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SOBER OPEN SETS IN SOBER TOPOLOGICAL SPACES

D. Jayanthi and S. Jafari*

Department of Mathematics,
Avinashilingam Institute for Home Science and
Higher Education for Women, Coimbatore, INDIA

E-mail : jayanthimaths2006@gmail.com

*Mathematical and Physical Science Foundation,4200 Slagelse, DENMARK

E-mail : jafaripersia@gmail.com

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Abstract: The aim of this paper is to introduce a new class of sets called sober open sets and investigate their basic properties. In this relation we establish a new type of topology called sober topology to further investigate sober open sets and related notions.

Keywords and Phrases: Sober open set, sober topological space, sober separation.

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1. Introduction

Geometry and analysis are built upon the foundation of general topology. Topology saw great advancement in the early 20th century. The significance of open and closed sets in topological spaces has been examined by numerous academicians. Soon-Mo Jung [11] analyzed a few features of interior and closure. The significance of topological spaces is explained by the continuities, connectedness, and separation axioms in a variety of domains. Additionally, Al-Shami [2] introduced the notions of somewhere dense sets and T_1 spaces, delving into aspects of somewhere dense continuity [3], compactness, and CS-dense sets [4] in collaboration with Noiri.These investigations collectively fortify the theoretical underpinnings of topology. Furthermore, this work presents a novel type of open sets known as sober open sets. Additionally, a sober closed set—the complement of a sober open set—is introduced, and some of its characteristics are examined. Additionally, we define sober continuity, separation, and connectivity and demonstrate their fundamental properties.It is demonstrated that, under certain circumstances, a sober continuity can transform into a strong continuity by comparing it to strong continuity. Moreover, sober connectedness is contrasted with connectedness, and it has been demonstrated by a counterexample that connectedness does not always entail sober connectedness.

2. Preliminaries

In this section we recall some basic definitions which help to shape this article.

Definition 2.1. [5, 6, 7] *A topology on a set X is a collection τ of subsets of X having the following properties:*

1. *ϕ and X are in τ*
2. *Arbitrary union of elements of τ belongs to τ*
3. *Finite intersection of elements of τ belongs to τ.*

A set X together with the topology τ is known as topological space and the elements of τ are called the open sets. The topological space is denoted by (X, τ).

Definition 2.2. [6] *Let (X, τ) be a topological space. If Y is a subset of X, the collection $\tau_Y = \{Y \cap U \in \tau\}$ is a topology on Y called the subspace topology and Y is called the subspace of X.*

Definition 2.3. [1] *For any set A in (X, τ), the interior and closure are defined by*

$$int(A) = \bigcup \{K/K \text{ is an open set in } X \,\&K \subseteq A\}$$
$$cl(A) = \bigcap \{L/L \text{ is a closed set in } X \,\&A \subseteq L\}$$

Definition 2.4. [9, 10] *Let X and Y be topological spaces. A function $f : X \to Y$ is said to be continuous if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X.*

Definition 2.5. [8] *A mapping $f : X \to Y$ is said to be strongly continuous if for every subset A of X, $f(cl(A)) \subseteq f(A)$.*

Levine further established that "f is strongly continuous if and only if each subset's inverse image is open (or closed)". Thus, it is clear that a mapping is strongly continuous if and only if each subset's inverse image is both open and closed (i.e., clopen).

Definition 2.6. [6] *Let X be a topological space. A separation of X is a pair U, V of disjoint non-empty open subsets of X whose union is X. The space X is said to be connected if there exists no separation of X.*

Definition 2.7. [6] *A topological space is said to be Hausdorff if each pair x_1, x_2 of distinct points of X, there exist neighborhoods U_1 and U_2 of x_1 and x_2 respectively, that are disjoint.*

Definition 2.8. [6] *Suppose that one-point sets are closed in X. Then X is said to be regular if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B respectively.*

Definition 2.9. [6] *Suppose that one-point sets are closed in X. Then X is said to be normal if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.*

3. Sober open sets

In this section the definitions of a new open set called sober open set and its complement sober closed set in topological space are introduced and their basic properties are discussed. Also a topological space called sober topological space is introduced to study further sober open sets.

