*South East Asian J. of Mathematics and Mathematical Sciences*
*Vol. 20, No. 3 (2024), pp. 279-286*

DOI: 10.56827/SEAJMMS.2024.2003.21 **ISSN (Online): 2582-0850**

**ISSN (Print): 0972-7752**

# 2-WOVEN FRAMES IN 2-HILBERT SPACES

**Chetna Mehra, Narendra Biswas and Mahesh C. Joshi**

Department of Mathematics
Kumaun University,
Nainital - 263002, Uttarakhand, INDIA

E-mail : chetnamehra2992@gmail.com, narendrabiswas@yahoo.com,
mcjoshi69@gmail.com

(**Received:** *Dec. 02, 2023* **Accepted:** *Dec. 12, 2024* **Published:** *Dec. 30, 2024)*

**Abstract:** In this paper, we introduce 2-woven frames in 2- Hilbert spaces and explore some of their properties. Also, 2-woven frame operators are defined and some related results for these operators are established in 2-Hilbert spaces.

**Keywords and Phrases:** 2-woven frame, 2-inner product, 2-Hilbert space, 2-woven frame operators.

**2020 Mathematics Subject Classification:** 42C15, 46C50.

## 1. Introduction

Frames in Hilbert space were introduced by Duffin and Schaeffer in their work on nonharmonic Fourier series [6], reintroduced by Daubechies et al. [4] in 1986 and since then work on frames is going on. Frame theory has left a distinct mark in applied mathematics and engineering and some properties of frames have made them an important part of functional analysis [7, 9].

The concept of 2-inner product space was first introduced by Diminnie et al. [5] and frames in 2-Hilbert space were introduced by Arefijamaal and Sadeghi [1].

The concept of woven frames was introduced to deal with some problems in wireless sensor networks and signal processing by Bemrose et al. [2] in 2015 and frame related operators for woven frames were recently defined by A. Rahimi et al. [8].

Throughout the paper, $I$ denotes a finite or countably infinite index set, $\mathbb{N}$ set of natural numbers and for $k \in \mathbb{N}, [k] = \{1, ..., k\}$, $[k]^c = \mathbb{N} \setminus [k] = \{k+1, k+2, ...\}$.

## 2. Preliminaries

**Lemma 2.1.** [1] *Let* $(X, \langle \cdot, \cdot | \cdot \rangle)$ *be a 2-inner product space and* $x, z \in X$. *Then,*

$$\|x, z\| = sup\{|\langle x, y|z\rangle|; y \in X, \|y, z\| = 1\}.$$

**Definition 2.1.** [1] *Let* $(X\langle \cdot, \cdot | \cdot \rangle)$ *be a 2- Hilbert space and* $\eta \in X$. *A sequence* $\{x_i\}_{i\in I}$ *of elements of* $X$ *is called a 2-frame associated to* $\eta$ *if there exist constants* $0 < C \leq D < \infty$ *such that for all* $x \in X$

$$C\|x, \eta\|^2 \leq \sum_{i\in I} |\langle x, x_i|\eta\rangle|^2 \leq D\|x, \eta\|^2.$$

*The constants* $C$ *and* $D$ *are called the lower and upper 2-frame bounds. If* $C = D$, *then* $\{x_i\}_{i\in I}$ *is called a tight 2-frame.*

**Remark 1.** *A sequence* $\{x_i\}_{i\in I}$ *satisfying the upper 2-frame condition i.e.,* $\sum_{i\in I} |\langle x, x_i|\eta\rangle|^2 \leq D\|x, \eta\|^2$, *is called a 2-Bessel sequence with Bessel bound* $D$.

