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SOLVABILITY OF NONLINEAR HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS VIA FIXED POINT THEOREM

Monkhum Khilak, Saifur Rahman*, Bhuban Chandra Deuri and Bipan Hazarika*****

Department of Mathematics,
Jawaharlal Nehru College Pasighat,
East Siang - 791103, Arunachal Pradesh, INDIA

E-mail : monkhum.khilak@rgu.ac.in

*Department of Mathematics,
Jamia Millia Islamia University,
Jamia Nagar, Okhla, New Delhi - 110025, INDIA

E-mail : saifur.rahman@rgu.ac.in

**Department of Mathematics,
Rajiv Gandhi University,
Rono Hills, Doimukh - 791112, Arunachal Pradesh, INDIA

E-mail : bhuban.deuri@rgu.ac.in

***Department of Mathematics,
Gauhati University,
Guwahati - 781014, Assam, INDIA

E-mail : bh_gu@gauhati.ac.in

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Abstract: In this article, we generalized Darbo's fixed point theorem and establish the existence of solutions of fractional hybrid differential equations. The proof relies on Darbo's fixed point theorem, and the solvability is investigated in the tempered sequence space ℓ_p^α. An illustrative example is provided to verify the applicability of our results.

Keywords and Phrases: Measure of noncompactness; fractional integral equations; fixed point theorem.

2020 Mathematics Subject Classification: 45G10, 45P05, 47H10.

1. Introduction

The concept of measure of noncompactness(MNC) was first proposed by Kuratowski [16] in the year 1930. It proves to be an important tools in studying the theory of fixed point, characterization of compact operators and its equations in Banach spaces. Darbo [6] developed his famous fixed point theorem by using the concept of measure of noncompactness which was actually a generalized version of Schuader's fixed point theorem [1]. Darbo's fixed theoerem proves to be a crucial tool in investigating the existence results of different class of integral and differential equations. Due to its widespread applications in the recent times it attracts many researchers. Many authors generalized the Darbo's fixed point theorem and employed it to study the existence results of wide class of equations. The interplay between fixed point theorems and the measure of noncompactness has been a cornerstone in the study of fractional integral equations. Several papers have extended Darbo's theorem to accommodate the specific requirements of fractional integral equations. For instance, [10] introduces a generalized version of Darbo's theorem that incorporates a new contraction operator, enabling the solution of nonlinear fractional integral equations. Similarly, in the reference [11] the authors established new fixed point theorems with a contractive condition based on the measure of noncompactness, generalizing Darbo's theorem and other results. In addition to generalizing the Darbo's fixed point, Mohiuddine et al. [17] investigated the solvability of infinite system of integral equation in the tempered sequence space. Das et al. [9] generalized the Darbo's fixed point theorem and examined the existence result of infinite system of weighted fractional integral equations of a function with regard to another function. Olszowy and Zajac [19] generalized Darbo and Sadovskii type fixed point theorems and studied the existence result of Volterra type integral equation. Recently, Haque et al. [15] generalized the Darbo's fixed point theorem and investigated the existence result of two variable fractional integral equation in the tempered sequence space c_0^α and ℓ_p^α. Bhujel and Hazarika [4] examined the existence result of nonlinear integral equation of Fredholm type satisfying the Hölder condition via Darbo's theorem. Das et al. [8] examined the existence result of hybrid differential and fractional hybrid differential equations in Banach space with the help of generalized Darbo's fixed point theorem. Recently, Mursaleen and Savas [18] constructed a new tempered space and solvability of infinite system of fractional differential equations involving p-laplacian operator was studied via Darbo's fixed point theorem.

Inspired and motivated by [5, 12, 14, 21, 22], in the context of an MNC in tempered sequence space. This paper is divided into five sections: The introduction contain a brief history of the measure of noncompactness and a literature review The second section covers some well known fixed point theorems which will be used later in construction of new result. The third section presents the main result, wherein a new Darbo-type fixed point theorem is constructed. The fourth section discusses a brief history of fractional calculus and the solvability of hybrid fractional differential equations has been studied. The conclusion summarizes the main findings and suggests possible future research directions.