Definition 3.1. *A non-empty set $A \neq X$ is said to be a sober open set in a topological space (X, τ), if there exist two distinct non-empty open sets $A_1 \neq X$ and $A_2 \neq X$ in X such that*

1. *$A \cup A_1$ is open and $A \cup A_1 \neq X$*
2. *$A \cap A_2$ is open and $A \cap A_2 \neq \phi$*

The complement A^c of a sober open set A is a sober closed set in (X, τ). The collection of all sober open (sober closed) sets is denoted by $B_o(X)(B_c(X))$ and the union of all sober open (sober closed) sets is denoted by $\mathbf{O_{sober}}(\mathbf{D_{sober}})$.

Example 3.2. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$, then (X, τ) is a topological space. Let $A = \{b, c\}$ be a non empty subset of X. Then A is a sober open set in X, since there exist two distinct non-empty open sets $A_1 = \{a, c\}$ and $A_2 = \{c\}$ such that $A \cup A_1 = \{a, b, c\} \neq X$ is open and $A \cap A_2 = \{c\} \neq \phi$ is open.
Here $B_o(X) = \{\{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
It is to be noted that the empty set and the whole set of X are not sober open.

Therefore sober open sets do not establish a topology.

Definition 3.3. *The sober complement of a sober open (res. sober closed) set A (res. C), denoted by ${}^c(A)$, is defined by ${}^c(A) = \mathbf{O_{sober}}$ - A (res. ${}^c(C) = \mathbf{D_{sober}}$ - C).*

Remark 3.4. *It is not necessary that the sober complement of a sober open set is sober closed and the sober complement of a sober closed set is sober open in (X, τ).*

Example 3.5. In Example 3.2, $A = \{b, c\}$ is a sober open set. Then the sober complement of A is ${}^c(A) = \mathbf{O_{sober}} - A = \{a, b, c\} - \{b, c\} = \{a\}$ is not a sober closed set. The other way around is analogous.

Remark 3.6. *Every open set in (X, τ) need not be sober open in general.*

Example 3.7. *Let $X = \{a, b\}$ and $\tau = \{\phi, \{a\}, X\}$, then (X, τ) is a topological space. Here $\{a\}$ is an open set but not a sober open set, as there exist no non-empty open sets $A_1 \neq A_2 \neq X$ satisfying Definition 3.1.*

Theorem 3.8. *An open set A in (X, τ) is a sober open set if and only if it is either a proper subset or a superset of a proper open set in (X, τ).*
Proof.
Sufficiency: Let A be an open set in (X, τ) and $A \subset B$ where B is a proper open set in X. Then since A itself is open such that $A \cup A = A$ and $A \cap B = A$ which are open in X, A is sober open in X.
If in the above case, $A \supset B$, then $A \cap B = B$, which is also open and the conditions to be a sober open set are satisfied.
Necessity: Let an open set A in X be sober open in X. Then there exist two proper open sets say A itself and B with required conditions. Then obviously either $A \supset B$ or $A \subset B$.

Theorem 3.9. *Any non-empty set $A \neq X$ is sober open if and only if it is a subset of a non-trivial open set and a superset of a non-trivial open set in X.*
Proof.
Sufficiency: Let A be a non-trivial subset of X and let A_1, A_2 be two non-trivial open sets in τ such that $A \subseteq A_1$ and $A \supseteq A_2$. Then $A \cup A_1 = A_1$ and $A \cap A_2 = A_2$. Hence A satisfies the definition of a sober open set and so A is sober open in X.
Necessity: Let A be sober open in X. Then by Definition 3.1, there exist two non-empty open sets $A_1 \neq A_2 \neq X$ such that $A \cup A_1$ and $A \cap A_2$ are open in X. Obviously $A \cup A_1 \supseteq A$ and $A \cap A_2 \subseteq A$.

