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ON I-STATISTICAL CONVERGENCE IN G-METRIC SPACES

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Abstract: This paper deals with the introduction of the concept of ideal convergence of sequences in generalized metric spaces. Since the investigation in the G-metric space deals with two sequences, so the ideal considered is that of NxN. We have investigated some basic properties of the introduced notion.

Keywords and Phrases: Ideal, G-metric space, I-Cauchy.

2020 Mathematics Subject Classification: 40A05, 40A35, 46E30.

1. Introduction

In this section, we present some basic definitions and results on I-convergence, statistical I-convergence in G-metric spaces. The concept of statistical convergence was introduced in the year 1951 by Fast [6] and Steinhaus [17] independently and established a relation with summability. It was further investigated from sequence space point of view by Fridy [7], Salat [18], and many others. Applications of statistical convergence in number theory and mathematical analysis can be found in the works due to [1, 3, 5, 6, 7, 13, 14, 17, 18, 19, 20].

A generalization of statistical convergence is I-convergence which was introduced by Kostyrko et. al. [14]. An ideal I of the set $\mathbb{N}$ was used to define this concept from the topology. For details of the article we refer [5, 12, 19, 20].

The notion of statistical convergence is based on the definition of natural density of a set $A \subseteq \mathbb{N}$ such that $\delta(A) = \lim_{n\to\infty} \frac{|A_n|}{n}$ where $A_n = \{k \in A : k \leq n\}$ and $|A_n|$ gives the cardinality of A_n.

We procure the following well known definitions from the literature.

Definition 1.1. *A sequence (x_k) is statistically convergent to L provided that for every $\varepsilon > 0$, $\delta(\{k \leq n : |x_k - L| \geq \varepsilon\}) = 0$. It is denoted by $st - \lim x_k = L$.*

Definition 1.2. *Let X is a non-empty set. A family of subsets $I \subset P(X)$ is called an ideal on X if and only if*
(a) for each $A, B \in I$ implies $A \cup B \in I$;
(b) for each $A \in I$ and $B \subset A$ implies $B \in I$.

An ideal I is called non-trivial if $I \neq \emptyset$ and $X \notin I$. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if $I \supset \{\{x\} : x \in X\}$.

The concept of the distance is a very important concept in different branches of science and engineering. A metric is a distance function. But now a days, due to the availability of very large and complex data sets, the definition of metric has been generalised from different aspects. Keeping this in mind many studies have been carried out by different researcher, one may refer to the works due to [2, 4, 6, 8, 9, 10, 11, 14,15, 16].

Definition 1.3. *Let X be a non-empty set. A function $D : X \times X \times X \to \mathbb{R}^+$satisfying the following conditions:*
D1) $D(x, y, z) \geq 0$ for all $x, y, z \in X$.
D2) $D(x, y, z) = 0$ if and only if $x = y = z$.
D3) $D(x, y, z) = D(x, z, y) = D(y, z, x) = \cdots$ (symmetry in all three variables)
D4) $D(x, y, z) \leq D(x, y, t) + D(x, t, z) + D(t, y, z)$ for all $x, y, z, t \in X$ (rectangle inequality)

It is called a D-metric on X and (X, D) is called a D-metric space.

Subsequently in Dhang [4] attempted to develop topological structures in these spaces. These developments inspired many researcher to define a more appropriate generalized metric space called G-metric. Which is defined as follows:

Definition 1.6. *Let X be a non-empty set. A function $G : X \times X \times X \to \mathbb{R}^+$satisfying the following properties is called generalized metric or G-metric on X.*
(G1) $G(x, y, z) = 0$ if and only if $x = y = z$ for all $x, y, z \in X$

(G2) $0 < G(x, y, z)$, *for all* $x, y, z \in X$ *with* $x \neq y$
(G3) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$, *with* $z \neq y$
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ *(symmetry in all three variables)*
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$ *(rectangle inequality)*

The pair (X, G) *is called by a G-metric space.*

These properties are satisfied when $G(x, y, z)$ is the perimeter of a triangle with vertices at x, y and z in $\mathbb{R}^2$. Also taking a in the interior of the triangle shows that (G5) is best possible. G-metric function is a distance function which generalizes the concept of distance between three points.

