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ON FERMATEAN FUZZY α-SEPARATION AXIOMS

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Abstract: The basis for investigating topological spaces is established by the compactness, connectedness, and separation axioms, all have an impact on notions in analysis, geometry, and other fields. They are crucial to both theoretical and computational mathematics since they support the formation of an extensive understanding of the structure and behavior of functions. Compactness, connectedness, and separation axioms in Fermatean fuzzy topology improve the conceptual structure and allow for greater uncertain modeling. They are significant in areas like artificial intelligence, decision-making, and systems modeling since they offer fundamental concepts to analyze fuzzy structures. In this study we explore FF α-separation axioms, FF α-connectedness, and FF α-compactness in Fermatean fuzzy topological spaces.

Keywords and Phrases: Fermatean fuzzy set, Fermatean fuzzy topology, Fermatean fuzzy α-set, Fermatean fuzzy α- separation axioms, Fermatean fuzzy α-connectedness, and Fermatean fuzzy α-compactness.

2020 Mathematics Subject Classification: 03E72, 03E75, 54A05, 54A40, 54D65.

1. Introduction

In the field of set theory, Zadeh L. A. [23] introduced fuzzy set(FS) in 1965 that constitutes an important breakthrough. These fuzzy sets can resolve imperfection and ambiguity that classical binary sets find challenging to tackle. Each element in a fuzzy set has a membership function, varying from 0 to 1, that specifies proportion of a belongingness. This method greatly replicates the real-life circumstances where the boundaries are often unclear and ambiguous.

In many real-world scenarios, simply determining whether an element belongs to a set or not is insufficient. Intuitionistic fuzzy set(IFS) introduced by Atanassov K. T. [4] extends the concept of fuzzy set by incorporating a degree of uncertainty alongside the degrees of association value(AV) and non-association value(NAV) whose value vary from 0 to 1 with the sum of AV and NAV lies between 0 and 1. While fuzzy sets have transformed how we approach uncertainty and imprecision, intuitionistic fuzzy sets take this a step further by acknowledging the complexity of human reasoning and the subtleties of real-world scenarios.

Yager R. R. [22] coined Pythagorean fuzzy set(PFS) that can deal uncertainty even more better than IFS by taking the values for AV and NAV between 0 and 1 individually with sum of squares of them lie between 0 and 1. To extend the range for AV and NAV, Senapati T. [17] introduced Fermatean fuzzy set(FFS) in which sum of cubes of AV and NAV lies between 0 and 1. Farid H. M. A. et al. [9] applied FF CODAS approach for selection of sustainable supplier. Revathy A. [16] used FFS in MCDM.

Chang C. L. [5] and Coker D. [7] studied the topological structures of FS, Olgun M. [14] investigated Pythagorean fuzzy topological spaces(PFTS), Ibrahim H. Z. [12] introduced Fermatean fuzzy topological space(FTS), and Turkarslan, E. [21] established q-Rung orthopair fuzzy topological spaces.

Lowen R. [13] and Coker D. [8] studied the compactness of FS in FTSs and IFTS. Ajmal N [2], Chaudhuri, A. K. [6], Turanli N. [20], Ozcag S. [15], Haydar, A. [11] investigated connectedness in FTS, IFTS, special IFTS, and PFTS. Singal M. K. [18] studied fuzzy α- sets, Ajay D. [1] investigated Pythagorean fuzzy α-continuity, and Revathy A. [3] investigated the generalisations of FF sets. Ghanim M. H. [10], and Srivastava R [19] studied seperation axioms in FTS and IFTS.

In this study we introduce and investigate FF α-separation axioms, FF α-connectedness, and FF α-compactness.

The contribution of this article is as follows

- Preliminaries are given in Section 2.
- In Section 3, FF α-separation axioms are discussed.

- In Section 4 FF α-compactness is defined and explored .
- Section 5 deals with FF α-connectedness.
- Section 6 ends up with conclusion.

The acronyms used in the current research are listed below.

Acronyms	**Expansion**
AV	Association value
NAV	Non association value
FS	Fuzzy set
FTS	fuzzy topological space
IFS	Intutionistic fuzzy Set
IFTS	intutionistic fuzzy topological space
PFS	Pythagorean fuzzy set
PFTS	Pythagorean fuzzy topological space
FFS	Fermatean fuzzy set
FF	Fermatean fuzzy
FFTS	Fermatean fuzzy topological space
FFSB	Fermatean fuzzy sub base
$FFOS$	Fermatean fuzzy open set
$FF\alpha OS$	Fermatean fuzzy α-open set
FFP	Fermatean fuzzy point
$FF\alpha C$	Fermatean fuzzy α continuous function
$FF\alpha I$	Fermatean fuzzy α irresolute function
$e-C$	e-continuous
$e^{'}-C$	$e^{'}$-continuous

2. Preliminaries

This section conveys some of the essential concepts utilised in this study.