**Definition 2.2.** [2] *Let* $F = \{f_{ij}\}_{i\in I}$ *for* $j \in [k]$ *be a family of frames for the Hilbert space* $H$. *If there exists universal constants* $0 < C \leq D < \infty$ *such that for every partition* $\{\sigma_j\}_{j\in[k]}$ *of* $I$, *the family* $\{f_{ij}\}_{i\in\sigma_j, j\in[k]}$ *is a frame for* $H$ *with same bounds* $C$ *and* $D$, *then* $F$ *is called a woven frame. For every* $j \in [k]$, *the frames* $\{f_{ij}\}_{i\in\sigma_j, j\in[k]}$ *are called weaving frames.*

*If* $C = D$ *then* $F = \{f_{ij}\}_{i\in I}$ *for* $j \in [k]$ *is called a tight woven frame.*

**Remark 2.** *If for every partition* $\{\sigma_j\}_{j\in[k]}$, *the family* $\{f_{ij}\}_{i\in\sigma_j, j\in[k]}$ *is a Bessel sequence then the family* $F = \{f_{ij}\}_{i\in I}$ *for* $j \in [k]$ *is called a Bessel woven.*

**Proposition 2.1.** [3] *Let* $H$ *be a Hilbert space and* $T : H \to H$ *be a operator satisfying* $\|\mathcal{I} - T\| < 1$, *then* $T$ *is invertible opetator, where* $\mathcal{I}$ *is an identity operator.*

## 3. Woven frames in 2-Hilbert space

In this section, 2-woven frame is introduced and some results are proved for 2-weaving families of vectors.

**Definition 3.1.** *Let* $(X, \langle \cdot, \cdot | \cdot \rangle)$ *be a 2-Hilbert space and* $F = \{f_{ij}\}_{i\in I, j\in[k]}$ *be a family of 2-frames for* $X$ *associated to* $\eta$. *Then,* $F$ *is said to be a 2-woven frame if for universal constants* $C$ *and* $D$ *with* $0 < C \leq D < \infty$, *the family* $\{f_{ij}\}_{i\in\sigma_j, j\in[k]}$ *for every partition* $\{\sigma_j\}_{j\in[k]}$ *of* $I$ *is a 2-frame for* $X$.
*The constants* $C$ *and* $D$ *are called lower and upper frame bounds respectively.*
*Further, each family* $\{f_{ij}\}_{i\in\sigma_j, j\in[k]}$ *is called a 2-weaving frame.*

**Proposition 3.1.** *Let* $F = \{f_{ij}\}_{i\in I, j\in[k]}$ *be a woven frame for Hilbert space* $X$. *Then* $F$ *is also a 2-woven frame for 2-Hilbert space with the standard 2-inner product associated to* $\eta \in X$ $(\|\eta\| = 1)$.

**Proof.** Since $F = \{f_{ij}\}_{i\in I, j\in[k]}$ is a woven frame, therefore each family $\{f_{ij}\}_{i\in\sigma_j, j\in[k]}$ is a frame. From Proposition (2.3) of [1], every frame for a Hilbert space is a 2-frame for 2-Hilbert space. Therefore each family $\{f_{ij}\}_{i\in\sigma_j, j\in[k]}$ is a 2-frame for every partition $\{\sigma_j\}_{j\in[k]}$ of $I$. Hence $F$ is a 2-woven frame.

The following example shows that the converse of the above proposition is not true.

**Example 1.** Let $\Phi = \{e_1 + e_2, e_2\}$ and $\Psi = \{e_2, e_1 + e_2\}$ be 2-frames associated to $\eta = e_1$, where $e_1$ and $e_2$ are mutually orthogonal vectors with unit norm. Then $\Phi$ and $\Psi$ are 2-woven frame for 2-Hilbert space $\mathbb{R}^2$ with the standard 2-inner product. But $\Phi$ and $\Psi$ are not woven frame for $\mathbb{R}^2$. Because for index set $I = \{1, 2\}$, taking $\sigma = \{1\}$, $\sigma^c = \{2\}$ we get

$$\{\Phi_i\}_{i\in\sigma} \cup \{\Psi_i\}_{i\in\sigma^c} = \{e_1 + e_2, e_1 + e_2\}.$$

Which is not a frame for $\mathbb{R}^2$.