Later, Abazani [1] defined statistical convergence in g-metric spaces where g-metric is a distance function defined between (n + 1) points.

2. Statistical Convergence in G-Metric Space

In this section we procure some known definitions.

Definition 2.1. *Let* $A \subset \mathbb{N}^2$ *and* $A(n) = \{i_1, i_2 \leq n : (i_1, i_2) \in A\}$, *then* $\rho_1(A) = \lim_{n\to\infty} \frac{2}{n^2}|A(n)|$ *is called 2-dimensional asymptotic (or natural) density of the set* A *.*

Definition 2.2. *Let* (x_i) *be a sequence in a G-metric space* (X, G). *For every* $\varepsilon > 0$, *if*

$$\lim_{n\to\infty} \frac{2}{n^2} \left|\{(i_1, i_2) \in A : i_1, i_2 \leq n, G\left(x, x_{i_1}, x_{i_2}\right) \geq \varepsilon\}\right| = 0,$$

then (x_i) *statistically converges to* x *in* X. *It is denoted by* $GS - \lim x_i = x$ *or* $x_i \xrightarrow{GS} x$.

Definition 2.3. *Let* (x_i) *be a sequence in a G-metric space* (X, G). *For every* $\varepsilon > 0$, *if*

$$\lim_{n\to\infty} \frac{2}{n^2} \left|\{(i_1, i_2) \in A : i_1, i_2 \leq n, G\left(x_n, x_{i_1}, x_{i_2}\right) \geq \varepsilon\}\right| = 0,$$

then (x_i) *statistically Cauchy in* X.

3. Main Results

In this section we introduce the following definitions and main results. Let (X, G) be a fixed G-metric space and I denotes a non-trivial admissible ideal of subsets of $\mathbb{N}$. Now we introduce some definitions.

Definition 3.1. *A sequence* (x_i) *of elements of a G-metric space* (X, G) *is said to be I-convergent to* $x \in X$ *if and only if for each* $\varepsilon > 0$, *the set*

$$A(\varepsilon) = \left\{(i_1, i_2) \in \mathbb{N}^2 : G\left(x, x_{i_1}, x_{i_2}\right) \geq \varepsilon\right\} \text{ belongs to } I_2.$$

The element x is called I-limit of the sequence (x_i). We write $I - \lim x_i = x$.

Definition 3.2. *A sequence (x_i) of elements of a G-metric space (X, G) is said to be I-null if $x = 0$. We write $I - \lim x_i = 0$.*

Definition 3.3. *Let I be an strongly admissible ideal of $\mathbb{N}$. A sequence $(x_i) \in X$ is said to be I-Cauchy if for each $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$, such that the set*

$$\left\{(i_1, i_2) \in \mathbb{N}^2 : i_1, i_2 \leq n_\varepsilon, G\left(x_{i_1}, x_{i_2}, x_{n_\varepsilon}\right) \geq \varepsilon\right\} \text{ belongs to } I.$$

Definition 3.4. *A sequence $(x_i) \in X$ is said to be I-bounded if there exists a non-negative real number M such that the set*

$$\left\{(i_1, i_2) \in \mathbb{N}^2 : G\left(x_i, x_{i_1}, x_{i_2}\right) > M\right\} \text{ belongs to } I.$$

Definition 3.5. *A sequence $(x_i) \in X$ is said to be I-statistically convergent to $x \in X$ if for every $\varepsilon > 0$ and $\delta > 0$, the set*

$$\left\{n \in \mathbb{N} : \frac{2}{n^2} \left|\left\{(i_1, i_2) \in A : i_1, i_2 \leq n, G\left(x, x_{i_1}, x_{i_2}\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \textit{ belongs to } I.$$

This situation is denoted by $GS_I - \lim x_i = x$. The set of all statistically convergent sequences in X is denoted by GS_I.