Definition 2.1. [17] *A set $\mathcal{S} = \{\langle a, \mu_S(a), \nu_S(a)\rangle : a \in A\}$ in the universe A is called FFs if $0 \leq (\mu_S(a))^3 + (\nu_S(a))^3 \leq 1$ where $\mu_S(a) : A \to [0,1]$, $\nu_S(a) : A \to [0,1]$ and $\pi = \sqrt[3]{1-(\mu_S(a))^3-(\nu_S(a))^3}$ are degree of AV, NAV and indeterminacy of a in S and its complement is $\mathcal{S}^c = \{\langle a, \nu_S(a), \mu_S(a)\rangle : a \in A\}$.*

Definition 2.1. [12] *A FF topological space (FFTS) ia s pair $(\mathcal{A}, \tau_{\mathcal{A}})$ if*

1. $1_{\mathcal{A}} \in \tau_{\mathcal{A}}$
2. $0_{\mathcal{A}} \in \tau_{\mathcal{A}}$
3. *arbitrary union of the elements of any sub-collection of $\tau_{\mathcal{A}}$ is in $\tau_{\mathcal{A}}$.*
4. *finite intersections of elements of of $\tau_{\mathcal{A}}$ is in $\tau_{\mathcal{A}}$.*

The members of $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{A}}^c$ are called FF open set($FFOS$) and FF closed set($FFCS$). FF interior of $\mathcal{A}$ (int $\mathcal{A}$) is the union of all $FFOS$s contained in $\mathcal{A}$ and FF closure of $\mathcal{A}$(cl $\mathcal{A}$) is the intersection of all $FFCS$s containing $\mathcal{A}$.

Definition 2.3. [3] *A FFS $\mathcal{S} = \{\langle a, \mu_S(a), \nu_S(a)\rangle : a \in A\}$ of a FFTS $(\mathcal{A}, \tau_{\mathcal{A}})$ is*

1. *a Fermatean fuzzy semi open set($FFSOS$) if $\mathcal{S} \subseteq cl\,(int\,(\mathcal{S}))$.*
2. *a Fermatean fuzzy pre open set($FFPOS$) if $\mathcal{S} \subseteq int\,(cl\,(\mathcal{S}))$.*
3. *a Fermatean fuzzy α-open set($FF\alpha OS$) if $\mathcal{S} \subseteq int\,(cl\,(int\,(\mathcal{S})))$.*

Their complements are Fermatean fuzzy semi closed set($FFSCS$), Fermatean fuzzy pre closed set($FFPCS$) and Fermatean fuzzy α-closed set($FF\alpha CS$) respectively. The FF α-closure of $\mathcal{S}$($cl_\alpha(\mathcal{S})$) is the intersection of all FF α-closed super sets of $\mathcal{S}$ and the FF α-interior of $\mathcal{S}$($int_\alpha(\mathcal{S})$) is the union of all FF α-open subsets of $\mathcal{S}$.

Definition 2.4. *A Fermatean fuzzy point (FFP) denoted by $P_{(\lambda,\,\mu)}$ is a FFS defined by*

$$P_{(\lambda,\mu)} = \begin{cases} (\lambda, \mu) & \text{if } x = p \\ (0, 1) & \text{otherwise.} \end{cases} \tag{2.1}$$

A FFP $P_{(\lambda,\,\mu)} \in \mathcal{F}$ if $\lambda \leq \alpha_{\mathcal{F}}(x)$ and $\mu \geq \beta_{\mathcal{F}}(x)$.