Proposition 3.10. *Any singleton set $A = \{x\}$ in X is sober open if and only if*

1. $A \in \tau$

2. $A \subset A_1$, where $A_1 \in \tau$

Proof.
Necessity: Let A be sober open. Then by Definition 3.1 there exist two non-empty open sets $C \neq D \neq X$ such that $A \cup C \in \tau$ and $A \cap D \in \tau$. These conditions imply that respectively $A \subset A \cup C = A_1$ (say) in τ and $A \cap D = \{x\} = A \in \tau$.
Sufficiency: Suppose A satisfies conditions (i) & (ii), then by Theorem 3.8, A is sober open in X.

Remark 3.11. *The union of two sober open sets need not be sober open in general.*

Example 3.12. Let $X = \{a, b, c, d, e\}$ and $\tau = \{\phi, \{e\}, \{a, e\}, \{c, e\}, \{a, c, e\}, \{a, b, c, e\}, \{a, c, d, e\}, X\}$. Then (X, τ) is a topological space. Let $A = \{b, c, e\}$ and $B = \{c, d, e\}$. Then A is sober open as there exist two distinct non-empty proper open sets $A_1 = \{a, e\}$ and $A_2 = \{e\}$ such that $A \cup A_1 = \{a, b, c, e\} \neq X$, is open and $A \cap A_2 = \{e\} \neq \phi$, is open. Also B is sober open since there exist two non-empty proper open sets $B_1 = \{a, e\}$ and $B_2 = \{a, c, e\}$ such that $B \cup B_1 = \{a, c, d, e\} \neq X$, is open and $B \cap B_2 = \{c, e\} \neq \phi$ is open. Now $A \cup B = \{b, c, d, e\}$ is not sober open as there exist no non-empty proper open sets which satisfy the conditions as in Definition 3.1.

Remark 3.13. *The intersection of two sober open sets need not be sober open in general.*

Example 3.14. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, X\}$. Then (X, τ) is a topological space. Let $A = \{a, b\}$ and $B = \{a, c\}$.Then A is sober open as it is open. Also B is sober open since there exist two non-empty open sets $B_1 = \{a, b\}$ and $B_2 = \{b, c\}$ such that $B \cup B_1 = \{a, b, c\} \neq X$, is open and $B \cap B_2 = \{c\} \neq \phi$ is open. Now $A \cap B = \{a\} \notin \tau$ and from Proposition 3.10, it is not sober open.

Remark 3.15. *Sober open sets in a topological space (X, τ) exist only when τ contains at least two proper open sets which are not disjoint.*

Hereafter we consider only the topology which has at least two proper open sets which are not disjoint. We call it as sober topology denoted by τ_{sober} and the corresponding space as the sober topological space denoted by X_{sober}.

Definition 3.16. *Let X_{sober} be a sober topological space with topology τ_{sober}. If Y is a subset of X, the collection $\tau_Y = \{Y \cap U / U \in \tau_{sober}\}$ on Y, is a topology called the subspace topology. Then (Y, τ_Y) is called a subspace of X_{sober}. It is to be noted that Y is also a sober topological space.*

Proposition 3.17. *In a sober topological space X_{sober} every proper open set is a* *sober open set.*

Proof. As in a sober topological space, every proper open set is either a proper subset or a super set of a proper open set, the Proposition holds good from Theorem 3.8.

Definition 3.18. *Let (X, τ_{sober}) be a topological space. Then the sober - interior and sober - closure of any proper set A in X_{sober} is defined by*

$$sober - int(A) = \bigcup \{K/K \text{ is a sober open set in } X \ \&K \subseteq A\},$$
$$sober - cl(A) = \bigcap \{K/K \text{ is a sober closed set in } X \ \&A \subseteq K\}.$$

Remark 3.19. *For any proper set A, $sober - int(A) \subseteq A$ is true only when $sober - int(A)$ exists. Also $A \subseteq B \Rightarrow sober - int(A) \subseteq sober - int(B)$ is true for any two proper sets A and B only when $sober - int(A)$ and $sober - int(B)$ exist.*

Example 3.20. Let $X = \{a, b, c, d\}$ and $\tau_{sober} = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, X\}$. Then (X, τ_{sober}) is a topological space. Here the collection of sober open sets $B_o(X) = \{\{b\}, \{c\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$. Let $A = \{a\}$ be a non-empty set. Then there exists no sober open set which is contained in A.

