_5}\mathring{e}_4 = 0, \tilde{\nabla}_{\mathring{e}_5}\mathring{e}_5 = 0.
\end{gathered}$$

In consequence of above equations, the manifold $\tilde{M}$ satisfies

$$\tilde{\nabla}_{X_1}\xi = -\alpha\phi^2 X_1,$$

and

$$(\tilde{\nabla}_{X_1}\phi)X_2 = \alpha(g(\phi X_1, X_2)\xi - \eta(X_2)\phi X_1),$$

The components of the Riemannian curvature tensor $\tilde{R}$ can be derived using the equation $\tilde{R}(X_1, X_2)X_3 = \tilde{\nabla}_{X_1}\tilde{\nabla}_{X_2}X_3 - \tilde{\nabla}_{X_2}\tilde{\nabla}_{X_1}X_3 - \tilde{\nabla}_{[X_1,X_2]}X_3$. This yields:

$$\begin{gathered}
\tilde{R}(\mathring{e_1}, \mathring{e_2})\mathring{e_2} = \tilde{R}(\mathring{e_1}, \mathring{e_3})\mathring{e_3} = \tilde{R}(\mathring{e_1}, \mathring{e_4})\mathring{e_4} = \tilde{R}(\mathring{e_1}, \mathring{e_5})\mathring{e_5} = -\alpha^2\mathring{e_1}, \\
\tilde{R}(\mathring{e_1}, \mathring{e_2})\mathring{e_1} = \alpha^2\mathring{e_2}, \quad \tilde{R}(\mathring{e_1}, \mathring{e_3})\mathring{e_1} = \tilde{R}(\mathring{e_2}, \mathring{e_3})\mathring{e_2} = \tilde{R}(\mathring{e_5}, \mathring{e_3})\mathring{e_5} = \alpha^2\mathring{e_3}, \\
\tilde{R}(\mathring{e_2}, \mathring{e_3})\mathring{e_3} = \tilde{R}(\mathring{e_2}, \mathring{e_4})\mathring{e_4} = \tilde{R}(\mathring{e_2}, \mathring{e_5})\mathring{e_5} = -\alpha^2\mathring{e_2}, \quad \tilde{R}(\mathring{e_3}, \mathring{e_4})\mathring{e} = -\alpha^2\mathring{e_3}, \\
\tilde{R}(\mathring{e_1}, \mathring{e_5})\mathring{e_2} = \tilde{R}(\mathring{e_1}, \mathring{e_5})\mathring{e_1} = \tilde{R}(\mathring{e_4}, \mathring{e_5})\mathring{e_4} = \tilde{R}(\mathring{e_3}, \mathring{e_5})\mathring{e_3} = \alpha^2\mathring{e_5}, \\
\tilde{R}(\mathring{e_1}, \mathring{e_4})\mathring{e_1} = \tilde{R}(\mathring{e_2}, \mathring{e_4})\mathring{e_2} = \tilde{R}(\mathring{e_3}, \mathring{e_4})\mathring{e_3} = \tilde{R}(\mathring{e_5}, \mathring{e_4})\mathring{e_5} = \alpha^2\mathring{e_4}.
\end{gathered}$$

Making use of (2.6) and performing direct calculations, we get

$$\tilde{R}(\mathring{e_1}, \mathring{e_2})\mathring{e_2} = \alpha^2[g(\mathring{e_1}, \mathring{e_2})\mathring{e_2} - g(\mathring{e_2}, \mathring{e_2})\mathring{e_1}] = -\alpha^2\mathring{e_2}.$$

$$\tilde{R}(\mathring{e_2}, \mathring{e_3})\mathring{e_2} = \alpha^2[g(\mathring{e_2}, \mathring{e_2})\mathring{e_3} - g(\mathring{e_3}, \mathring{e_2})\mathring{e_2}] = \alpha^2\mathring{e_3}.$$

$$\tilde{R}(\mathring{e_2}, \mathring{e_5})\mathring{e_5} = \alpha^2[g(\mathring{e_2}, \mathring{e_5})\mathring{e_5} - g(\mathring{e_5}, \mathring{e_5})\mathring{e_2}] = -\alpha^2\mathring{e_2}.$$

$$\tilde{R}(\mathring{e_4}, \mathring{e_5})\mathring{e_4} = \alpha^2[g(\mathring{e_4}, \mathring{e_4})\mathring{e_5} - g(\mathring{e_5}, \mathring{e_4})\mathring{e_4}] = \alpha^2\mathring{e_5}.$$

Similarly, all other components satisfy the above conditions.
Hence, from the above curvature tensor expressions, we finally conclude that $\tilde{M}^5$ constitutes an α-cosymplectic manifold.