Example 3.1. Let $X = \mathbb{R}$ and G be the metric defined as follows:

$$G : \mathbb{R}^3 \to \mathbb{R}^+$$
$$G(x, y, z) = \max\{|x - y|, |y - z|, |z - x|\}.$$

Consider the sequences $I_1 = \mathbb{N} - \{1, 4, 9, 16, 25, ...\}$ and
$I_2 = \mathbb{N} - \{1, 8, 27, 64, 125, ...\}$. Consider the sequence (x_n) defined by
$x_n = \frac{1}{n}$, *if* $n \in I_1$ *or* I_2.
$= n$, *otherwise*.

Then it can be easily verified that $G(x, xi_1, x_{i_2}) = 0$ for $n \in I_1$ or I_2. So $GS_I - limx_n = 0$.

Definition 3.6. *A sequence $(x_i) \in X$ is said to be G-statistically Cauchy to $x \in X$, if for every $\varepsilon > 0$ and $\delta > 0$, the set*

$$\left\{n \leq \mathbb{N} : \frac{2}{n^2} \left|\left\{(i_1, i_2) \in A : i_1, i_2 \leq n, G\left(x_n, x_{i_1}, x_{i_2}\right) \geq \varepsilon\right\}\right| \geq \delta\right\} \text{ belongs to } I.$$

Theorem 3.1. *In a G-metric space every convergent sequence is statistically convergent.*

Proof. Let (x_i) be a sequence in a G-metric space (X, G) which converges to x. Then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $i_1, i_2 \geq n_0$,

$$G\left(x, x_{i_1}, x_{i_2}\right) < \varepsilon.$$

Let,

$$A(n) = \left\{(i_1, i_2) \in \mathbb{N}^2, i_1, i_2 \leq n : G\left(x, x_{i_1}, x_{i_2}\right) < \varepsilon\right\}.$$

Then,

$$|A(n)| \geq^{\binom{n-n_0}{2}} C_2.$$

Therefore, $\lim_{n\to\infty} \frac{2|A(n)|}{n^2} \geq \lim_{n\to\infty} \frac{2}{n^2} \frac{(n-n_0)!}{2!(n-n_0-2)!}$

$$\begin{aligned}
&= \lim_{n\to\infty} \frac{2}{n^2} \frac{(n-n_0)(n-n_0-1)}{2} \\
&= \lim_{n\to\infty} \frac{2}{2} \left\{1 - \frac{n_0}{n}\right\} \left\{1 - \frac{n_0}{n} - \frac{1}{n}\right\} \\
&= 1.
\end{aligned}$$

Thus, we have $GS - \lim x_i = x$.

The following example shows that the converse of the above theorem is not true.

Example 3.2. Let $X = \mathbb{R}$ and G be the metric defined as follows:

$$G : \mathbb{R}^3 \to \mathbb{R}^+$$
$$G(x, y, z) = \max\{|x-y|, |y-z|, |z-x|\}.$$

Consider the sequence (x_k) defined by

$$x_k = \begin{cases} k, & \text{if } k \text{ is a square} \\ 0, & \text{otherwise} \end{cases}$$

Then, it can be verified that (x_k) is statistically convergent, but it is not convergent.

Theorem 3.2. *Let (x_i) be a sequence in a G-metric space (X, G) such that $x_i \xrightarrow{GS_I} x$ and $x_i \xrightarrow{GS_I} y$. Then $x = y$.*

Proof. Let, $\varepsilon > 0$ be given and $\delta > 0$. Let

$$A_1 = \left\{(i_1, i_2) \in \mathbb{N}^2 : i_1, i_2 \leq n, \frac{2}{n^2} \left|\left\{(i_1, i_2) : G\left(x, x_{i_1}, x_{i_2}\right) \geq \frac{\varepsilon}{4}\right\}\right| \geq \frac{\delta}{2}\right\} \in I,$$

and

$$A_2=\left\{(i_1,i_2)\in\mathbb{N}^2 : i_1,i_2\le n,\ \frac{2}{n^2}\left|\left\{(i_1,i_2) : G\left(y,x_{i_1},x_{i_2}\right)\ge\frac{\varepsilon}{4}\right\}\right|\ge\frac{\delta}{2}\right\}\in I.$$