2. Preliminaries

Let us first get acquaint with the measure of noncompactness definition(see [2]). Let $(\Theta, \| \,.\, \|)$ represents a Banach space.

- $\mathfrak{M}_\Theta$ represents the family of bounded and non-empty subset of Θ and $\mathfrak{N}_\Theta$ denotes the subfamily of all relatively compact set of Θ.
- If $\phi \neq \Omega \subset \Theta$ then the convex closure and closure of Ω will be denote by ConΩ and $\overline{\Omega}$ respectively.
- $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}$ =set of natural number.

Definition 2.1. [2] *In the space Θ a measure of noncompactness is a function $\pounds\colon \mathfrak{M}_\Theta \to \mathbb{R}_+$ satisfying all the below conditions:*

(i) $for\ all\ Q \in \mathfrak{M}_\Theta,\ \pounds Q = 0$ *implies that* $Q \in \mathfrak{N}_\Theta$.

(ii) $ker\, \pounds = \{Q \in \mathfrak{M}_E : \pounds\,(Q) = 0\} \neq \phi$;

(iii) $Q_1 \subset Q_2$ *implies that* $\pounds\,(Q_1) \leq \pounds\,(Q_2)\,;$

(iv) $\pounds\left(\bar{Q_1}\right) = \pounds\,(Q_1)\,;$

(v) $\pounds\,(conQ_1) = \pounds\,(Q_1)\,;$

(vi) $\pounds\,(\hat{\kappa} Q_1 + (1 - \hat{\kappa})\, Q_2) \leq \hat{\kappa} \pounds\,(Q_1) + (1 - \hat{\kappa})\, \pounds\,(Q_2)$ *for* $\hat{\kappa} \in [0, 1]\,;$

(vii) *if* $Q_{\hat{n}} \in \mathfrak{M}_\Theta$, $Q_{\hat{n}} = \overline{Q_{\hat{n}}}$, $Q_{\hat{n}+1} \subset Q_{\hat{n}}$, *for* $\hat{n} \in \mathbb{N}$ *and* $\lim\limits_{\hat{n} \to +\infty} \pounds\,(Q_{\hat{n}}) = 0$, *then* $\bigcap\limits_{\hat{n}=1}^{+\infty} Q_{\hat{n}}$ *is nonempty.*

Definition 2.2. [2, 16] *If $\bar{\mathfrak{D}}$ is a bounded subset of a metric space X, the Kuratowski's measure of noncompactness $\alpha\left(\bar{\mathfrak{D}}\right)$ on $\bar{\mathfrak{D}}$ is given as*

$$\alpha\left(\bar{\mathfrak{D}}\right) = \inf\left\{\varepsilon > 0 : \bar{\mathfrak{D}} \subset \bigcup_{\bar{j}=1}^{\hat{n}} \bar{\mathfrak{D}}_{\bar{j}},\ diam\left(\bar{\mathfrak{D}}_{\bar{j}}\right) < \varepsilon \text{ for } \bar{j} = 1, 2, 3, \ldots, \hat{n}; \hat{n} \in \mathbb{N}\right\}.$$

Definition 2.3. [3] *If $\bar{\mathfrak{D}}$ is a bounded subset a metric space X, the Hausdorff measure of noncompactness $\chi(\bar{\mathfrak{D}})$ on $\bar{\mathfrak{D}}$ is given as*

$$\chi(\bar{\mathfrak{D}}) = \inf\left\{\hat{\varepsilon} > 0 : \bar{\mathfrak{D}} \subset \bigcap_{\hat{j}=1}^{\hat{n}} \Delta(\omega_{\hat{j}}, d_{\hat{j}}), \omega_{\hat{j}} \in \bar{\mathfrak{D}}, d_{\hat{j}} < \hat{\varepsilon}, 1 \leq \hat{j} \leq \hat{n}; \hat{n} \in \mathbb{N}\right\}.$$

Here $\Delta(\omega_{\hat{j}}, d_{\hat{j}})$ represents the open ball having $\omega_{\hat{j}}$ as center and $d_{\hat{j}}$ as radius.

