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ON HYPERBOLIC KENMOTSU MANIFOLDS WITH THE GENERALIZED SYMMETRIC METRIC CONNECTION

N. V. C. Shukla and Komal Anand

Department of Mathematics and Astronomy,
University of Lucknow, Lucknow, Uttar Pradesh, INDIA

E-mail : nvcshukla72@gmail.com, koomalanand210@gmail.com

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Abstract: In this paper, we define Hyperbolic Kenmotsu manifolds and the generalized symmetric metric connection on this manifold. Further we discuss curvature tensor and Ricci curvature tensor with respect to the generalized symmetric metric connection. We also study Ricci semi-symmetric 3-dim Hyperbolic Kenmotsu manifold with the generalized symmetric metric connection and Projectively flat manifold with respect to the generalized symmetric metric connection.

Keywords and Phrases: Hyperbolic Kenmotsu manifold, Generalized symmetric metric connection.

2020 Mathematics Subject Classification: 53C15, 53C20.

1. Introduction

A linear connection $\bar{\nabla}$ is said to be the generalized symmetric metric connection if its torsion tensor T is of the form

$$T(X,Y) = \alpha\{\eta(Y)X - \eta(X)Y\} + \beta\{\eta(Y)\phi X - \eta(X)\phi Y\}, \tag{1.1}$$

for any vector fields X, Y on a manifold, where α and β are smooth functions. ϕ is a tensor of type (1,1) and η is a 1-form associated with a non-vanishing smooth non-null unit vector field ξ. Moreover, the connection $\bar{\nabla}$ is said to be the generalized symmetric metric connection if there is a Riemannian metric g in M such that $\bar{\nabla}g = 0$, otherwise it is non metric.

In the equation (1.1), the generalized symmetric metric connection reduces to a semi-symmetric and quarter-symmetric respectively when (α,β)= (1,0) and (α,β) = (0,1).

Thus , it can be suggested that the generalized symmetric metric connection came in the idea by the generalized semi-symmetric and quarter-symmetric connection. This two connection play an important role for both the geometry study and applications to physics. H. A. Hayden [7] introduced a metric connection with non-zero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and non-metric connection have been studied by many authors (see [1, 3, 5]). The idea of quarter-symmetric linear connections in a differential manifold was introduced by S. Golab [6]. Sharfuddin and Hussian [11] defined a semi-symmetric metric connection in an almost contact manifold.

On the other hand, the notion of almost contact Hyperbolic (f, g, η, ξ)-structure was introduced by Upadhyay and Dube [12]. Further, it was studied by number of authors [2, 8]. A non-zero vector field $v \in T_p(M)$ is said to be timelike (respectively., null, space-like, and non-space-like) if it satisfies $g_p(v, v) < 0$ $(respectively., = 0, > 0$ and $\leq 0)$ [10], where $T_p(M)$ denotes the tangent space of M at point p. Let $\{e_1, e_2..., e_{2n}, e_{2n+1}= \xi\}$ be a local orthonormal basis of vector fields in a (2n+1)-dimensional semi-Riemannian manifold. Thus, the Ricci tensor S and the scalar curvature τ of a (2n+1)-dimensional almost Hyperbolic contact metric manifold M_{2n+1} endowed with the semi-Riemannian metric g are, respectively, defined as follows :

$$\begin{aligned} S(X, Y) &= \sum_{i=1}^{2n+1} \varepsilon_i g(R(e_i, X)Y, e_i) \\ &= \sum_{i=1}^{2n} g(R(e_i, X)Y, e_i) - g(R(\xi, X)Y, \xi), \end{aligned}$$

$$\tau = \sum_{i=1}^{2n+1} \varepsilon_i S(e_i, e_i) = \sum_{i=1}^{2n} \varepsilon_i S(e_i, e_i) - S(\xi, \xi) \tag{1.2}$$

for all vector fields X and Y, where $\varepsilon_i = g(e_i, e_i)$, ξ is the unit timelike vector field, that is $g(\xi, \xi)$ = -1 and R represents the curvature tensor of M_{2n+1} [10].
In 1972, Kenmotsu [9] studied a class of contact Riemannian manifolds satisfying some special conditions named as Kenmotsu manifold. Kenmotsu proved that a locally Kenmotsu manifold is a warped product $I \times_f M$ of an interval I and a Kaehler manifold M with wraping function$f(t) = se^t$; where s is a non-zero

constant. This is expressed by the condition

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{1.3}$$

for all vector fields X and Y.
In this paper, we studied the properties of 3-dimensional Hyperbolic Kenmotsu manifolds with the generalized symmetric metric connection.