Proposition 3.21. *If A is sober open, then $sober-int(A) = A$ but not conversely in (X, τ_{sober}).*

Proof. Let $\{A_i\}$ be a collection of sober open sets contained in A where A is a sober open set. Then $\cup A_i \subseteq A$ and as A itself is a sober open set, $sober-int(A) = A \cup (\cup A_i) = A$.

Example 3.22. In Example 3.12, $B_o(X) = \{\{e\}, \{a, e\}, \{b, e\}, \{c, e\}, \{d, e\}, \{a, b, e\}, \{a, c, e\}, \{a, d, e\}, \{b, c, e\}, \{c, d, e\}, \{a, b, c, e\}, \{a, c, d, e\}\}$ and $sober-int(A) = A$ for $A = \{b, d, e\}$. But A is not a sober open set in (X, τ_{sober}).

Theorem 3.23. *If A and B are sober open sets in a topological space X_{sober}, then the following are satisfied:*

1. $sober - int(A \cup B) \subseteq sober - int(A) \cup sober - int(B)$
2. $sober - int(A \cap B) \subseteq sober - int(A) \cap sober - int(B)$
3. $A \subseteq B$ implies $sober - int(A) \subseteq sober - int(B)$
4. $sober - int(sober - int(A)) = sober - int(A)$

Proof.

1. If A and B are sober open, then by Proposition 3.21, $sober-int(A) = A$ and $sober-int(B) = B$. Now $sober-int(A \cup B) \subseteq A \cup B = sober-int(A) \cup sober-int(B)$

2. is similar to (1) and (3) & (4) are obvious.

Remark 3.24. *For any proper set A in (X, τ_{sober}), if $sober-int(A)$ exists, then $int(A) \subseteq sober-int(A)$.*

Theorem 3.25. *The following are true in a sober topological space (X, τ_{sober}) for any two proper sets A and B whose sober - closure exist:*

1. $A \subseteq sober-cl(A)$ and $sober-cl(A) \subseteq cl(A)$

2. If A is sober closed, then $A = sober-cl(A)$

3. $A \subseteq B \Rightarrow sober-cl(A) \subseteq sober-cl(B)$

4. ${}^{c}(A) \neq A^{c}$ in general and ${}^{c}(A) = A^{c}$ if and only if $\mathbf{O_{sober}} = X$ or $\mathbf{D_{sober}} = X$

Proof. Straightforward.

4. Sober continuous mappings and sober separation

In this section continuous mapping between two sober topological spaces has been established and the properties are investigated. Separation and connectedness based on sober open sets are introduced (called as sober separation and sober connected) and some results on sober connectedness are analyzed.

Definition 4.1. *A mapping $f : X_{sober} \to Y_{sober}$ is said to be sober continuous, if $f^{-1}(A)$ is sober open in X_{sober} for every sober open set A in Y_{sober}.*

Example 4.2. Let (X, τ_{sober}) and (Y, σ_{sober}) be two sober topological spaces, where $X = \{a, b, c\}$ and $Y = \{a, b, c\}$, $\tau_{sober} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma_{sober} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Here $B_o(X_{sober}) = \{\{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and $B_o(Y_{sober}) = \{\{a\}, \{b\}, \{a, b\}, \{b, c\}\}$.

Define a mapping $f : X_{sober} \to Y_{sober}$ by $f(a) = b, f(b) = a$ and $f(c) = c$. Then f is a sober continuous mapping.

It is very important to note that the sober continuous mappings are different from other continuous mappings as ϕ and X are not sober open sets. But a strongly continuous mapping can be made into a sober continuous mapping under some circumstance. Following proposition explains it for better understanding.