Theorem 2.4. (Schauder [1]) *A continuous mapping $\hat{\mathcal{T}} : \bar{\mathfrak{D}} \to \bar{\mathfrak{D}}$ possess at least one fixed point in $\bar{\mathfrak{D}}$ where $\bar{\mathfrak{D}}$ is a bounded, nonempty, compact and convex subset ($\mathcal{BNCCS}$) of a Banach space Θ.*

Theorem 2.5. (Darbo [6]) *Consider $\bar{\mathfrak{D}}$ is a $\mathcal{BNCCS}$ of a Banach space Θ. A continuous mapping $\hat{\mathcal{T}} : \bar{\mathfrak{D}} \to \bar{\mathfrak{D}}$ has a fixed point if a constant $\hat{\kappa} \in [0, 1)$ exists and satisfying the condition*

$$£(\hat{\mathcal{T}}\mathcal{Q}) \leq \hat{\kappa}£(\mathcal{Q}),\ \ \mathcal{Q} \subseteq \bar{\mathfrak{D}}.$$

3. Fixed Point Theorem

Here, we mentioned some definitions and results that will help us to prove the main theorems.

Definition 3.1. *Suppose $\hat{\mathcal{T}}_m$ is the collection of all functions $m : \mathbb{R}_+ \to [1, +\infty)$ which fulfill the condition $\lim_{n\to+\infty} m(v_n) = 1$ implies that $\lim_{n\to+\infty} v_n = 0$ for all $v_n \in \mathbb{R}_+$.*

Definition 3.2. [13] *Let $\hat{\mathcal{T}}_j$ represents the set of all class of functions $j : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which satisfies all the following conditions*
i) $\max\{\mu_1, \mu_2\} \leq j(\mu_1, \mu_2)$ *for* $\mu_1, \mu_2 \geq 0$
ii) j is continuous and non decreasing
iii) $j(\mu_1 + \nu_1, \mu_2 + \nu_2) \leq j(\mu_1, \mu_2) + j(\nu_1, \nu_2)$.

For example if $j(\mu + \nu) = \mu + \nu$ then $j \in \hat{\mathcal{T}}_j$.

Theorem 3.3. *If $\mathfrak{B}$ is a $\mathcal{BNCCS}$ of a Banach space Θ and a continuous function $\hat{\mathcal{T}} : \mathfrak{B} \to \mathfrak{B}$ fulfill the condition*

$$m[j(£(\hat{\mathcal{T}}\mathcal{V}), f(£(\hat{\mathcal{T}}\mathcal{V})))] \leq [m\{j(£\mathcal{V}, f(£\mathcal{V}))\}]^{\hat{k}},$$

where $\emptyset \neq \mathcal{V} \subset \mathfrak{B}$, $j \in \hat{\mathcal{T}}_j, m \in \hat{\mathcal{T}}_m$ *and* $0 \leq \hat{k} < 1$ *and* $f : \mathbb{R}_+ \to \mathbb{R}_+$, *such that* f *is a continuous function, and* £ *is an arbitrary MNC. Then* $\hat{\mathcal{T}}$ *possess at least one fixed point.*

Proof. First we develop a sequence $(\mathcal{V}_n)$ where $\mathcal{V}_1 = \mathcal{V}$ and $\mathcal{V}_{n+1} = Con(\hat{\mathcal{T}}\mathcal{V}_n)$ for $n \geq 1$. Then $\hat{\mathcal{T}}\mathcal{V}_1 = \hat{\mathcal{T}}\mathcal{V} \subseteq \mathcal{V} = \mathcal{V}_1, \mathcal{V}_2 = Con(\hat{\mathcal{T}}\mathcal{V}_1) \subseteq \mathcal{V} = \mathcal{V}_1$ and moving in the same way we obtain $\mathcal{V}_1 \supseteq \mathcal{V}_2 \supseteq \mathcal{V}_3 \supseteq ...... \supseteq \mathcal{V}_n \supseteq \mathcal{V}_{n+1} \supseteq ....$

If there exist $\hat{n}_i \in \mathbb{N}$ such that $£(\mathcal{V}_{\hat{n}_i}) = 0$ then the theorem is proved. Let $£(\mathcal{V}_n) > 0$ for all $n \in \mathbb{N}$, then the sequence $\{£\mathcal{V}_n\}$ is a non negative, decreasing and bounded below sequence.

