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A STUDY ON CONJUGACY GRAPHS

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Abstract: In this paper, we introduce the notion of an equivalence graph based on equivalence relation defined on a group. Furthermore, restricting ourselves to conjugacy relation, a special type of equivalence graph called a conjugacy graph is also defined. In addition, a graph theoretical expression for the class equation is established followed by related results.

Keywords and Phrases: Equivalence graph, conjugacy relation, conjugacy graph.

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1. Introduction

Recall that a *group* G is a set together with a binary operation $*$ defined on it such that it satisfies closure property, associativity, the existence of an identity element, and the existence of the inverse. In order to avoid ambiguity we denote the identity element of the group by i_G. The *order* of a group, denoted by $o(G)$, is the total number of elements present in the group. Let H be a non-empty subset of G. Then, for any $a \in G$, the set $aH = \{ah : h \in H\}$ is called a *left coset* of H in G and the set $Ha = \{ha : h \in H\}$ is called a *right coset* of H in G. The *center* of a group G, denoted by $Z(G)$, is the set of elements in G that commute with every element in G. The *normalizer* of an element $a \in G$, denoted by $N(a)$, is the set of those elements in the group G that commute with a. For terms and definitions in group theory, we refer to [13, 16].

A graph is a discrete structure consisting of vertices and edges, which can be used to represent any binary relation. For the terminology in graph theory, we refer to [21]. In this discussion, we denote a group by G and the corresponding graph by Γ_G. The graph may be denoted by Γ if the context is clear.

For a graph Γ, its subgraph Γ' is said to be an induced subgraph if Γ' is a graph such that $V(\Gamma') \subseteq V(\Gamma)$ and $E(\Gamma')$ is the set of all edges in Γ whose end vertices are in Γ'. A *singleton component* of a graph is a component of the graph consisting of exactly one vertex in it. A *non-singleton component* of a graph is a component consisting of two or more vertices in it.

The terms like isomorphism, homomorphism and automorphism are not restricted to algebraic structures but to graphs as well. This ignites an intuitive thought that there can be a relation between graphs and algebraic structures as both of them are structure-oriented concepts.

Many graphs like Cayley graphs, circulant graphs, transitive graphs, etc. (see [12]) lay the foundation for algebraic graph theory as a separate field of study. An insight into other such studies includes graphs associated with conjugacy classes of groups (see [6-17, 17, 19, 20]), inverse graphs on finite groups (see [23]), non-commuting graphs (see [1]), order sum graphs (see [5]), prime graphs (see [22]), non-inverse graphs (see [4]) and the list keeps on increasing. The approach in these studies gives a beautiful structural insight into the constructed graphs and a detailed study of different parameters of coloring, domination, connectivity and many other structural aspects related to the constructed graph. The spectral analysis of some algebraic graphs can be viewed in [2, 3, 14, 15, 18].

2. Equivalence Graphs of Groups

Recall that an equivalence relation defined on a set A is a relation that is reflexive, symmetric and transitive. Based on an equivalence relation defined on a group G, the notion of an equivalence graph is defined as follows:

Definition 2.1. *Let $(G, *)$ be a finite group with an equivalence relation R defined on G. The equivalence graph of G, denoted by $\pi(G)$, with respect to R is the graph such that*

(i) $V(\pi(G)) = G$.

(ii) For $u, v \in V(\pi(G))$, $uv \in E(\pi(G))$ if and only if $u * v \in R$.

The following theorem is a generalisation for partitioning the vertices of an equivalence graph in terms of cliques.