Proposition 2.1. *A FFS in A is the union of all FFP belonging to A.*

Definition 2.5. *Let (A, τ_A) be a FFTS and $B \subseteq A$. Then (B, τ_B) is called a FF subspace of (A, τ_A).*

Definition 2.6. *Let (A, τ_A) and (B, τ_B) be FFTSs. A function $f : (A, \tau_A) \to (B, \tau_B)$ is a Fermatean fuzzy α-continuous function($FF\alpha C$) if the inverse image of each $FFOS$ in (B, τ_B) is a $FF\alpha OS$ in (A, τ_A).*

3. Fermatean Fuzzy α-Separation Axioms

Definition 3.1. *A FFTS (A, τ_A) is a $FF\alpha T_0$ if for every pair of FFPs $x = a_{1(\lambda_1,\,\mu_1)}$ and $y = a_{2(\lambda_2,\,\mu_2)}$ with different supports there exists a $FF\alpha OS$, U such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.*

Theorem 3.1. *A FFTS (A, τ_A) is a $FF\alpha T_0$ if and only if any two FFPs with different support have disjoint $FF\alpha$-closure.*

Proof. Let (A, τ_A) be a $FF\alpha T_0$ and $x = a_{1(\lambda_1,\,\mu_1)}$ and $y = a_{2(\lambda_2,\,\mu_2)}$ be two FFPs with supports a_1 and a_2 respectively with $a_1 \neq a_2$. Since (A, τ_A) is a $FF\alpha T_0$ there exists a $FF\alpha OS$, U such that either $x \in U$, $y \notin U$ or $y \in U$, $x \notin U$. If $x \in U$ and $y \notin U$ then $y \notin cl_\alpha(y)$ also $cl_\alpha(y) \nsubseteq U$. Since $x \notin U^c$, $x \notin (cl_\alpha(y))^c$. But

$x \in cl_{\alpha}(x)$. Therefore $cl_{\alpha}(x) \neq cl_{\alpha}(y)$.
Conversely, let x and y be two FFPs with different supports a_1 and a_2 respectively. Let x_1 and y_1 be FFPs such that $x_1(a_1) = y_1(a_2) = 1$. By assumption $Cl_{\alpha}(x_1) \neq Cl_{\alpha}(y_1)$. But $x \leq x_1$ implies $x^c \geq x_1^c \geq (Cl_{\alpha}(x_1))^c$. Thus $(Cl_{\alpha}(x_1))^c$ is a $FF\alpha OS$ such that $y \nsubseteq Cl_{\alpha}(x_1)$, $x \subseteq Cl_{\alpha}(x_1)$. Hence (A, τ_A) is a $FF\alpha T_0$.

Theorem 3.2. *Let f be a FF injective $FF\alpha I$, $f : (A, \tau_A) \to (B, \tau_B)$. If (B, τ_B) is a $FF\alpha T_0$ space then (A, τ_A) is a $FF\alpha T_0$.*
Proof. Let $x = a_{1(\lambda_1,\, \mu_1)}$ and $y = a_{2(\lambda_2,\, \mu_2)}$ be two FFPs with different supports a_1 and a_2 in (A, τ_A), then $f(x)$ and $f(y)$ are two FFPs with different supports in (B, τ_B). Since (B, τ_B) is a $FF\alpha T_0$ space then there exists a $FF\alpha OS$, U such that $f(x) \subseteq U, f(y) \nsubseteq U$ or $f(y) \subseteq U, f(x) \nsubseteq U$. If $f(x) \subseteq U, f(y) \nsubseteq U$ then $x \subseteq f^{-1}(U), y \nsubseteq f^{-1}(U)$ where $f^{-1}(U)$ is a $FF\alpha OS$ in (A, τ_A). Therefore (A, τ_A) is a $FF\alpha T_0$ space.

Theorem 3.3. *If $f : (A, \tau_A) \to (B, \tau_B)$ is a FF injective $FF\alpha C$ and (B, τ_B) is a $FF\alpha T_0$ space then (A, τ_A) is a $FF\alpha T_0$ space.*
Proof. Let $x = a_{1(\lambda_1,\, \mu_1)}$ and $y = a_{2(\lambda_2,\, \mu_2)}$ are two FFPs with different supports a_1 and a_2 in (A, τ_A), then $f(x)$ and $f(y)$ are two FFPs with different supports in (B, τ_B). Since (B, τ_B) is a $FF\alpha T_0$ space then there exists a $FFOS$, U such that $f(x) \subseteq U, f(y) \nsubseteq U$ or $f(y) \subseteq U, f(x) \nsubseteq U$. If $f(x) \subseteq U, f(y) \nsubseteq U$ then $x \subseteq f^{-1}(U), y \nsubseteq f^{-1}(U)$ where $f^{-1}(U)$ is a $FF\alpha OS$ in (A, τ_A). Therefore (A, τ_A) is a $FF\alpha T_0$ space.