Now

$$\begin{aligned} m[j(£(\mathcal{V}_{n+1}), f(£(\mathcal{V}_{n+1})))] &= m[j(£(Con(\mathcal{T}\mathcal{V}_n)), f(£(Con(\mathcal{T}\mathcal{V}_n)))] \\ &= m[j(£(\hat{\mathcal{T}}\mathcal{V}_n), f(£(\hat{\mathcal{T}}\mathcal{V}_n)))] \\ &\leq [m\{j(£(\mathcal{V}_n), f(£(\mathcal{V}_n)))\}]^{\hat{k}} \\ &\leq [m\{j(£(\mathcal{V}_{n-1}), f(£(\mathcal{V}_{n-1})))\}]^{\hat{k}^2} \\ &. \\ &. \\ &. \\ &\leq [m\{j(£(\mathcal{V}_1), f(£(\mathcal{V}_1)))\}]^{\hat{k}^n}. \end{aligned}$$

It is obvious from the above that as $n \to +\infty$, then $m[j(£(\hat{\mathcal{T}}\mathcal{V}_n), f(£(\hat{\mathcal{T}}\mathcal{V}_n)))] = 1$. Then by the definition of m we have

$$\lim_{n\to+\infty} j(£(\mathcal{V}_n), f(£(\mathcal{V}_n))) = 0.$$

implies that

$$\lim_{n\to+\infty} £(\mathcal{V}_n) = 0 = \lim_{n\to+\infty} f(£(\mathcal{V}_n)).$$

Now $\mathcal{V}_n \supseteq \mathcal{V}_{n+1}$ and by the definition of £ we infer that $\mathcal{V}_{+\infty} = \bigcap_{n=1}^{+\infty} \mathcal{V}_n$ is nonempty, closed and convex subset of $\mathcal{V}$ and also under $\hat{\mathcal{T}}$, $\mathcal{V}_{+\infty}$ is invariant. So applying Schauder's theorem it brings to the conclusion that $\hat{\mathcal{T}}$ possess at least one fixed point in $\mathcal{V}_{+\infty} \subseteq \mathcal{V}$.

Corollary 3.4. *Let* $\hat{\mathcal{T}} : \mathfrak{B} \to \mathfrak{B}$ *be an operator,* $m \in \hat{\mathcal{T}}_m$, *then* $\hat{\mathcal{T}}$ *has at least one fixed point if*

$$m[£(\hat{\mathcal{T}}\mathcal{V}) + f(£(\hat{\mathcal{T}}\mathcal{V}))] = [m(£\mathcal{V} + f(£\mathcal{V}))]^{\hat{k}}.$$

Proof. In Theorem 3.3 by putting $j(\mu_1, \mu_2) = \mu_1 + \mu_2$ we obtain our required result.

Corollary 3.5. *Let $\hat{\mathcal{T}} : \mathfrak{B} \to \mathfrak{B}$ be an operator, $m \in \hat{\mathcal{T}}_m$, then $\hat{\mathcal{T}}$ has at least one fixed point if*

$$£(\hat{\mathcal{T}}\mathcal{V}) + f(£(\hat{\mathcal{T}}\mathcal{V})) = \hat{k}(£\mathcal{V} + f(£\mathcal{V})).$$

Proof. Putting $m(t) = e^t$ in the Corollary 3.4 we obtain our required result.

Remark 3.6. *Darbo's theorem can be obtained by putting $f(t) = 0$ in Corollary 3.5.*

4. Applications

Fractional calculus provides a powerful framework in modeling various problems in the field of heat transfer, signal analysis, fluid mechanics and control theory. Recently Das et al. [7] investigated the existence result of the system (4.1) in the space $m(\theta)^\beta$ of tempered sequence. In our study we did not find much literature related to the existence result of fractional hybrid differential equation in tempered sequence spaces. Which motivate us to examined the existence result of system (4.1) in the tempered sequence space ℓ_p^α with initial conditions by converting it to integral equation (4.2). In this present study we will examine the solvability of equation (4.1) in the tempered sequence space ℓ_p^α via Darbo's theorem. The tempered sequence space ℓ_p^α is a Banach space and was introduced in [20]. The norm on the Banach space ℓ_p^α is define as