Theorem 2.2. *Let R be an equivalence relation defined on a group G and $\mathscr{E}$ be the collection of equivalence classes $\mathscr{E}_i$ of G with respect to R. Then, the equivalence graph $\pi(G)$ is a disjoint union of cliques.*

Proof. Consider an equivalence class $\mathscr{E}_i$ of G with respect to the relation R. Because of the symmetry of R, any two elements in $\mathscr{E}_i$ are related to each other, implying any two elements belonging to $\mathscr{E}_i$ are adjacent to each other. Thus, each $\mathscr{E}_i$ of G induces a clique, say $Q(\mathscr{E}_i)$ in $\pi(G)$. Moreover, R partitions G into mutually disjoint equivalence classes, the corresponding partition makes $\pi(G)$ a disjoint union of corresponding cliques. Thus, $V(\pi(G)) = \bigcup\limits_{\mathscr{E}_i \in \mathscr{E}} V(Q(\mathscr{E}_i))$. This completes the proof.

It is to be noted that any arbitrary vertex belonging to a particular component can be treated as a representative vertex of that component. The following theorem provides an alternate graph theoretical proof for Lagrange's theorem.

Theorem 2.3. [Lagrange's Theorem] *Let $(G, *)$ be a finite group and $(H, *)$ be a subgroup of G. Then, $o(H)|o(G)$.*

Proof. Let $\sim_H$ denote the equivalence relation on G such that $a \sim_H b$ if and only if $a^{-1}b \in H$, where $a, b \in G$. Let $G_H = \{a_1H, a_2H, \ldots, a_nH : a_i \in G\}$ denote the set of all distinct left cosets of H in G.

Let $\mathcal{C}(G)$ be a graph defined as follows:

(i) $V(\mathcal{C}(G)) = G$.

(ii) For $g, h \in V(\mathcal{C}(G))$, $gh \in E(\mathcal{C}(G))$ if and only if , $g \sim_H h$ in G.

By Theorem 2.2, each equivalence class a_iH in G induces a clique, say $\langle a_iH\rangle$ in $\mathcal{C}(G)$, and $\mathcal{C}(G)$ is a disjoint union of cliques $\langle a_iH\rangle$. Hence, for $1 \leq i \neq j \leq n$, $\langle a_iH\rangle$ and $\langle a_jH\rangle$ are disjoint components in $\mathcal{C}(G)$.

Since, a_iH induces a clique and $|a_iH| = |H|$ for any i, we have $|\langle V(a_iH)\rangle| = |\langle V(a_jH)\rangle| = |\langle V(H)\rangle|$ and hence $\mathcal{C}(G)$ is a regular graph.

Therefore, $|V(\mathcal{C}(G)| = |\ddot{\bigcup\limits_{a_i \in \mathcal{C}(G), i \in [n]}} V(\langle a_iH\rangle)| = \sum \bigcup\limits_{a_i \in \mathcal{C}(G) i \in [n]} |\langle V(a_iH)\rangle| = q|\langle V(H)\rangle|$, where $n = o(G) = |V(\mathcal{C}(G))|$. Therefore, as q is an integer, we have $|\langle V((H)\rangle|$ divides $|V(\mathcal{C}(G))|$. By the definition of $\mathcal{C}(G)$, $V(\mathcal{C}(G)) = G$ and $\langle H\rangle$ corresponds to the subgroup H in G. Thus, the result holds true for group $(G, *)$ with its subgroup $(H, *)$. That is, $o(H)$ divides $o(G)$.

The following corollary is an immediate consequence of Theorem 2.3 as the elements in the vertex set of the graph is same as that of the group and the elements in the vertex set of the subgraph is same as that in the subgroup.

Corollary 2.4. *Let Γ_G be a graph with $V(\Gamma_G) = G$ and H' be its subgraph with $V(H') = H$, then $|V(H')|$ divides $|V(\Gamma_G)|$.*

3. Graphs on Conjugacy Classes of Finite Groups

Let G be a finite group and $a, b \in G$. Then, b is said to be conjugate of a if there exists an element $c \in G$ such that $b = c^{-1}ac$ and it is denoted by $b \sim a$. The relation '$\sim$' is termed as *conjugacy relation*. For an element a in the group G, the conjugate class of a is denoted by $C(a)$ and defined as $C(a) = \{b \in G : b \sim a\}$. It can be observed that the conjugacy relation is an equivalence relation.