Definition 1.1. *A hyperbolic Kenmotsu manifold M is said to be the generalized η–Einstein manifold* [13] *if its Ricci tensor S is of the form*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) + cg(\phi X, Y). \tag{1.4}$$

where a, b, c are scalar functions.

2. Preliminaries

Let M be a $(2n + 1)$-dimensional almost Hyperbolic contact manifold with a fundamental tensor field ϕ of type (1,1), ξ is a vector field, η is a 1-form. Then the structure (ϕ,η,ξ) satisfies,

$$\phi^2 = I + \eta \otimes \xi, \qquad \phi(\xi) = 0, \qquad \eta(\phi) = 0 \tag{2.1}$$

$$rank(\phi) = 2n$$

$$\eta(\xi) = -1 \tag{2.2}$$

where I is the identity endomorphism of the tangent bundle of M_{2n+1}. An almost Hyperbolic contact manifold M_{2n+1} is said to be an almost Hyperbolic contact metric manifold if the semi-Riemannian metric g of M_{2n+1} satisfies

$$g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

$$g(\phi X, Y) = -g(X, \phi Y) \tag{2.4}$$

$$\nabla_X \xi = -X - \eta(X)\xi \tag{2.5}$$

$$(\nabla_X \eta)(Y) = g(\phi X, \phi Y) = -g(X, Y) - \eta(X)\eta(Y) \tag{2.6}$$

$$g(X, \xi) = \eta(X) \tag{2.7}$$

$$\nabla_X fY = (Xf)Y + f\nabla_X Y \tag{2.8}$$

where $X, Y \in \chi(M)$ and ∇ is the Levi- Civita connection.
The structure (ϕ, ξ, η, g) on M_{2n+1} is called almost Hyperbolic contact metric structure. In a 3-dimensional Hyperbolic Kenmotsu manifold M, the following relations are hold

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.9}$$

$$R(X,\xi)\xi = -X - \eta(X)\xi \tag{2.10}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X \tag{2.11}$$

$$S(X,\xi) = 2\eta(X) \tag{2.12}$$

$$S(\xi,\xi) = -2 \tag{2.13}$$

$$Q\xi = 2\xi \tag{2.14}$$

$$g(\xi,\xi) = -1 \tag{2.15}$$

where R is the Riemannian curvature tensor and S is Ricci tensor defined by $S(X,Y) = g(QX,Y)$, where Q is Ricci operator.

3. Generalized Symmetric Metric Connection in a Hyperbolic Kenmotsu Manifold

Let ∇ be Levi-Civita connection and $\tilde{\nabla}$ be a linear connection in Hyperbolic Kenmotsu manifold M. The linear connection $\tilde{\nabla}$ satisfying

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X,Y) \tag{3.1}$$

for all vector fields X, $Y \in \mathfrak{X}(M)$, is known to be the generalized symmetric metric connection $\tilde{\nabla}$. Here H is $(1,2)$-type tensor such that

$$H(X,Y) = \frac{1}{2}\big[T(X,Y) + \tilde{T}(X,Y) + \tilde{T}(Y,X)\big] \tag{3.2}$$

where T is torsion tensor of $\tilde{\nabla}$ and

$$g(\tilde{T}(X,Y),W) = g(T(W,X),Y) \tag{3.3}$$

In view of (1.1), (3.3) and (3.2, we have

$$\tilde{T}(X,Y) = \alpha\big[\eta(X)Y - g(X,Y)\xi\big] + \beta\big[\eta(X)\phi Y - g(\phi X,Y)\xi\big] \tag{3.4}$$

and hence

$$H(X,Y) = \alpha\eta(Y)X - \beta\eta(X)\phi Y + \beta g(\phi X,Y)\xi \tag{3.5}$$