$$\| u \|_{\ell_p^\alpha} = \left(\sum_{n=1}^{\infty} \alpha_n^p |u_n|^p \right)^{\frac{1}{p}},$$

where $u = (u_n)$ is a real(or complex) sequence. $\alpha = (\alpha_n)$ is non-increasing real sequence which is fixed, $\alpha_n > 0$, for all $n \in \mathbb{N}$ and $p \in [0, +\infty)$. The Hausdorff Measure of noncompacteness χ in the space ℓ_p^α is define as

$$\chi_{\ell_p^\alpha}(\hat{D}) = \lim_{n \to \infty} \left[\sup_{u \in \hat{D}} \left(\sum_{\hat{k} \geq n} \alpha_{\hat{k}}^p |u_{\hat{k}}|^p \right)^{\frac{1}{p}} \right], \ \hat{D} \in \mathfrak{M}_{\ell_p^\alpha}.$$

We shall now recall some of the basic definition of fractional calculus.
For a function $h : \mathbb{R}_+ \to \mathbb{R}$, the order λ Riemann-Liouville fractional integral is expressed as

$$I^a h(t) = \frac{1}{\Gamma(\lambda)} \int_0^t (t - \omega)^{\lambda - 1} h(\omega) d\omega,$$

where $\lambda > 0$, provided that the integral exists. The Riemann-Liouville fractional derivative of order λ can similarly be expressed as

$$D^\lambda h(t) = \frac{1}{\Gamma(\varpi - \lambda)} \left(\frac{d}{dt}\right)^\varpi \int_0^t \frac{1}{(t-\omega)^{a-\varpi+1}} h(\omega) d\omega,$$

where $\varpi = [\lambda] + 1$. I and D obeys the following relations for $\lambda_1, \lambda_2 > 0$.

$$I^{\lambda_1+\lambda_2} h(t) = I^{\lambda_1} I^{\lambda_2} h(t), \;\; D^{\lambda_1} I^{\lambda_2} h(t) = h(t).$$

Let us consider the infinite system of fractional hybrid differential equations with initial conditions and involving Riemann-Liouville fractional differential operator of order $0 < \rho < 1$

$$D^\rho [z_n(\vartheta) - \hat{s}_n(\vartheta, z_n(\vartheta))] = \hat{g}_n(\vartheta, z_n(\vartheta)), \; \vartheta \in [0, a], \tag{4.1}$$

with $z_n(\vartheta_0) = 0$, where $z(\vartheta) = (z_n(\vartheta))_{n=1}^\infty$, $\hat{g}_n, \hat{s}_n \in C(I \times \mathbb{R}, \mathbb{R})$ and $z_n(\vartheta) \in C(I, \mathbb{R})$, $n \in \mathbb{N}$.

The equation (4.1) can be prove that it is equivalent to the infinite system of nonlinear hybrid integral equations which is

$$z_n(\vartheta) = -\hat{s}_n(\vartheta_0, 0) + \hat{s}_n(\vartheta, z_n(\vartheta)) + \frac{1}{\Gamma(\rho)} \int_0^\vartheta (\vartheta - w)^{\rho-1} \hat{g}_n(w, z_n(w)) dw, \tag{4.2}$$

where $n \in \mathbb{N}$ and $\vartheta \in I$.

4.1. Solvability in tempered sequence space ℓ_p^α

Let us consider the following assumptions

(T.1) The functions G_n are well defined as

$$G_n : I \times C(I, \ell_p^\alpha) \to \mathbb{R}, \quad n \in \mathbb{N}$$

and the operator $G : C(I, \ell_p^\alpha) \to C(I, \ell_p^\alpha)$ is defined as

$$(\vartheta, z(\vartheta)) \to (Gz)(\vartheta) = (G_n(\vartheta, z(\vartheta)))_{n\in\mathbb{N}},$$

where

$$G_n(\vartheta, z(\vartheta)) = \hat{s}_n(\vartheta, z_n(\vartheta)) + \frac{1}{\Gamma(q)} \int_0^\vartheta (\vartheta - w)^{\rho-1} \hat{g}_n(w, z_n(w)) dw.$$

Moreover for all points in the space $C(I, \ell_p^\alpha)$ the family $((Gz)(\vartheta))_{\vartheta \in I}$ is equicontinuous.