Owing to the concepts of conjugacy relation and conjugate classes defined on a finite group, a new class of algebraic graphs, namely conjugacy graph, is defined as follows:

Definition 3.1. *Let $(G, *)$ be a finite group with the conjugacy relation R defined on G. The conjugacy graph of G, denoted by $\Gamma(G)$, with respect to R is the graph such that*

(i) $V(\Gamma(G)) = G$.

(ii) For $a, b \in V(\Gamma(G))$, $ab \in E(\Gamma(G))$ if and only if $a \sim b$.

Theorem 3.2. [Structure Theorem] *The following results hold true for the conjugacy graph Γ of a finite group G.*

(i) The vertex i_G is always an isolated vertex contributing to a singleton component in Γ.

(ii) Γ is a disconnected graph with $C(a)$ being its component, where a is a representative vertex from each component.

(iii) Let $\mathscr{F}$ be the family of disjoint conjugate classes in the group G. Then, $V(\Gamma) = \bigcup\limits_{C(a_i)\in\mathscr{F}}^{\cdot\cdot} V(C(a_i))$, where a_i is a representative vertex of the component $C(a_i)$.

(iv) Each component induced by $C(a)$ is a clique with $|C(a)|$ vertices in it.

(v) Let $d(a)$ denote the degree of $a \in V(\Gamma)$ in $C(a)$, then $d(a) + 1 = |C(a)|$.

(vi) The elements belonging to the center of a group $Z(G)$ are isolated vertices in Γ. Hence elements of $Z(G)$ always result in singleton components in Γ.

Proof. For any group G with identity element i_G, $C(i_G) = \{i_G\}$. Thus, $C(i_G)$ induces singleton component in Γ. In other words, i_G is always an isolated vertex in Γ. Since $\sim$ is an equivalence relation, it will result in a partition of elements. This implies that any two conjugate classes in the group G are either identical or disjoint and also we have i_G to be an isolated vertex. Thus, Γ has at least two components.

Since the disjoint conjugate classes form a partition of the elements, they will also induce distinct components and partition of the vertices in Γ. Therefore, $V(\Gamma)$ is a disjoint union of vertices of components $C(a)$.Consider a conjugate class of an element a. All the elements in $C(a)$ are conjugate to each other implying that they are adjacent to each other in the component induced by $C(a)$ in Γ. Thus, inducing a clique with its order equal to the order of $C(a)$.

Let $d(a)$ denote the degree of the representative vertex a in the component induced by $C(a)$. Since $C(a)$ induces a complete graph, $d(a)$ counts all the elements of $C(a)$ except a. Therefore, $d(a) + 1 = |C(a)|$. The center $Z(G)$ consists of all the elements that commute with every element in G. If $x \in Z(G)$, then $c^{-1}xc = x$ for any $c \in G$, implying that $C(x) = \{x\}$. Hence, conjugate class of elements belonging to $Z(G)$ always induce singleton components in Γ and thus are always isolated vertices in Γ, completing the proof.

Theorem 3.3. [Class Equation for graph] *For any conjugacy graph Γ of a group G, $|V(\Gamma)| = \ell + \sum\limits_{a\in C(a),\ |C(a)|>1} (d(a)+1)$, where summation runs over one representative vertex a from each non-singleton component and ℓ denotes the number of isolated vertices in Γ.*

Proof. By Theorem 3.2, we know that Γ is a disconnected graph consisting of singleton components and non-singleton components. Hence, counting all the vertices in the components would give us the order of the graph. Let ℓ denote the total number of distinct singleton components in Γ. The number of vertices in each non-singleton component is $d(a)+1$, where a is a representative vertex chosen from each non-singleton component. Thus, $\sum\limits_{a\in C(a),\ |C(a)|>1} (d(a)+1)$ gives the total number of vertices in non-singleton components in Γ. Summing up the total number of vertices in singleton and non-singleton components gives us the order the graph Γ. Therefore, $|V(\Gamma)| = \ell + \sum\limits_{a\in C(a),\ |C(a)|>1} (d(a)+1)$.