Let $X$ be a Hilbert space. For the subspace $Y_\eta$ generated by a fixed element $\eta \in X$, $Z_\eta$ denotes the orthogonal complement of $Y_\eta$.

**Theorem 3.1.** *Let $X$ be a Hilbert space and $(X, \langle\cdot,\cdot|\cdot\rangle)$ be a 2-Hilbert space. Then the family $F = \{f_{ij}\}_{i\in I, j\in[k]}$ is 2-woven associated to $\eta(\|\eta\| = 1)$ if and only if it is a woven frame for the Hilbert space $Z_\eta$ with same bounds.*

**Proof.** Let $F = \{f_{ij}\}_{i\in I, j\in[k]}$ be a 2-woven frame associated to $\eta$ with bounds $C$ and $D$, i.e. for every partition $\{\sigma_j\}_{j\in[k]}$ of $I$ and $x \in X$,

$$C\|x, \eta\|^2 \le \sum_{j=1}^{k} \sum_{i\in\sigma_j} |\langle x, f_{ij}|\eta\rangle|^2 \le D\|x, \eta\|^2.$$

By simple calculation, we get

$$C\|x - \langle x, \eta\rangle\eta\|^2 \le \sum_{j=1}^{k} \sum_{i\in\sigma_j} |\langle x - \langle x, \eta\rangle\eta, f_{ij}\rangle|^2 \le D\|x - \langle x, \eta\rangle\eta\|^2.$$

Since $x - \langle x, \eta\rangle\eta \in Z_\eta$. Therefore $F$ is a woven frame for $Z_\eta$.
Conversely. For any $x \in X$, $x - \langle x, \eta\rangle\eta \in Z_\eta$.
Since $F = \{f_{ij}\}_{i\in I, j\in[k]}$ be a woven frame with frame bounds $C$ and $D$ for $Z_\eta$, then

$$C\|x - \langle x, \eta\rangle\eta\|^2 \le \sum_{j=1}^{k} \sum_{i\in\sigma_j} |\,\langle x - \langle x, \eta\rangle\eta, f_{ij}\rangle|^2 \le D\|x - \langle x, \eta\rangle\eta\|^2$$

$$C\langle x, x|\eta\rangle \leq \sum_{j=1}^{k}\sum_{i\in\sigma_j} |\langle x, f_{ij}|\eta\rangle|^2 \leq D\langle x, x|\eta\rangle$$

$$C\|x, \eta\|^2 \leq \sum_{j=1}^{k}\sum_{i\in\sigma_j} |\langle x, f_{ij}|\eta\rangle|^2 \leq D\|x, \eta\|^2.$$

Hence $F = \{f_{ij}\}_{i\in I, j\in[k]}$ is a 2-woven frame for $X$ associated to $\eta$.

## 4. Operators for 2-woven frames

Let $l^2(I)$ with inner product $\langle\cdot,\cdot\rangle$/standard 2-inner product $\langle\cdot,\cdot|\cdot\rangle$ be a Hilbert/2-Hilbert space of square summable sequences.
Following [8], we construct 2-Hilbert space as:
For each family of subspaces $\{(l^2(I))_j\}_{j\in[k]}$ of $l^2(I)$, we have

$$(l^2(I))_j = \{\{a_{ij}\}_{i\in\sigma_j} \,|\, a_{ij} \in \mathbb{C},\ \sigma_j \subset I, \sum_{i\in\sigma_j} |a_{ij}|^2 < \infty\},\ \forall j \in [k].$$

Now, taking

$$(\sum_{j\in[k]} \oplus (l^2(I))_j)_{l_2} = \{\{a_{ij}\}_{i\in I, j\in[k]} \mid \{a_{ij}\}_{i\in\sigma_j} \in (l^2(I))_j\}, \forall j \in [k]$$