Let, $C=A_1\cup A_2$. Then, it follows that $C\in I$.
Suppose $(i_1,i_2)\in C^c$, then we have

$$\begin{aligned}
G(x,y,y) \le& G\left(x,x_{i_1},x_{i_1}\right)+G\left(x_{i_1},y,y\right)\\
=& G\left(x_{i_1},x_{i_1},x\right)+G\left(y,y,x_{i_1}\right)\\
\le& G\left(x,x_{i_1},x_{i_1}\right)+G\left(y,x_{i_1},x_{i_2}\right)\\
\le& G\left(x,x_{i_2},x_{i_2}\right)+G\left(x_{i_2},x_{i_1},x_{i_1}\right)+G\left(y,x_{i_1},x_{i_2}\right)\\
\le& G\left(x,x_{i_1},x_{i_2}\right)+G\left(x_{i_2},y,y\right)+G\left(y,x_{i_1},x_{i_1}\right)\\
&+G\left(y,x_{i_1},x_{i_2}\right)\\
=& G\left(x,x_{i_1},x_{i_2}\right)+G\left(y,y,x_{i_2}\right)+G\left(x_{i_1},x_{i_1},y\right)\\
&+G\left(y,x_{i_1},x_{i_2}\right)\\
\le& G\left(x,x_{i_1},x_{i_2}\right)+G\left(y,x_{i_1},x_{i_2}\right)+G\left(x_{i_1},x_{i_2},y\right)\\
&+G\left(y,x_{i_1},x_{i_2}\right)\\
=& 2G\left(x,x_{i_1},x_{i_2}\right)+2G\left(y,x_{i_1},x_{i_2}\right)\\
<& \varepsilon
\end{aligned}$$

Since, ε is arbitrary, we have $G(x,y,y)=0$.

This shows that $x=y$.

Theorem 3.3. *For any sequence (x_i), $st-\lim x_i=x$ implies that $GS_I-\lim x_i=x$.*
Proof. Let $st-\lim x_i=x$. Then for each $\varepsilon>0$, the set $A(\varepsilon)=\{(i_1,i_2)\in\mathbb{N}^2 : G\left(x,x_{i_1},x_{i_2}\right)\ge\varepsilon\}$ has natural density zero, i.e., $\lim_{n\to\infty}\frac{2}{n^2}|A(\varepsilon)|=0$.

Therefore, for every $\varepsilon>0$ and $\delta>0$, the set $\left\{(i_1,i_2)\in\mathbb{N}^2 : i_1,i_2\le n,\frac{2}{n^2}\left|\{(i_1,i_2) : G\left(x,x_{i_1},x_{i_2}\right)\ge\varepsilon\}\right|\ge\delta\right\}$ is finite set and belongs to I since I is an admissible ideal.

Hence, $GS_I-\lim x_i=x$.

Theorem 3.4. *If (x_i), be a sequence, then $I-\lim x_i=x$ implies that $GS_I-\lim x_i=x$.*
Proof. $I-\lim x_i=x$ shows that for every $\varepsilon>0$, the set $A(\varepsilon)=\{(i_1,i_2)\in\mathbb{N}^2 : G\left(x,x_{i_1},x_{i_2}\right)\ge\varepsilon\}\in I$.
But for a given $\delta>0$, the set

$$\left\{(i_1,i_2)\in\mathbb{N}^2 : i_1,i_2\le n,\ \frac{2}{n^2}\left|\{(i_1,i_2) : G\left(x,x_{i_1},x_{i_2}\right)\ge\varepsilon\}\right|\ge\delta\right\}\subset A(\varepsilon).$$

This shows that $GS_{I_2} - \lim x_i = x$ since I is admissible ideal.