(T.2) $\hat{s}_n,\ \hat{g}_n : I \times \mathbb{R}$ are continuous. Also

$$|\hat{s}_n(\vartheta, z_n(\vartheta))|^p \leq m_n(\vartheta)|z_n(\vartheta)|^p,$$

where $m_n(\vartheta) : I \to \mathbb{R}$ is continuous for all $n \in \mathbb{N}$ such that the sequence $(m_n(\vartheta))$ is equibounded on the whole I and let

$$\bar{M} = \sup_{n\in\mathbb{N},\vartheta\in I} m_n(\vartheta).$$

Also

$$|\hat{g}_n(\vartheta, z_n(\vartheta))|^p \leq a_n(\vartheta) + b_n(\vartheta)|z_n(\vartheta)|^p,$$

where the functions $a_n(\vartheta), b_n(\vartheta) : I \to \mathbb{R}_+$ such that $\sum\limits_{n\geq 1} \alpha_n^p |a_n(\vartheta)|$ converges uniformly on $I = [0, \vartheta]$. Moreover, we write

$$\bar{M} = \sup_{n\in\mathbb{N},\vartheta\in I} m_n(\vartheta), \quad \bar{A}(\vartheta) = \sum_{n\geq 1} \alpha_n^p a_n(\vartheta), \quad A = \sup_{\vartheta\in I} \bar{A}(\vartheta),$$
$$B = \sup_{n\in\mathbb{N},\vartheta\in I} b_n(\vartheta).$$

Moreover,

$$0 < 2^p \bar{M}^p + \frac{B a^{p\rho} 2^p}{(p\rho - p + 1)(\Gamma(\rho))^p} < 1, \text{where } p\rho - p + 1 \neq 0.$$

Theorem 4.1. *The tempered sequence space ℓ_p^α contains a solution for the equation (4.1) if all the given above assumptions are fulfilled.*
Proof. Consider the operator $\hat{\mathcal{T}} : \ell_p^\alpha \to \ell_p^\alpha$ define as

$$(\hat{\mathcal{T}} z)(\vartheta) = \{(\hat{\mathcal{T}}_n z)(\vartheta)\}_{n=1}^{+\infty} = \{G_n(\vartheta, z(\vartheta))\}_{n=1}^{+\infty}.$$

Then

$$(\hat{\mathcal{T}}_n z_n)(\vartheta) = \hat{s}_n(\vartheta, z_n(\vartheta)) + \frac{1}{\Gamma(\rho)} \int_0^\vartheta (\vartheta - w)^{\rho-1} \hat{g}_n(w, z_n(w))dw.$$

$$\begin{aligned}
\| \hat{\mathcal{T}}(z) \|_{\ell_p^\alpha}^p &= \left(\sum_{n=1}^\infty \alpha_n^p |\hat{\mathcal{T}}(z_n)(\vartheta)|^p \right) \\
&= \sum_{n\geq 1} \alpha_n^p \left[\left| \hat{s}_n(\vartheta, z_n(\vartheta)) + \frac{1}{\Gamma(\rho)} \int_0^\vartheta (\vartheta - w)^{\rho-1} \hat{g}_n(w, z_n(w))dw \right|^p \right] \\
&\leq 2^p \sum_{n\geq 1} \alpha_n^p \big| \hat{s}_n(\vartheta, z_n(\vartheta)) \big|^p + 2^p \sum_{n\geq 1} \alpha_n^p \left| \frac{1}{\Gamma(\rho)} \int_0^\vartheta (\vartheta - w)^{\rho-1} \hat{g}_n(w, z_n(w))dw \right|^p
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^p\sum_{n\geq 1}\alpha_n^p(m_n(\vartheta))^p|z_n(\vartheta)|^p+\frac{\vartheta^{p-1}2^p}{(\Gamma(\rho))^p}\sum_{n\geq 1}\alpha_n^p\int_0^\vartheta(\vartheta-w)^{p\rho-p}|\hat{g}_n(w,z_n(w))|^p dw\\
&\leq 2^p\bar{M}^p\parallel z_n(\vartheta)\parallel_{\ell_p^\alpha}^p+\frac{\vartheta^{p-1}2^p\vartheta^{p\rho-p+1}}{(p\rho-p+1)(\Gamma(\rho))^p}\sum_{n\geq 1}\alpha_n^p|\hat{g}_n(w,z_n(w)|^p\\
&\leq 2^p\bar{M}^p\parallel z_n(\vartheta)\parallel_{\ell_p^\alpha}^p+\frac{\vartheta^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}\sum_{n\geq 1}\alpha_n^p(a_n(\vartheta)+b_n(\vartheta)|z_n(\vartheta)|^p)\\
&\leq 2^p\bar{M}^p\parallel z_n(\vartheta)\parallel_{\ell_p^\alpha}^p+\frac{A\vartheta^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}+\frac{B\vartheta^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}\sum_{n\geq 1}\alpha_n^p|z_n(\vartheta)|^p\\
&\leq 2^p\bar{M}^p\parallel z_n(\vartheta)\parallel_{\ell_p^\alpha}^p+\frac{A\vartheta^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}+\frac{B\vartheta^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}\parallel z_n(\vartheta)\parallel_{\ell_p^\alpha}^p\\
&=\frac{A\vartheta^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}+\left(2^p\bar{M}^p+\frac{B\vartheta^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}\right)\parallel z_n(\vartheta)\parallel_{\ell_p^\alpha}^p.
\end{aligned}
$$