Theorem 3.4. *If $f : (A, \tau_A) \to (B, \tau_B)$ is a FF injective $FF\alpha^* C$ and (B, τ_B) is a $FF\alpha T_0$ space then (A, τ_A) is a FFT_0 space.*
Proof. Let $x = a_{1(\lambda_1,\, \mu_1)}$ and $y = a_{2(\lambda_2,\, \mu_2)}$ are two FFPs with different supports a_1 and a_2 in (A, τ_A), then $f(x)$ and $f(y)$ are two FFPs with different supports in (B, τ_B). Since (B, τ_B) is a $FF\alpha T_0$ space then there exists a $FFOS$, U such that $f(x) \subseteq U, f(y) \nsubseteq U$ or $f(y) \subseteq U, f(x) \nsubseteq U$. If $f(y) \subseteq U, f(x) \nsubseteq U$ then $y \subseteq f^{-1}(U), x \nsubseteq f^{-1}(U)$ where $f^{-1}(U)$ is a $FFOS$ in (A, τ_A). Therefore (A, τ_A) is a FFT_0 space.

Definition 3.2. *A FFTS (A, τ_A) is $FF\alpha T_1$ if for every pair of $FFPs$ $x = a_{1(\lambda_1,\, \mu_1)}$ and $y = a_{2(\lambda_2,\, \mu_2)}$ with different supports there exists $FF\alpha OSs$, U and V such that $x \in U$ and $y \notin U$ or $y \in V$ and $x \notin V$.*

Theorem 3.5. *A $FFTS$ (A, τ_A) is a $FF\alpha T_1$ space if and only if every FFP is a $FF\alpha CS$.*
Proof. Let (A, τ_A) be a $FF\alpha T_1$ space and $x_0 = a_{0(\lambda_0,\, \mu_0)}$ be a FFP with support a_0. For any FFP $x = a_{(\lambda,\, \mu)}$ with support a in (A, τ_A) such that $a \neq a_0$, there

exists a $FF\alpha OS$s, U and V such that $x_0 \subseteq U, x \nsubseteq U$ and $x \subseteq V, x_0 \nsubseteq V$. Since $x_0 \nsubseteq V$, $x_0^c = \bigcup_{(x \subseteq x_0^c)} V$ and x_0^c is a $FF\alpha OS$. Therefore x_0 is a $FF\alpha CS$.
Conversely, let $x_1 = a_{1(\lambda_1,\, \mu_1)}$ and $x_2 = a_{2(\lambda_2,\, \mu_2)}$ be two FFPs with different supports a_1 and a_2 respectively, such that $y_1(a_1) = y_2(a_2) = 1$. The FFSs y_1^c and y_2^c are $FF\alpha OS$s and satisfy $x_1 \subseteq y_2^c$ and $x_2 \nsubseteq y_2^c$ and $x_2 \subseteq y_2^c$,$x_1 \nsubseteq y_1^c$. Hence (A, τ_A) is a $FF\alpha T_1$ space.

Remark 3.1. *Every $FF\alpha T_1$ space is $FF\alpha T_0$ space but the converse need not be true as seen from the following example.*

Example 3.1. Let $A = \{a_1, a_2\}$ and A_1, A_2 be FFSs on A defined as $A_1 = \{\langle A : (a_1, 0.1, 0.8), (a_2, 0.1, 0.8)\rangle\}$ and $A_2 = \{\langle A : (a_2, 0.8, 0.1), (a_2, 0.8, 0.1)\rangle\}$. Let $\tau_A = \{0_A, 1_A, A_1, A_2\}$. Then the space (A, τ_A) is a $FF\alpha T_0$ space but not $FF\alpha T_1$.

Theorem 3.6. *Let $f : (A, \tau_A) \to (B, \tau_B)$ be FF injective $FF\alpha I$. If (B, τ_B) is a $FF\alpha T_1$ space then (A, τ_A) is also a $FF\alpha T_1$ space.*
Proof. Let $x = a_{1(\lambda_1,\, \mu_1)}$ and $y = a_{2(\lambda_2,\, \mu_2)}$ be two FFPs with different supports a_1 and a_2 in (A, τ_A), then $f(x)$ and $f(y)$ are two FFPs with different supports in (B, τ_B). Since (B, τ_B) is a $FF\alpha T_1$ space then there exists a $FF\alpha OS$ U and V such that $f(x) \subseteq U, f(y) \nsubseteq U$ and $f(y) \subseteq V, f(x) \nsubseteq V$. If $f(x) \subseteq U, f(y) \nsubseteq U$ then $x \subseteq f^{-1}(U), y \nsubseteq f^{-1}(U)$ and $y \subseteq f^{-1}(V), x \nsubseteq f^{-1}(V)$ where $f^{-1}(U)$ and $f^{-1}(V)$ are $FF\alpha OS$ in (A, τ_A). Therefore (A, τ_A) is a $FF\alpha T_1$ space.