Theorem 3.5. *If each subsequence of (x_i) is I-statistically convergent to x then (x_i) is I-statistically convergent to x.*

Proof. Suppose, the sequence (x_i) is no I-statistically convergent to x, then there exists $\varepsilon > 0$ and $\delta > 0$ such that

$B = \{(k,l) \in \mathbb{N}^2 : k,l \leq n, 2n^{-1} \mid (k,l) : G(x,x_k,x_l) \geq \varepsilon\} \mid \geq \delta\} \notin I.$

Since, I is admissible ideal so B must be an infinite set.

Let, $B = \{(k_1,l_1) < (k_2,l_2) < (k_3,l_3) < \ldots\}$.

Let $y_j = x_{k_j}$ or x_{l_j} for $j \in \mathbb{N}$. Then (y_j) is a subsequence of (x_i) which is not I-statistically convergent to x. Thus we have a contradiction.

Thus, the result is established.

Theorem 3.6. *Let (x_i) and (y_i) be two sequences,*

(i) if $GS_I - \mathrm{limx}\, x_i = x$ and $GS_I - \lim y_i = y$ then $GS_I - \lim (x_i + y_i) = x + y$.

(ii) If $GS_I - \lim x_i = x$ and $c \in \mathbb{R}$, then $GS_I - \mathrm{limcx}_i = cx$.

Proof. (i) Let

$$A_1 = \left\{(l,m) \in \mathbb{N}^2 : l,m \leq n, \frac{2}{n^2} \left|\left\{(l,m) : G(x,x_l,x_m) \geq \frac{\varepsilon}{2}\right\}\right| < \frac{\delta}{2}\right\} \notin I,$$

and

$A_2 = \left\{(l,m) \in \mathbb{N}^2 : l,m \leq n, \frac{2}{n^2} \left|\left\{(l,m) : G(y,y_l,y_m) \geq \frac{\varepsilon}{2}\right\}\right| < \frac{\delta}{2}\right\} \notin I.$

Clearly, $A_1 \cap A_2 \neq \emptyset$. Therefore for all $n \in A_1 \cap A_2$ we have,

$$\begin{aligned} \frac{2}{n^2}\Big| &\{(l,m) : G\,(x+y, x_l+y_l, x_m+y_m) \geq \varepsilon\} \,| \\ &\leq \frac{2}{n^2}\left|\left\{(l,m) : G(x,x_l,x_m) \geq \frac{\varepsilon}{2}\right\}\right| + \frac{2}{n^2}\left|\left\{(l,m) : G(y,y_l,y_m) \geq \frac{\varepsilon}{2}\right\}\right| \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta \end{aligned}$$

Hence, we have,

$$\left\{(l,m) \in \mathbb{N}^2 : l,m \leq n, \frac{2}{n^2}\left|\{(l,m) : G(x+y, x_l+y_l, x_m+y_m) \geq \varepsilon\}\right| < \delta\right\} \notin I$$

Therefore, $GS_I - \lim (x_i + y_i) = x + y$

(ii) If $c = 0$, then it is obvious.

Let, $c \neq 0$, then we have $\frac{2}{n^2}\left|\{(l,m) : l,m \leq n, G\left(cx, cx_l, cx_m\right) \geq \varepsilon\}\right|$,

$$= \frac{2}{n^2}\left|\{(l,m) : l,m \leq n, |c| G\left(x, x_l, x_m\right) \geq \varepsilon\}\right|$$
$$\leq \frac{2}{n^2}\left|\left\{(l,m) : l,m \leq n, G\left(x, x_l, x_m\right) \geq \frac{\varepsilon}{|c|}\right\}\right| < \delta.$$

Hence, $GS_I-\lim cx_i = cx$.

4. Conclusion

We have considered the generalized metric space and investigated the I-statistical convergence of sequences in G-metric space. Looking at the nature the ideal that we have considered is on the class of subsets of NxN. We have established some results. It is hoped that the work done will be applied by many for further investigations and applications.

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