Thus we conclude

Corollary. *In a Hyperbolic Kenmotsu manifold, the generalized symmetric metric connection* $\tilde{\nabla}$ *of type* (α,β) *is given by*

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha\eta(Y)X + \beta\big[g(\phi X,Y)\xi - \eta(X)\phi Y\big] \tag{3.6}$$

which is obtained by using (3.1) and (3.5).
The generalized symmetric connection reduces to a semi-symmetric and quarter-symmetric respectively when (α,β)= (1,0) and (α,β) = (0,1).

4. Curvature Tensor on Hyperbolic Kenmotsu Manifold with the Generalized Symmetric Metric Connection

The curvature tensor $\tilde{R}$ of the generalized symmetric metric connection $\tilde{\nabla}$ in M is defined as

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z \tag{4.1}$$

By virtue of equations (2.1), (2.2), (2.5), (3.6) and (4.1), we obtain a relation between the curvature tensor $\tilde{R}$ of the generalized symmetric metric connection $\tilde{\nabla}$ and the curvature tensor R of Levi-Civita connection ∇ as

$$\begin{aligned}\tilde{R}(X,Y)Z =& R(X,Y)Z \\ &+ \alpha\big[g(Y,Z)X - g(X,Z)Y\big] + \alpha(1+\alpha)\big[\eta(Y)X - \eta(X)Y\big]\eta(Z) \\ &+ \beta(1+\alpha)\big[g(\phi Y,Z)X - g(\phi X,Z)Y\big] \\ &- \beta^2\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big]\xi \\ &+ 2\beta(1+\alpha)g(\phi X,Y)\eta(Z)\xi \\ &+ \beta\big[g(\phi Y,Z)\eta(X) - g(\phi X,Z)\eta(Y)\big]\xi,\end{aligned} \tag{4.2}$$

where $X, Y, Z \in \chi(M)$.
Taking inner product with W in (4.2), we get

$$\begin{aligned}\tilde{R}((X,Y)Z,W) =& R((X,Y)Z,W) \\ &+ \alpha\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] \\ &+ \alpha(1+\alpha)\big[\eta(Y)g(X,W) - \eta(X)g(Y,W)\big]\eta(Z) \\ &+ \beta(1+\alpha)\big[g(\phi Y,Z)g(X,W) - g(\phi X,Z)g(Y,W)\big] \\ &- \beta^2\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big]g(\xi,W) \\ &+ 2\beta(1+\alpha)\big[g(\phi X,Y)\eta(Z)\big]g(\xi,W) \\ &+ \beta\big[g(\phi Y,Z)\eta(X) - g(\phi X,Z)\eta(Y)\big]g(\xi,W).\end{aligned} \tag{4.3}$$

Taking inner product with ξ in (4.2), we have

$$g(\tilde{R}(X,Y)Z,\xi) = \eta(\tilde{R}(X,Y)Z) = \big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big].$$

Contracting (4.2) with respect to X, we have

$$\begin{aligned}\tilde{S}(Y,Z) =& S(Y,Z) + (2\alpha - \beta^2)g(Y,Z)\\ &+ (2\alpha(1+\alpha) + \beta^2)\eta(Y)\eta(Z)\\ &+ \beta(4(1+\alpha)+1)g(\phi Y,Z). \end{aligned} \tag{4.4}$$

Again using (4.2), we get

$$\begin{aligned}\tilde{R}(X,Y)\xi =& (1-\alpha^2)\big[\eta(Y)X - \eta(X)Y\big]\\ &- 2\beta(1+\alpha)g(\phi X,Y)\xi. \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}\tilde{R}(\xi,X)Y =& (1+\alpha+\beta^2)g(Y,Z)\xi + (\alpha^2-1)\eta(Z)Y\\ &+ (\alpha+\alpha^2+\beta^2)\eta(Y)\eta(Z)\xi\\ &+ \alpha\beta g(\phi Y,Z)\xi. \end{aligned} \tag{4.6}$$