Theorem 3.7. *Let $f : (A, \tau_A) \to (B, \tau_B)$ be FF injective $FF\alpha C$. If (B, τ_B) is a $FF\alpha T_1$ space then (A, τ_A) is also a $FF\alpha T_1$ space.*
Proof. Proof is similar to the proof of theorem 3.7.

Theorem 3.8. *If $f : (A, \tau_A) \to (B, \tau_B)$ is a FF injective, $FF\alpha^*C$ and (B, τ_B) is a $FF\alpha T_1$ space then (A, τ_A) is a $FF\alpha T_1$.*
Proof. Proof is similar to the proof of theorem 3.7.

Definition 3.3. *A FFTS (A, τ_A) is a $FF\alpha$-Hausdorff(FFαT_2) if for every pair of FFPs, $x = a_{1(\lambda_1,\, \mu_1)}$, $y = a_{2(\lambda_2,\, \mu_2)}$ with different supports, there exists two $FF\alpha OS$, U and V such that $x \subseteq U$, $y \nsubseteq U$, $y \subseteq V$, $x \nsubseteq V$ and $U \nsubseteq V$.*

Example 3.2. Let $A = \{a_1, a_2\}$ and A_1 be FFS on A defined as $A_1 = \{\langle A : (a_1, 1, 0), (a_2, 0, 1)\rangle\}$. Let $\tau_A = \{0_A, 1_A, A_1\}$. Then the space (A, τ_A) is a $FF\alpha T_0$, $FF\alpha T_1$ and $FF\alpha T_2$.

Remark 3.2. *Every subspace of a $FF\alpha T_2$ is $FF\alpha T_2$ space.*

Theorem 3.9. *A $FFTS$ (A, τ_A) is a $FF\alpha T_2$ space if for every pair of FFPs $x = a_{1(\lambda_1,\, \mu_1)}$ and $y = a_{2(\lambda_2,\, \mu_2)}$ with different supports, there exists a $FF\alpha OS$, U such that $x \subseteq U \subseteq cl_\alpha(U) \subseteq y^c$.*

Proof. Let $x = a_{1(\lambda_1,\,\mu_1)}$ and $y = a_{2(\lambda_2,\,\mu_2)}$ are two FFPs with different supports. Since (A, τ_A) is a $FF\alpha T_2$ space then there exists $FF\alpha OS$s, U and V such that $x \subseteq U, y \nsubseteq U$ and $y \subseteq V, x \nsubseteq V$ with $U \nsubseteq V$. Then $x \subseteq U \subseteq cl_\alpha(U), y \nsubseteq cl_\alpha(U)$. Thus $int_\alpha(cl_\alpha(U)) \subseteq cl_\alpha(U)$. Let $X = int_\alpha(cl_\alpha(U))$ is a $FF\alpha OS$. Hence $x \subseteq X \subseteq cl_\alpha(X) \subseteq y^c$.

Theorem 3.10. *Let $f : (A, \tau_A) \to (B, \tau_B)$ be FF injective, $FF\alpha I$. If (B, τ_B) is a $FF\alpha T_2$ space then (A, τ_A) is also a $FF\alpha T_2$ space.*

Proof. Let $x = a_{1(\lambda_1,\,\mu_1)}$ and $y = a_{2(\lambda_2,\,\mu_2)}$ are two FFPs with different supports a_1 and a_2 in (A, τ_A), then $f(x)$ and $f(y)$ are two FFPs with different supports in (B, τ_B). Since (B, τ_B) is a $FF\alpha T_2$ space then there exists $FF\alpha OS$s, U and V such that $f(x) \subseteq U, f(y) \nsubseteq U,\ f(y) \subseteq V, f(x) \nsubseteq V$ and $U \nsubseteq V$. Then $x \subseteq f^{-1}(U), y \nsubseteq f^{-1}(U)$, $y \subseteq f^{-1}(V), x \nsubseteq f^{-1}(V)$ and $f^{-1}(U) \nsubseteq f^{-1}(V)$ where $f^{-1}(U)$ and $f^{-1}(V)$ are $FF\alpha OS$ in (A, τ_A). Therefore (A, τ_A) is a $FF\alpha T_2$ space.