Proposition 4.3. *Every strongly continuous mapping $f : X_{sober} \to Y_{sober}$ is sober continuous if the inverse image of every subset of Y is a proper subset of X.*
Proof. Let $f : X_{sober} \to Y_{sober}$ be a strongly continuous mapping. Let A be a sober open set in Y. Then by hypothesis, its inverse image $f^{-1}(A)$ is a proper clopen set in X. That is $f^{-1}(A)$ is a proper open set in X and hence it is a sober open set in X. This implies that f is a sober continuous mapping.

Example 4.4. Let (X, τ_{sober}) and (Y, σ_{sober}) be two sober topological spaces, where $X = \{a, b, c\}$ and $Y = \{a, b, c\}$, $\tau_{sober} = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $\sigma_{sober} = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$. Here $B_o(X_{sober}) = \{\{a\}, \{a, b\}, \{a, c\}\}$ and $B_o(Y_{sober}) = \{\{b\}, \{a, b\}, \{b, c\}\}$.

Define a mapping $f : X_{sober} \to Y_{sober}$ by $f(a) = b, f(b) = a$ and $f(c) = c$. Then f is a sober continuous mapping but it is not strongly continuous as, $\{a, c\}$ is a subset of Y, but $f^{-1}(\{a, c\}) = \{b, c\}$ is not open in X.

Theorem 4.5. *Let X_{sober} and Y_{sober} be sober topological spaces and $f : X_{sober} \to Y_{sober}$ a mapping. Then the following are equivalent:*

a. f is sober continuous

b. For every sober closed set B of Y_{sober}, the set $f^{-1}(B)$ is sober closed in X_{sober}

Proof. $(a) \Leftrightarrow (b)$ is obvious as $f^{-1}(B^c) = (f^{-1}(B))^c$ in X_{sober}.

Proposition 4.6. *Let $f : X_{sober} \to Y_{sober}$ be a mapping. Then the following are true:*

(a) For every sober closed set A of X_{sober} , $f(sober - cl(A)) \subset sober - cl(f(A))$

(b) For every sober open set A of X_{sober} , $sober - int(f(A)) \subset f(sober - int(A))$

Proof.

(a) Let A be a sober closed set of X_{sober} , then $f(sober - cl(A)) = f(A) \subseteq sober - cl(f(A))$.

(b) is obvious by taking complement in (a).

Proposition 4.7. *Let $f : X_{sober} \to Y_{sober}$ be a sober continuous mapping. Then for each $x \in X$ and each sober open set V containing $f(x)$, there is a sober open set U containing x such that $f(U) \subset V$.*
Proof. Let V be a sober open set in Y_{sober} such that $f(x) \in V$. Then since f is sober continuous, $f^{-1}(V)$ is a sober open set in X_{sober} and $x \in f^{-1}(V)$. Let

$U = f^{-1}(V)$, then $f(U) = f(f^{-1}(V)) \subseteq V$.

Definition 4.8. *Let X_{sober} be a sober topological space. A sober separation of X_{sober} is a pair A and B of disjoint sober open sets of X_{sober} such that $X_{sober} = A \cup B$. The space X_{sober} is said to be sober connected if there exist no sober separation of X_{sober}.*

It is not necessary to mention non-emptiness in the above Definition 4.8, as a sober open set is always non-empty.

Example 4.9. Let (X, τ_{sober}) be a sober topological space where $X = \{a, b, c\}$ and $\tau_{sober} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$.
Here $B_o(X_{sober}) = \{\{a\}, \{b\}, \{a, b\}\}$. Then X_{sober} is sober connected as there exist no disjoint sober open sets A and B such that $X_{sober} = A \cup B$.

Theorem 4.10. *A space X_{sober} is sober connected if and only if there exists no proper subset of X_{sober} which is both sober open and sober closed in X_{sober}.*
Proof. Assume X_{sober} is sober connected. Suppose A is a subset of X_{sober} such that A is both sober open and sober closed. Then $C = A$ and $D = X - A$ are disjoint sober open sets and $X_{sober} = C \cup D$. This implies that X_{sober} is not sober connected. A contradiction to our hypothesis arises.