Now, let us consider the three-dimensional submanifold N^3 of $\tilde{M}^5(\phi, \xi, \eta, g)$ given by the isometric immersion $\psi : N \to \tilde{M}$ defined by $\psi(\tilde{x}_1, \tilde{x}_2, t) = (\tilde{x}_1, \tilde{x}_2, 0, 0, t)$. It is clear that $N = \{(\tilde{x}_1, \tilde{x}_2, t) \in \mathbb{R}^3\}$, where $(\tilde{x}_1, \tilde{x}_2, t)$ represent standard coordinates in $\mathbb{R}^3$, and it forms a 3-dimensional submanifold of $\tilde{M}$. The vector fields $\{\mathring{e_1}, \mathring{e_2}, \mathring{e_5}\}$ are given by

$$\mathring{e_1} = \alpha e^{\alpha t}\frac{\partial}{\partial \tilde{x}_1}, \quad \mathring{e_2} = \alpha e^{\alpha t}\frac{\partial}{\partial \tilde{x}_2}, \quad \mathring{e_5} = -\frac{\partial}{\partial t}.$$

Let us define metric g_1 as

$$g_1(\mathring{e_i}, \mathring{e_j}) = \begin{cases} 1, & \text{if} \quad i = j, \\ 0, & \text{if} \quad i \neq j, \quad \textit{where} \quad i, j = 1, 2, 5. \end{cases}$$

Taking $\mathring{e_5} = \xi$, we observe that the 1-form $\eta_1(\mathring{e_5}) = \tilde{g}_1(\xi, \mathring{e_5}) = 1$ and ϕ_1 denote the (1, 1)-tensor field is defined as

$$\phi_1(\mathring{e_1}) = -\mathring{e_2}, \quad \phi_2(\mathring{e_2}) = \mathring{e_1}, \quad \phi_3(\mathring{e_5}) = 0.$$

Utilizing the equations above, it is straightforward to show that

$$\phi_1^2 X_1 = -X_1 + \eta_1(X_1)\xi, \quad \eta_1(\mathring{e_5}) = 1,$$
$$g_1(\phi_1 X_1, \phi_1 X_2) = g_1(X_1, X_2) - \eta_1(X_1)\eta_1(X_2),$$

for any X_1, X_2 on N. This indicates that $N(\phi_1, \xi, \eta_1, g_1)$ constitutes a submanifold of $\tilde{M}$.
Suppose ∇ represent the Levi-civita connection induced by the metric g_1. Utilizing Koszul's formula, we obtain the following:

$$\nabla_{\mathring{e_1}}\mathring{e_1} = -\alpha\mathring{e_5}, \nabla_{\mathring{e_1}}\mathring{e_2} = 0, \nabla_{\mathring{e_1}}\mathring{e_5} = \alpha\mathring{e_1},$$
$$\nabla_{\mathring{e_2}}\mathring{e_1} = 0, \nabla_{\mathring{e_2}}\mathring{e_2} = -\alpha\mathring{e_5}, \nabla_{\mathring{e_2}}\mathring{e_5} = \alpha\mathring{e_2},$$
$$\nabla_{\mathring{e_5}}\mathring{e_1} = 0, \nabla_{\mathring{e_5}}\mathring{e_2} = 0, \nabla_{\mathring{e_5}}\mathring{e_5} = 0.$$

The above obtained results satisfy $\nabla_{X_1}\xi = -\alpha\phi^2 X_1$. The Riemannian curvature tensor R utilizing the formula $R(X_1, X_2)X_3 = \nabla_{X_1}\nabla_{X_2}X_3 - \nabla_{X_2}\nabla_{X_1}X_3 - \nabla_{[X_1,X_2]}X_3$ is given by

$$R(\mathring{e_1}, \mathring{e_2})\mathring{e_2} = R(\mathring{e_1}, \mathring{e_5})\mathring{e_5} = -\alpha^2\mathring{e_1},$$
$$R(\mathring{e_1}, \mathring{e_2})\mathring{e_1} = R(\mathring{e_2}, \mathring{e_5})\mathring{e_5} = \alpha^2\mathring{e_2}$$
$$R(\mathring{e_1}, \mathring{e_5})\mathring{e_2} = R(\mathring{e_1}, \mathring{e_5})\mathring{e_1} = \alpha^2\mathring{e_5},$$

By direct calculations and using (2.18), we get

$$R(\mathring{e_1}, \mathring{e_2})\mathring{e_2} = -\alpha^2\mathring{e_1}, \quad R(\mathring{e_2}, \mathring{e_5})\mathring{e_5} = \alpha^2\mathring{e_2}, \quad R(\mathring{e_1}, \mathring{e_5})\mathring{e_1} = \alpha^2\mathring{e_5}.$$

Thus, $N(\phi_1, \xi, \eta_1, g_1)$ forms a 3-dimensional α-cosymplectic manifold.
With the help of the above results we get the components of the Ricci tensor as follows:

$$S(\mathring{e_1}, \mathring{e_1}) = S(\mathring{e_2}, \mathring{e_2}) = S(\mathring{e_3}, \mathring{e_3}) = -2\alpha^2. \tag{4.2}$$

From (3.29) for $n = 3$, we get

$$S(\mathring{e_3}, \mathring{e_3}) = -\lambda - \mu - 2\alpha^2. \tag{4.3}$$

Equating (4.2) and (4.3), we obtain

$$\lambda = -\mu.$$

Hence λ and μ satisfies the equation (3.31) for $n = 3$ and, so g defines a $*$-η-Ricci soliton on a 3-dimensional α-cosymplectic manifold, which verifies Theorem (3.4).

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