Theorem 3.11. *Let $f : (A, \tau_A) \to (B, \tau_B)$ be FF injective $FF\alpha C$. If (B, τ_B) is a $FF\alpha T_2$ space then (A, τ_A) is also a $FF\alpha T_2$ space.*

Proof. Proof is similar to the proof of theorem 3.10.

Theorem 3.12. *If $f : (A, \tau_A) \to (B, \tau_B)$ is a FF injective, $FF\alpha^* C$ and (B, τ_B) is a $FF\alpha T_2$ space then (A, τ_A) is a $FF\alpha T_2$.*

Proof. Proof is similar to the proof of theorem 3.10.

4. Fermatean Fuzzy α-Compactness

In this section we introduce and study the concept of Fermatean Fuzzy α-compactness.

Definition 4.1. *In a FFTS (A, τ_A), a family C of FF subsets of A, is called a $FF\alpha$-covering of A if and only if C covers A and $C \subset FF\alpha(A)$, where $FF\alpha(A)$ is the collection of FF α-open sets of A.*

Definition 4.2. *A FFTS (A, τ_A) is said to be $FF\alpha$-compact if every $FF\alpha$-cover of A has a finite FF sub cover.*

Definition 4.3. *Let (A, τ_A) and (B, τ_B) be FFTSs and let τ_{A_e} be a FFT on A that has $FF\alpha(A)$ as a Fermatean fuzzy sub base(FFSB). A mapping $p : A \to B$ is said to be FF e-continuous($FFeC$) if $p : (A, \tau_{A_e}) \to (B, \tau_B)$ is FFC and p is said to be FF e'-continuous $(FFe'C)$ if $p : (A, \tau_{A_e}) \to (B, \tau_{B_e})$ is FFC.*

Theorem 4.1. *Let (A, τ_A) and (B, τ_B) be FFTSs and let τ_{A_e} be a FFT on A that has $FF\alpha(A)$ as a FFSB. If $p : (A, \tau_A) \to (B, \tau_B)$ is $FF\alpha C$, then p is $FFeC$.*

Proof. Let $U \in \tau_B$. If p is $FF\alpha C$ then $p^{-1}(U) \in FF\alpha(A)$ will imply $p^{-1}(U) \in \tau_{A_e}$. Hence p is $FFeC$.

Theorem 4.2. *Let (A, τ_A) and (B, τ_B) be FFTSs. Let τ_{A_e} and τ_{B_e} be respectively the FFTSs on A and B that have$FF\alpha(A)$ and $FF\alpha(B)$ as FFSBs. If $p : (A, \tau_A) \to (B, \tau_B)$ is $FF\alpha$-irresolute then p is $FFe^{'}C$.*

Proof. Let $U \in \tau_{B_e}$ and p be $FF\alpha$-irresolute. Then

$$U = \bigcup_n \left(\bigcap_{n=1}^{k} B_n\right) \textit{ where } B_n \in FF\alpha(B)$$

$$p^{-1}(U) = p^{-1}\left(\bigcup_n \left(\bigcap_{n=1}^{k} B_n\right)\right)$$

$$= \bigcup_n \left(\bigcap_{n=1}^{k} p^{-1}(B_n)\right)$$

since p is $FF\alpha$-irresolute, $p^{-1}(B_n) \in FF\alpha(A)$. Therefore $p^{-1}(U) \in \tau_{A_e}$. Hence p is $FFe^{'}C$.

Remark 4.1. *A FFTS A is $FF\alpha$-compact iff every family of $FF\alpha$-closed subsets of A with finite intersection property has non empty intersection.*

Theorem 4.3. *Let (A, τ_A) be FFTS and τ_{A_e} be a FFT on A that has $FF\alpha(A)$ as FFSB. Then (A, τ_A) is $FF\alpha$-compact iff (A, τ_{A_e}) is FF compact.*

Proof. Let (A, τ_{A_e}) be FF compact. Since $FF\alpha(A) \subset \tau_{A_e}$, (A, τ_A) is $FF\alpha$-compact.

Theorem 4.4. *Let (A, τ_A) be FFTS which is $FF\alpha$-compact. Then every FF τ_{A_e} closed set of A is $FF\alpha$-compact.*

Proof. Let U be FF τ_{A_e} closed set in A. Let $V_{K_j} : K_j \in J$ be a FF τ_{A_e} open cover of U. Since U^c is FF τ_{A_e} open, $V_{K_j} : K_j \in J \bigcup U^c$ is a FF τ_{A_e} open cover of A. Since A is τ_{A_e}-compact by here exists a finite FF subset $J_0 \subset J$ such that $A = \bigcup\{V_{K_j} : K_j \in J_0\} \cup U^c$ which implies $U \subset \bigcup\{V_{K_j} : K_j \in J_0\}$. Therefore U is $FF\alpha$-compact.