Proposition 3.4. *The size of Γ is given by $m(\Gamma) = \sum\limits_{a\in C(a)} \frac{(d(a)+1)d(a)}{2}$, where the sum runs over one representative vertex a from each component.*

Proof. Consider the conjugacy graph Γ, the singleton components are edgeless and hence do not contribute any edge. By Theorem 3.2, we have that the non-singleton components are complete graphs and we known that, $|E(K_n)| = \frac{n(n-1)}{2}$. Also, we can have more than one non-singleton component each of order $d(a)+1$. Thus, the total number of edges in each non-singleton component is equal to $\frac{(d(a)+1)d(a)}{2}$. Summing up all the edges from each non-singleton component, we get $m(\Gamma) = \sum\limits_{a \in C(a)} \frac{(d(a)+1)d(a)}{2}$.

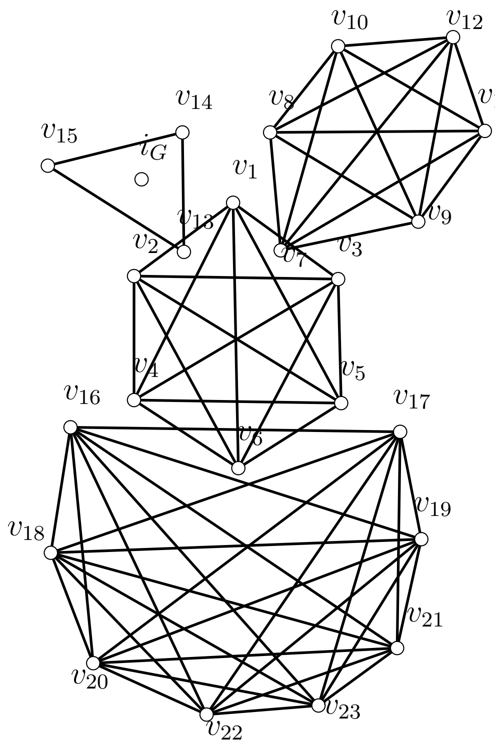

Figure 1: Conjugacy graph of symmetric group S_4

4. Applications of Structure Theorem

Theorem 4.1. [16] *If $o(G) = p^n$, where p is a prime number, then the center $Z(G)$ has at least one non-identity element in it.*

Theorem 4.2. *For a conjugacy graph Γ of a group G, if $|V(\Gamma)| = p^n$, then $\ell \geq 2$, where ℓ denotes the number of isolated vertices in Γ.*

Proof. Consider the class equation $|V(\Gamma)| = \ell + \sum\limits_{a \in C(a), |C(a)|>1} (d(a)+1)$, where summation runs over one arbitrary representative vertex a from each non-singleton component, where ℓ denotes the number of isolated vertices in Γ. By the hypothesis, we have $|V(\Gamma)| = p^n$, where p is a prime number. We need to prove that $\ell \geq 2$. Further, it suffices to prove that $p \mid (V(\Gamma) - \sum_{a \in C(a)} (d(a)+1))$. We know that, $|G| = |C(a)|\,|N(a)|$, this implies that $\frac{|G|}{|C(a)|} = |N(a)|$. Since $|N(a)|$ is an integer, $|C(a)|$ must divide $|G| = p^n$. Thus, $|C(a)| = p^k$, where k is a positive integer and $k \leq n$. By Theorem 3.2, we have $|C(a)| = d(a) + 1 = p^k$. This implies that $\sum_{a \in C(a)} (d(a)+1) = k_1 p^k$, where k_1 is some positive integer. Thus, we have $p^n = \ell + k_1 p^k$, implying $(p^n - k_1 p^k) = \ell$. Considering the term $(p^n - k_1 p^k)$ we have, $p \mid p^n$ and $p \mid p^k$ which implies $p \mid k_1 p^k$. Therefore $p \mid (p^n - k_1 p^k)$. Thus, p must divide ℓ. Hence, $\ell \geq 2$.