This suggests that the operator $\hat{\mathcal{T}}$ is bounded for all $p>1$. Also $(\hat{\mathcal{T}}_n z)(0)=0$ which implies that $(\hat{\mathcal{T}}_n z)(\vartheta)$ satisfies the initial condition.
Now if we take the set $Q\subset\ell_p^\alpha$, where Q is define as

$$Q=\{z\in\ell_p^\alpha:\parallel z\parallel\leq r\}.$$

Then the set Q is bounded, closed and convex. Also the constant r satisfies the condition

$$\frac{A\vartheta^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}+\left(2^p\bar{M}^p+\frac{B\vartheta^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}\right)\parallel z_n(\vartheta)\parallel_{\ell_p^\alpha}^p\leq r^p.$$

By assumption (T.1) we conclude that $\hat{\mathcal{T}}$ is continuous and bounded operator on $C(I,\ell_p^\alpha)$.

Now

$$
\begin{aligned}
\chi_{\ell_p^\alpha}(\hat{\mathcal{T}}(Q))&=\lim_{n\to\infty}\sup_{z\in Q}\Bigg(2^p\bar{M}^p\sum_{k\geq n}\alpha_k^p|z_k(\vartheta)|^p\\
&\qquad+\frac{\vartheta^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}\sum_{k\geq n}\alpha_k^p(a_k(\vartheta)+b_k(\vartheta)|z_k(\vartheta)|^p)\Bigg)\\
&\leq\lim_{n\to\infty}\sup_{z\in Q}\left(2^p\bar{M}^p\sum_{k\geq n}\alpha_k^p|z_k(\vartheta)|^p+\frac{B\vartheta^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}\sum_{k\geq n}\alpha_k^p|z_k(\vartheta)|^p\right)\\
&\leq\left(2^p\bar{M}^p+\frac{B\vartheta^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}\right)\lim_{n\to\infty}\sup_{z\in Q}\left(\sum_{k\geq n}\alpha_k^p|z_k(\vartheta)|^p\right)
\end{aligned}
$$

$$= \left(2^p \bar{M}^p + \frac{B\vartheta^{p\rho}2^p}{(p\rho - p + 1)(\Gamma(\rho))^p}\right)\chi_{\ell_p^\alpha}(Q).$$

This implies that

$$\chi_{C(I,\ell_p^\alpha)}(\hat{\mathcal{T}}(Q)) \leq \left(2^p \bar{M}^p + \frac{B\vartheta^{p\rho}2^p}{(p\rho - p + 1)(\Gamma(\rho))^p}\right)\chi_{C(I,\ell_p^\alpha)}(Q).$$

i.e., $\chi_{C(I,\ell_p^\alpha)}(\hat{\mathcal{T}}(Q)) \leq \left(2^p \bar{M}^p + \frac{Ba^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p}\right)\chi_{C(I,\ell_p^\alpha)}(Q)$.