Theorem 4.5. *If a FFTS A is $FF\alpha$-compact then every family of $FF\tau_{A_e}$-closed subsets of A with finite intersection property has non empty intersection.*

Proof. Let A be $FF\alpha$-compact. Let $U = \{V_{K_j} : K_j \in J\}$ be family of FFτ_{A_e}-closed subsets of A with finite intersection property. If $\bigcap\{V_{K_j} : K_j \in J\} = \phi$ then $\{V_{K_j}^c : K_j \in J\}$ is a FF τ_{A_e} open cover A. So it must contain a FF finite subcover $\{V_{K_{j_m}}^c : K_{j_m} \in m = 1, 2, 3, ...n\}$ of A. This gives that $\{V_{K_{j_m}} : m = 1, 2, 3, ...n\} = \phi$, which is a contradiction to the assumption.

Theorem 4.6. *Let (A, τ_A) and (B, τ_B) be FFTSs and let $p : (A, \tau_A) \to (B, \tau_B)$ be $FFe^{'}C$. If a FF subset H of A is $FF\alpha$-compact relative to A then $f(H)$ is*

$FF\alpha$-compact relative to B.

Proof. Let $\{V_{K_j} : K_j \in J\}$ be FF cover of $f(H)$ by τ_{B_e} open FFSs in B. Then $\{p^{-1}(V_{K_j}) : K_j \in J\}$ be FF cover of H by τ_{A_e} open FFSs in A. H is $FF\alpha$-compact relative to A. Then H is FF τ_{A_e} compact. Thus there exists a finite FF subset $J_0 \subset J$ such that $H \subset \bigcup\{p^{-1}\{V_{K_j} : K_j \in J_0\}$ and so $p(H) \subset \{\{V_{K_j} : K_j \in J_0\}$. Therefore $f(H)$ is τ_{B_e}-compact relative to B and hence $f(H)$ is $FF\alpha$-compact relative to B.

Corollary 4.1. *If $p : (A, \tau_A) \to (B, \tau_B)$ is a $FFe^{'}C$ subjective function and A is $FF\alpha$-compact then B is $FF\alpha$-compact.*

Corollary 4.2. *If $p : (A, \tau_A) \to (B, \tau_B)$ is a $FF\alpha$-irresolute surjective function and A is $FF\alpha$-compact then B is $FF\alpha$-compact.*

Theorem 4.7. *Let G and H be FF subsets of a FFTS (A, τ_A) such that G is $FF\alpha$-compact relative to A and H is FF τ_{A_e} closed in A. Then $G \cap H$ is $FF\alpha$-compact relative to A.*

Proof. Let $\{V_{K_j} : K_j \in J\}$ be FF cover of $G \cap H$ by τ_{A_e} open FF subsets of A. Since H^c is a τ_{A_e} open FF subset, $\{V_{K_j} : K_j \in J\} \cup H^c$ is a FF cover of G. G is $FF\alpha$-compact and so τ_{A_e}-compact relative to A. Therefore there exists a finite FF subset $J_0 \subset J$ such that $G \subset \{V_{K_j} : K_j \in J\} \cup H^c$. Therefore $G \cap H \subset \bigcup\{V_{K_j} : K_j \in J_0\}$. Hence $G \cap H$ is FF τ_{A_e}-compact. Therefore $G \cap H$ is $FF\alpha$-compact relative to A.

5. Fermatean Fuzzy α-Connectedness

Definition 5.1. *Two non empty FF subsets M and N of a FFTS (A, τ_A) is $FF\alpha$-separated if M and N neither contain α-limit point of the other. i.e., $M \cap FFcl_\alpha(N) = FFcl_\alpha(M) \cap N = \phi$.*

Definition 5.2. *A FF subset S of a FFTS (A, τ_A) is said to be $FF\alpha$-connected space in A if S is not the union of two $FF\alpha$-separated sets in A.*

Definition 5.3. *Let (A, τ_A) be a FFTS. A $FF\alpha$-separation on A is a pair of non empty proper $FF\alpha OSs$, M and N such that $M \cap N = \phi$ and $M \cup N = A$. The FFTS (A, τ_A) is said to be $FF\alpha$-disconnected space if it has $FF\alpha$-separation.*

Theorem 5.1. *If a FFTS is $FF\alpha$-connected space between FFS M and N then $M \neq \phi \neq N$.*

Proof. If any FF subset $M = \phi$ then ϕ being $FF\alpha$-clopen over A, (A, τ_A) cannot be $FF\alpha$-connected space between M and N, which is a contradiction to the assumption. Hence the proof.