Recall the following result, which is a consequence of Theorem 4.1.

Corollary 4.3. [16] *If $o(G) = p^2$, where p is a prime number, then G is Abelian.*

In view of the result mentioned above, we have the following result.

Theorem 4.4. *For a conjugacy graph Γ, if $|V(\Gamma)| = p^2$, where p is a prime number, then $\ell = V(\Gamma)$, where ℓ denotes the number of isolated vertices in Γ.*

Proof. Let $H = C(a)$ denote the component corresponding to the conjugate class of a in the graph $\Gamma = \Gamma(G_H)$. Consider the class equation $|V(\Gamma(G_H))| = \ell + \sum_{a \in C(a)} (d(a)+1)$. By Theorem 3.2, we have that $\ell = |Z(G)|$, where $Z(G)$ is the center of the group G. By Corollary 2.4, $\ell = |Z(G)|$ must divide $|V(\Gamma(G_H))| = p^2$. Therefore, $\ell = |Z(G)|$ must be either 1 or p or p^2. Now, since each non-singleton component H also induces a subgraph in $\Gamma(G_H)$, $|H| = d(a) + 1$ must divide $|V(\Gamma(G_H))| = p^2$. Therefore, $d(a) + 1$ must be 1 or p or p^2.

Case 1: Let $d(a) + 1 = 1$. This implies $d(a) = 0$, and thus all the vertices in $\Gamma(G_H)$ are isolated vertices. Therefore, $\ell = |V(\Gamma(G_H))|$. The case becomes trivial.

Case 2: Let $d(a) + 1 = p^2$. This implies that $d(a) + 1 = |V(\Gamma(G_H))|$. Thus, $\ell = 0$ is a contradiction to Theorem 4.2. Hence, this case is not possible.

Case 3: Let $d(a) + 1 = p$. Let us consider a vertex a such that a is not an isolated vertex in $\Gamma(G_H)$. This implies that $a \notin Z(G)$. We also have $Z(G) \subset N(a) \subset G$ as $a \notin Z(G)$, where $N(a)$ is a normalizer of an element $a \in G$. Now, in the graph $\Gamma(G_H)$, $Z(G)$ induces a proper subgraph of the subgraph induced by $N(a)$ in $\Gamma(G_H)$. Thus, by the corollary of Theorem 2.3, $\ell = |Z(G)|$ must divide $|V(N(a)|$. Hence, $|V(N(a))|$ is either p or p^2, but since a is not an isolated vertex, $|V(N(a))|$ must be greater than $\ell(= p)$. Moreover, we have, $|G| = |C(a)|\,|N(a)|$; that is, $|V(\Gamma(G_H))| = |V(C(a))|\,|V(N(a))|$. Thus, we have $\frac{|V(\Gamma(G_H))|}{|V(C(a))|} = |V(N(a))|$, which is $\frac{p^2}{d(a)+1} = |V(N(a))|$. Hence, $d(a) + 1 = p$ is not possible. This completes the proof.

5. Conclusion

In this paper, we have defined a particular graph called the equivalence graph derived from groups using the terminology of equivalence relation. We have also constructed the conjugacy graphs as a special case of equivalence graphs based on conjugacy relation and related observations. Further, we have also given graph theoretical expression and proof for class equation and related results. As a further scope of the study, the isomorphism of introduced graphs with any existing graphs can be investigated. Representing the graphs introduced in the paper as intersection graphs and related study in structural aspects would be promising. The applications of the class equation in group theory can be investigated in graph-theoretical aspects.

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