Putting $Z = \xi$ in (4.4), we get

$$\tilde{S}(Y,\xi) = S(Y,\xi) - 2(\alpha^2+\beta^2)\eta(Y). \tag{4.7}$$

Using (2.12) in above equation

$$\tilde{S}(Y,\xi) = 2(1-\beta^2-\alpha^2)\eta(Y). \tag{4.8}$$

5. Ricci Semi-symmetric 3-dimensional Hyperbolic Kenmotsu Manifolds with the Generalized Symmetric Metric Connection

A Hyperbolic kenmotsu manifold with the generalized symmetric metric connection is called Ricci semi-symmetric if $\bar{R}.\bar{S} = 0$, Then

$$\bar{S}(\bar{R}(X,Y)Z,W) + \bar{S}(Z,\bar{R}(X,Y)W) = 0. \tag{5.1}$$

Using (4.3) and (4.4) in (5.1), we get

$$\begin{aligned}& S(R(X,Y)Z,W) + S(Z,R(X,Y)W)\\ &\quad + (2\alpha-\beta^2)[g(R(X,Y)Z,W) + g(Z,R(X,Y)W)]\\ &\quad + 2\alpha(1+\alpha)\beta^2[\eta(R(X,Y)Z)\eta(W) + \eta(Z)\eta(R(X,Y)W)]\\ &\quad + \beta((1+\alpha)4+1)[g(\phi R(X,Y)Z,W) + g(\phi Z,R(X,Y)W)]\\ &\quad + [\alpha g(Y,Z) + \alpha(1+\alpha)\eta(Y)\eta(Z) + \beta(1+\alpha)g(\phi Y,Z)]\bar{S}(X,W)\\ &\quad + [-\alpha g(X,Z) - \alpha(1+\alpha)\eta(X)\eta(Z) - \beta(1+\alpha)g(\phi X,Z)]\bar{S}(Y,W)\\ &\quad + \big[2\beta(1+\alpha)[g(\phi X,Y)\eta(Z)]\end{aligned}$$

$$
\begin{aligned}
&+ \beta[g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y)] \\
&- \beta^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\big]\bar{S}(\xi, W) \\
&+ [\alpha g(Y, W) + \alpha(1+\alpha)\eta(Y)\eta(W) + \beta(1+\alpha)g(\phi Y, W)]\bar{S}(Z, X) \\
&+ [-\alpha g(X, W) - \alpha(1+\alpha)\eta(X)\eta(W) - \beta(1+\alpha)g(\phi X, W)]\bar{S}(Z, Y) \\
&+ \big[2\beta(1+\alpha)[g(\phi X, Y)\eta(W)] \\
&+ \beta[g(\phi Y, W)\eta(Z) - g(\phi X, W)\eta(Y)] \\
&- \beta^2[g(Y, W)\eta(X) - g(X, W)\eta(Y)]\big]\bar{S}(Z, \xi) = 0.
\end{aligned} \tag{5.2}
$$

If a 3-dimensional Hyperbolic Kenmotsu manifold with the generalized symmetric metric connection becomes Ricci semi-symmetric, then