Example 5.1.

1. If (A, τ_A) is discrete FFTS on A then (A, τ_A) is not $FF\alpha$-connected space.
2. If (A, τ_A) is in discrete FFTS on A then (A, τ_A) is always $FF\alpha$-connected space.

Definition 5.4. *A FF subspace (B, τ_B) of FFTS (A, τ_A) is said to be $FF\alpha$-open(resp $FF\alpha$-closed) subspace if $M \in FF\alpha OS(A)$(resp $M \in FF\alpha CS(A)$) where M is $FF\alpha$-connected.*

Proposition 5.1. *Let (B, τ_B) be a FF semi connected subspace of FFTS (A, τ_A) such that $G \cap H \in FFSOS(A)$, G is FF subspace of B and $N \in FFSOS(A)$. If A has a FF semi separations M and N then either $G \subseteq M$ or $G \subseteq N$.*

Theorem 5.2. *If (A, τ_A) is $FF\alpha$-connected space and τ_B is FF coarser than τ_A then (A, τ_B) is also $FF\alpha$-connected space.*
Proof. Let M and N be a $FF\alpha$-separation on (A, τ_B). Then $M, N \in \tau_B$ and $\tau_B \in \tau_A$ imply $M, N \in \tau_A$ such that M, N is $FF\alpha$-separation on (A, τ_A) which is a contradiction with the $FF\alpha$-connectedness of (A, τ_A). Hence (A, τ_B) is $FF\alpha$-connected space.

Theorem 5.3. *A FF subspace (B, τ_B) of $FF\alpha$-disconnected space (A, τ_A) is $FF\alpha$-disconnected if $M \cap N \in FF\alpha OS(A)$ for every $N \in FF\alpha OS(A)$ and M is a FF subset of B.*
Proof. Let (B, τ_B) be a $FF\alpha$-connected space. Since (A, τ_A) is $FF\alpha$-disconnected there exist $FF\alpha$-separation M, N on (A, τ_A). By hypothesis, $M \cap H \in FF\alpha OS(A)$, $N \cap H \in FF\alpha OS(A)$ and $[M \cap H] \cup [M \cap H] = H$, which is a contradiction with the $FF\alpha$-connectedness of (B, τ_B). Therefore (B, τ_B) is $FF\alpha$-disconnected space.

Remark 5.1. *$FF\alpha$-disconnected property is not hereditary.*

Theorem 5.4. *Let (A, τ_A) and (B, τ_B) be FFTSs and $f : (A, \tau_A) \to (B, \tau_B)$ be a $FF\alpha$-irresolute surjective function. If (A, τ_A) is $FF\alpha$-connected space then (B, τ_B) is also $FF\alpha$-connected space.*
Proof. Let (B, τ_B) be a $FF\alpha$-disconnected space. Then there exists M, N of non-empty proper $FF\alpha$-open sets of B such that $M \cap N = \phi$ and $M \cup N = B$. f is $FF\alpha$-irresolute function, then $f^{-1}(M)$ and $f^{-1}(N)$ are pair of non null proper $FF\alpha O$ subsets of A such that $f^{-1}(M) \cap f^{-1}(N) = f^{-1}(M \cap N) = f^{-1}(\phi) = \phi$ and $f^{-1}(M) \cup f^{-1}(N) = f^{-1}(M \cup N) = f^{-1}(A) = A$ imply $f^{-1}(M)$ and $f^{-1}(N)$ forms a $FF\alpha$-separation of A, which is a contradiction with the $FF\alpha$-connectedness of (A, τ_A). Therefore (B, τ_B) is $FF\alpha$-connected space.

6. Conclusion and Future work: The adoption of the FF separation, FF connectedness, and FFcompactness axioms in Fermatean fuzzy topology has many advantages in various disciplines, enhancing the ability to develop, analyze, and handle complex problems that entail apprehension. Such concepts can be used to build trustworthy systems and structures which more precisely reflect the intricate nature of the real world. In the future work FF compactness and FF connectedness will be applied to assure the system efficiency and dependability in uncertain contexts by means of fuzzy controllers.

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