Since $2^p \bar{M}^p + \frac{Ba^{p\rho}2^p}{(p\rho-p+1)(\Gamma(\rho))^p} < 1$ and $\hat{\mathcal{T}}$ fulfill all the assumptions of Theorem 2.5, which implies that $\hat{\mathcal{T}}$ has a solution in Q. Consequently there exists a solution for (4.1) in $C(I, \ell_p^\alpha)$.

Example 4.2. Let us consider the following numerical problem

$$D^{\frac{3}{4}}\left[z_n(\vartheta) - \frac{e^\vartheta z_n(\vartheta)}{5+n^2}\right] = \frac{\vartheta\sin(n\vartheta)}{n^3} + \sum_{i\geq n}\frac{z_n(\vartheta)\ \ln(1+\vartheta)}{(4+\vartheta)^2 i^2}, \tag{4.3}$$

where $z_n(0) = 0$.
Here $\hat{s}_n(\vartheta, z_n(\vartheta)) = \frac{e^\vartheta z_n(\vartheta)}{5+n^2}, \hat{g}_n(\vartheta, z_n(\vartheta)) = \frac{\vartheta\sin(n\vartheta)}{n^3} + \sum_{i\geq n}\frac{z_n(\vartheta)\ \ln(1+\vartheta)}{(2+\vartheta)^2 i^2}$ $, n \in \mathbb{N}$ $, \vartheta \in I = [0,1]$. We have

$$\begin{aligned}|\hat{s}_n(\vartheta, z_n(\vartheta))| &= \left|\frac{e^\vartheta z_n(\vartheta)}{5+n^2}\right| \\ &\leq \left|\frac{e^\vartheta}{5+n^2}\right||z_n(\vartheta)|.\end{aligned}$$

This gives

$$|\hat{s}_n(\vartheta, z_n(\vartheta))|^2 \leq \left|\frac{e^\vartheta}{5+n^2}\right|^2|z_n(\vartheta)|^2$$

and

$$|\hat{g}_n(\vartheta, z_n(\vartheta))| = \left|\frac{\vartheta\sin(n\vartheta)}{n^3} + \sum_{i\geq n}\frac{z_n(\vartheta)\ \ln(1+\vartheta)}{(4+\vartheta)^2 i^2}\right|$$

$$\begin{aligned}\Rightarrow |\hat{g}_n(\vartheta, z_n(\vartheta))|^2 &\leq 2\left|\frac{\vartheta\sin(n\vartheta)}{n^3}\right|^2 + 2\left|\sum_{i\geq n}\frac{z_n(\vartheta)\ \ln(1+\vartheta)}{(4+\vartheta)^2 i^2}\right|^2 \\ &\leq \frac{2|\vartheta|^2}{n^6} + \frac{|\ln(1+\vartheta)|^2|z_n(\vartheta)|^2\pi^4}{288}.\end{aligned}$$

Here $m_n(\vartheta) = \left|\frac{e^{\vartheta}}{5+n^2}\right|$ therefore we have $M = .02718$, $a_n(\vartheta) = \frac{2|\vartheta|^2}{n^6}$, $b_n(\vartheta) = \frac{|ln(1+\vartheta)|^2\pi^4}{288}$ implies that $B = \frac{|\ln(2)|^2\pi^4}{288}$.
Therefore $2^p\bar{M}^p + \frac{Ba^{pq}2^p}{(pq-p+1)(\Gamma(q))^p} = 2^2(.02718)^2 + 0.86472 \approx 0.86767$.
Thus the system (4.3) satisfies all the hypothesis of Theorem 4.1. Thus we conclude that there exist a solution for system (4.3) in the space $C(I, \ell_2^{\alpha})$.

5. Conclusion

This study discussed many further modified Darbo-type fixed point findings for the concept of a family of contraction operators applying various control functions in Banach spaces. The measure of noncompactness has proven to be an indispensable tool in the study of fractional hybrid differential equations in tempered sequence spaces. Using the Darbo-type fixed point theorems, we could establish solutions under various conditions, significantly advancing our understanding of these fractional hybrid differential equations.

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