Conversely assume that X_{sober} is not sober connected. Then there exist two disjoint sober open sets C and D such that $X_{sober} = C \cup D$ where C is both sober open and sober closed as $C = X - D$.

Theorem 4.11. *The sober continuous image of a sober connected space is sober connected.*
Proof. Let $f : X_{sober} \to Y_{sober}$ be a sober continuous map and let X_{sober} be sober connected. Then it is to be proved that $f(X_{sober})$ is sober connected. Suppose $f(X_{sober})$ is not sober connected, then there exist two disjoint sober open sets A and B such that $f(X_{sober}) = A \cup B$. As f is sober continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are sober open in X_{sober} and $X_{sober} = f^{-1}(A) \cup f^{-1}(B)$ where both $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint. A contradiction arises to our assumption. Hence $f(X_{sober})$ is sober connected.

Lemma 4.12. *If two sets A and B form a separation of a sober topological space X_{sober} and if Y is a sober connected subspace of X_{sober}, then either $Y \subset A$ or $Y \subset B$.*
Proof. Let A and B form a separation of X_{sober}, then $X_{sober} = A \cup B$ where A and B are non-empty disjoint open sets in X_{sober} and let Y be a sober connected subspace of X_{sober}. Suppose $y_i \in A$ and $y_j \in B$ where $y_i, y_j \in Y$ $(i \neq j \& i, j \in I)$, then $Y = (Y \cap A) \cup (Y \cap B)$ where $Y \cap A$ and $Y \cap B$ are disjoint non-empty open

sets in Y. Since Y is a sober topological space, they are sober open sets in Y hence form a sober separation of Y. A contradiction to our hypothesis arises. Therefore Y must lie either in A or in B.

Proposition 4.13. *Every sober connected space is connected but not conversely in general.*

Proof. Let X_{sober} be sober connected. Suppose it is not connected, then there exist two non-empty disjoint open sets A and B such that $X_{sober} = A \cup B$. As every proper open set is sober open in X_{sober}, A and B are sober open in X_{sober}. This implies X_{sober} is not sober connected. A contradiction arises. Hence X_{sober} is connected.

Example 4.14. Let $X = \{a, b, c, d\}$ and $\tau_{sober} = \{\phi, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, X\}$. Then (X, τ) is connected but it is not sober connected as $B_o(X_{sober}) = \{\{a\}, \{c\}, \{a, b\}, \{b, c\}, \{c, d\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}\}$ and $X_{sober} = \{a, b\} \cup \{c, d\}$ which are disjoint sober open sets.

Theorem 4.15. *Let A be a sober connected subspace of X_{sober}. If $A \subset B \subset sober - cl(A)$, then B is connected.*

Proof. Let A be sober connected and let $A \subset B \subset sober - cl(A)$. By Proposition 4.13, A is connected and since $sober - cl(A) \subseteq cl(A)$, B is obviously connected [6].

Theorem 4.16. *Let A be a sober connected subspace of X_{sober}. If $A \subset B \subset cl(A)$, then B is connected.*

Proof. Let A be sober connected and let $A \subset B \subset cl(A)$. Suppose B is not connected, then there exist two non-empty disjoint open sets C and D in B such that $B = C \cup D$. Then by Lemma 4.10, A must lie either in C or in D. Suppose $A \subset C$, then $cl(A) \subset cl(C)$ where $cl(C)$ and D are disjoint. Now as $B \subset cl(A) \subset cl(C)$, $B \cap D \subset cl(C) \cap D = \phi$. That is $B \cap D = \phi$ and this contradicts the fact that D is a non-empty subset of B. Hence B is connected.

Remark 4.17. *A sober topological space X_{sober} with respect to sober open sets is not a Hausdorff ($= T_2$) space in general. But it may be a T_2 space when τ_{sober} contains some disjoint open sets together with at least two proper open sets which are not disjoint.*

Example 4.18. Let $X = \{a, b, c, d\}$ and $\tau_{sober} = \{\phi, \{b\}, \{c\}, \{d\}, \{a, b\}, \{b, c\}, \{c, d\}, \{b, d\}, \{a, b, d\}, \{a, b, c\}, \{b, c, d\}, X\}$. Here $B_o(X_{sober}) = \{\{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{c, d\}, \{a, b, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$.
Then (X, τ_{sober}) is a T_2 space with respect to sober open sets.