$$
S(R(X, Y)Z, W) + S(Z, R(X, Y)W) = 0. \tag{5.3}
$$

Using (5.3), (5.2) reduces to

$$
\begin{aligned}
&(2\alpha - \beta^2)[g(R(X, Y)Z, W) + g(Z, R(X, Y)W)] \\
&+ 2\alpha(1+\alpha)\beta^2[\eta(R(X, Y)Z)\eta(W) + \eta(Z)\eta(R(X, Y)W)] \\
&+ \beta((1+\alpha)4 + 1)[g(\phi R(X, Y)Z, W) + g(\phi Z, R(X, Y)W)] \\
&+ [\alpha g(Y, Z) + \alpha(1+\alpha)\eta(Y)\eta(Z) + \beta(1+\alpha)g(\phi Y, Z)]\bar{S}(X, W) \\
&+ [-\alpha g(X, Z) - \alpha(1+\alpha)\eta(X)\eta(Z) - \beta(1+\alpha)g(\phi X, Z)]\bar{S}(Y, W) \\
&+ \big[2\beta(1+\alpha)[g(\phi X, Y)\eta(Z)] \\
&+ \beta[g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y)] \\
&- \beta^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\big]\bar{S}(\xi, W) \\
&+ [\alpha g(Y, W) + \alpha(1+\alpha)\eta(Y)\eta(W) + \beta(1+\alpha)g(\phi Y, W)]\bar{S}(Z, X) \\
&+ [-\alpha g(X, W) - \alpha(1+\alpha)\eta(X)\eta(W) - \beta(1+\alpha)g(\phi X, W)]\bar{S}(Z, Y) \\
&+ \big[2\beta(1+\alpha)[g(\phi X, Y)\eta(W)] \\
&+ \beta[g(\phi Y, W)\eta(Z) - g(\phi X, W)\eta(Y)] \\
&- \beta^2[g(Y, W)\eta(X) - g(X, W)\eta(Y)]\big]\bar{S}(Z, \xi) = 0.
\end{aligned} \tag{5.4}
$$

Putting W=ξ in (5.4), we obtain

$$
\begin{aligned}
&[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)][2\beta^2(1-\beta^2) + \alpha(2 - 5\alpha\beta^2 - 4\beta^2)] \\
&+ \eta(X)g(\phi Y, Z)(\beta(4(1+\alpha) + 1)(1-\alpha^2) + 2\alpha\beta(1 - \alpha^2 - \beta^2)) \\
&+ \eta(Y)g(\phi X, Z)(2(1 - \alpha^2 - \beta^2) - \beta(4(1+\alpha) + 1)(1-\alpha^2) \\
&+ \alpha^2[\eta(X)\bar{S}(Z, Y) - \eta(Y)\bar{S}(Z, X)] \\
&- 4\beta(1+\alpha)(2 - 2\alpha^2 - 2\beta^2)\eta(Z)g(\phi X, Y) = 0.
\end{aligned} \tag{5.5}
$$

Again putting $X{=}\xi$ in (5.5), we get

$$\begin{aligned}&-(g(Y,Z)+\eta(Y)\eta(Z))[2\beta^2(1-\beta^2)+\alpha(2-5\alpha\beta^2-4\beta^2)]\\&-g(\phi Y,Z)[\beta(4(1+\alpha)+1)(1-\alpha^2)+2\alpha\beta(1-\alpha^2-\beta^2)]\\&+\alpha^2[-\bar{S}(Z,Y)-(2-2\alpha^2-2\beta^2)\eta(Y)\eta(Z)]=0.\end{aligned} \tag{5.6}$$

which gives

$$\begin{aligned}\bar{S}(Y,Z)=&\frac{1}{-\alpha^2}\Big\{[2\beta^2(1-\beta^2)+\alpha(2-5\alpha\beta^2-4\beta^2)]g(Y,Z)\\&+[2\beta^2(1-\beta^2)+2\alpha^2(1-\alpha^2)+2\alpha^2-7\alpha^2\beta^2-4\alpha\beta^2]\eta(Y)\eta(Z)\\&+[\beta(4(1+\alpha)+1)(1-\alpha^2)+2\alpha\beta(1-\alpha^2-\beta^2)]g(\phi Y,Z)\Big\}.\end{aligned} \tag{5.7}$$

Thus we have

Theorem 5.1. *A Ricci semi-symmetric 3-dimensional Hyperbolic Kenmotsu Manifold with the generalized symmetric metric connection is generalized η-Einstein manifold given as*

$$\bar{S}(Y,Z)=Ag(Y,Z)+B\eta(Y)\eta(Z)+Cg(\phi Y,Z).$$

Where
$A=\frac{1}{-\alpha^2}[2\beta^2(1-\beta^2)+\alpha(2-5\alpha\beta^2-4\beta^2)]$,
$B=\frac{1}{-\alpha^2}[2\beta^2(1-\beta^2)+2\alpha^2(1-\alpha^2)+2\alpha^2-7\alpha^2\beta^2-4\alpha\beta^2]$
and
$C=\frac{1}{-\alpha^2}[\beta(4(1+\alpha)+1)(1-\alpha^2)+2\alpha\beta(1-\alpha^2-\beta^2)]$.

6. Projectively flat 3-dimensional Hyperbolic Kenmotsu manifolds with the Generalized Symmetric metric connection

In this section, we study Projectively flat 3-dimensional Hyperbolic Kenmotsu manifolds with respect to the generalized symmetric metric connection. In a 3-dimensional Hyperbolic Kenmotsu manifold, the Projective curvature tensor with respect to the generalized symmetric metric connection is given by

$$\bar{P}(X,Y)Z=\bar{R}(X,Y)Z-\frac{1}{2}\{\bar{S}(Y,Z)X-\bar{S}(X,Z)Y\}. \tag{6.1}$$

If $P=0$, then the manifold M is called Projectively flat manifold with respect to the generalized symmetric metric connection .

Let M be a Projectively flat manifold with respect to the generalized symmetric metric connection. From (6.1), we have

$$\bar{R}(X,Y)Z = \frac{1}{2}\{\bar{S}(Y,Z)X - \bar{S}(X,Z)Y\}. \tag{6.2}$$

Using (4.3) and (4.4) in (6.2), we get

$$\begin{aligned} & R((X,Y)Z,W) \\ & + \alpha\big[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\big] \\ & + \alpha(1+\alpha)\big[\eta(Y)g(X,W) - \eta(X)g(Y,W)\big]\eta(Z) \\ & + \beta(1+\alpha)\big[g(\varphi Y,Z)g(X,W) - g(\varphi X,Z)g(Y,W)\big] \\ & - \beta^2\big[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\big]g(\xi,W) \\ & + 2\beta(1+\alpha)\big[g(\varphi X,Y)\eta(Z)\big]g(\xi,W) \\ & + \beta\big[g(\varphi Y,Z)\eta(X) - g(\varphi X,Z)\eta(Y)\big]g(\xi,W) \\ & = \frac{1}{2}\{[S(Y,Z)g(X,W) - S(X,Z)g(Y,W)] \\ & + (2\alpha - \beta^2)[(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + (2\alpha(1+\alpha) + \beta^2)[\eta(Y)\eta(Z)g(X,W) - g(\eta(X)\eta(Z)g(Y,W)] \\ & + (\beta(4(1+\alpha)+1)[g(\phi Y,Z)g(X,W) - g(\phi X,Z)g(Y,W)]\}. \end{aligned} \tag{6.3}$$

Now putting $W = \xi$ in (6.3), we have

$$\begin{aligned} & (1 + \frac{3}{2}\beta^2)[\eta(X)g(Y,Z) - \eta(Y)g(X,Z)] \\ & - (\frac{5}{2}\beta + \alpha\beta)[\eta(X)g(\phi Y,Z) - \eta(Y)g(\phi X,Z)] \\ & - 2\beta(1+\alpha)g(\phi X,Y)\eta(Z) \\ & = \frac{1}{2}[\eta(X)S(Y,Z) - \eta(Y)S(X,Z)]. \end{aligned} \tag{6.4}$$

Again putting $X = \xi$ in (6.4), we get

$$\begin{aligned} S(Y,Z) = & (2 + 3\beta^2)g(Y,Z) + 3\beta^2\eta(Y)\eta(Z) \\ & - (5\beta + \frac{1}{2}\alpha\beta)g(\phi Y,Z). \end{aligned} \tag{6.5}$$

Thus, we have

Theorem 6.1. *A Projectively flat 3-dimensional Hyperbolic Kenmotsu manifold with the generalized symmetric metric connection is the generalized η-Einstein manifold with respect to Levi-Civita connection is given by*

$$\begin{aligned} S(Y,Z) &= (2+3\beta^2)g(Y,Z)+3\beta^2\eta(Y)\eta(Z) \\ &\quad -(5\beta+\frac{1}{2}\alpha\beta)g(\phi Y,Z). \end{aligned}$$

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