Proposition 4.19. *The disjoint sober closed sets A and B are sober separated in X_{sober}.*

Proof. Let A and B be disjoint sets. Then $A \cap B = \phi$. Since A and B are sober closed sets, $sober - cl(A) = A$ and $sober - cl(B) = B$. Hence $sober - cl(A) \cap B = A \cap B = \phi$ and $A \cap sober - cl(B) = A \cap B = \phi$.

Proposition 4.20. *The disjoint sober open sets A and B whose sober - closure exist are sober separated in X_{sober}.*

Proof. Let A and B be disjoint sober open sets whose sober - closure exist. Then $A \cap B = \phi$. This implies that $A \subset X - B$ and $sober - cl(A) \subset sober - cl(X - B) = X - B$, as $X - B$ is sober closed. We have $sober - cl(A) \subset X - B$ and hence $sober - cl(A) \cap B = \phi$. Similarly $A \cap sober - cl(B) = \phi$.

Definition 4.21. *Suppose that one-point sets are closed in X_{sober}. Then X_{sober} is said to be sober regular if for each pair consisting of a point x and a sober closed set B disjoint from x, there exist disjoint sober open sets containing x and B respectively.*

Proposition 4.22. *If a sober topological space X_{sober} is sober regular, then given a point $x \in X$ and a sober open set U of x, there is a sober open set V of x such that $sober - int(V) \subset U$.*

Proof. Assume that X_{sober} is sober regular and $x \in X$. Given that U is a sober open set containing x. Let $B = X - U$, then B is sober closed which is disjoint from x. By hypothesis, there exist disjoint sober open sets V and W such that $x \in V$ and $B \subset W$. We have $V \cap B = \phi$, since if $y \in V \cap B$, then $y \in V$ and $y \in B \subset W$ where V and W are disjoint, which is a contradiction. Hence $V \cap (X - U) = sober - int(V) \cap (X - U) = \phi$ and therefore $sober - int(V) \subset U$.

Theorem 4.23. *A sober topological space X_{sober} is sober regular, then for a given point $x \in X$ and a sober open set N of x, there is a sober open set M of x whose sober - closure exists such that $sober - cl(M) \subset N$.*

Proof. Let $x \in X$ and N, a sober open set such that $x \in N$. Then $X - N$ is sober closed and $x \notin X - N$. By hypothesis, there exist two disjoint sober open sets say L and M such that $X - N \subset L$ and $x \in M$. We have $L \cap M = \phi$. This implies that $M \subset X - L$ and $sober - cl(M) \subset sober - cl(X - L) = (X - L)$, since $X - L$ is sober closed. Also $X - N \subset L \Rightarrow X - L \subset N$. Therefore $sober - cl(M) \subset X - L \subset N$. That is $sober - cl(M) \subset N$.

Definition 4.24. *Suppose that one-point sets are closed in X_{sober}. Then X_{sober} is said to be sober normal if for each pair A, B of disjoint sober closed sets of X_{sober}, there exist disjoint sober open sets containing A and B, respectively.*

Proposition 4.25. *If a sober topological space X_{sober} is sober normal, then given a sober closed set A and a sober open set $U \supset A$, there is a sober open set $V \supset A$*

such that $sober - int(V) \subset U$.

Proof. Assume that X_{sober} is sober normal and given that U is a sober open set containing A where A is sober closed in X_{sober}. Then $B = X - A$ is sober closed and it is not containing A. By hypothesis, there exist two disjoint sober open sets say V and W such that $A \subset V$ and $B \subset W$. We have $V \cap B = \phi$, since if $y \in V \cap B$, then $y \in V$ and $y \in B \subset W$ where V and W are disjoint, which is a contradiction. Hence $V \cap (X - U) = sober - int(V) \cap (X - U) = \phi$ and therefore $sober - int(V) \subset U$.

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