and inner product

$$\langle\{x_{ij}\}_{i\in I,j\in[k]}, \{y_{ij}\}_{i\in I,j\in[k]}|\{z_{ij}\}_{i\in I,j\in[k]}\rangle = \det\begin{vmatrix} \langle\{x_{ij}\}_{i\in I,j\in[k]}, \{y_{ij}\}_{i\in I,j\in[k]}\rangle & \langle\{x_{ij}\}_{i\in I,j\in[k]}, \{z_{ij}\}_{i\in I,j\in[k]}\rangle \\ \langle\{z_{ij}\}_{i\in I,j\in[k]}, \{y_{ij}\}_{i\in I,j\in[k]}\rangle & \langle\{z_{ij}\}_{i\in I,j\in[k]}, \{z_{ij}\}_{i\in I,j\in[k]}\rangle \end{vmatrix},$$

we can easily prove that $(\sum_{j\in[k]} \oplus (l^2(I))_j)_{l_2}$ with above inner product is a 2-Hilbert space.

Throughout rest of the paper, we shall denote a 2-Hilbert space by $X$ and 2-woven frame $\{f_{ij}\}_{i\in I,j\in[k]}$ by $F$.
Like 2-frames and its extensions, we can characterize a 2-woven frame in terms of its operators.

**Theorem 4.1.** *Let $X$ be a 2-Hilbert space and $F$ be a 2-woven frame associated to $\eta$ with frame bounds $C$ and $D$. Then for $F$ the 2-pre woven frame (synthesis) operator*

$$T_{F\eta} : (\sum_{j\in[k]} \oplus (l^2(I))_j)_{l_2} \to X, where$$

$$T_{F\eta}\{a_{ij}\}_{i\in I,j\in[k]} = \sum_{i\in I,j\in[k]} a_{ij}f_{ij},$$

*is well defined and bounded.*

**Proof.** Let $\{a_{ij}\}_{i\in I,j\in[k]} \in (\sum_{j\in[k]} \oplus(l^2(I))_j)_{l_2}$. By Lemma 2.1,

$$\begin{aligned}\|T_{F\eta}\{a_{ij}\}_{i\in I,j\in[k]},\eta\|^2 &= \|\sum_{i\in I,j\in[k]} a_{ij}f_{ij},\eta\|^2\\ &= sup\{|\langle\sum_{i\in I,j\in[k]} a_{ij}f_{ij}, y|\eta\rangle|^2,\, y\in X,\, \|y,\eta\|=1\}\\ &\leq \sum_{i\in I,j\in[k]} |a_{ij}|^2 sup\{\sum_{i\in I,j\in[k]} |\langle f_{ij}, y|\eta\rangle|^2,\, y\in X,\, \|y,\eta\|=1\}\\ &\leq D\sum_{i\in I,j\in[k]} |a_{ij}|^2.\end{aligned}$$

Therefore $\sum_{i\in I,j\in[k]} a_{ij}f_{ij} \in X$ is well defined and $\|T_{F\eta},\eta\| \leq \sqrt{D}$ i.e. $T_{F\eta}$ is bounded.

**Remark 3.** *The adjoint of $T_{F\eta}$ defined as*

$$T^*_{F\eta} : X \to (\sum_{j\in[k]} \oplus(l^2(I))_j)_{l_2},$$

$$T^*_{F\eta}x = \{\langle x, f_{ij}|\eta\rangle\}_{i\in I,j\in[k]},\ \ \forall x\in X$$

*is called analysis operator of $F$. Clearly, $T^*_{F\eta}$ is well defined and bounded operator.*

**Definition 4.1.** *Let $F$ be a 2-woven frame associated to $\eta$ for the 2-Hilbert space $X$. Then the operator $S_{F\eta} : X\to X$ is called 2-woven frame operator for $F$, defined as*

$$\begin{aligned}S_{F\eta}x &= T_{F\eta}T^*_{F\eta}x\\ &= T_{F\eta}\{\langle x, f_{ij}|\eta\rangle\}_{i\in I}\\ &= \sum_{i\in I,j\in[k]} \langle x, f_{ij}|\eta\rangle f_{ij}.\end{aligned}$$

*Clearly, $S_{F\eta} = T_{F\eta}T^*_{F\eta}$ is bounded. Moreover $\|S_{F\eta},\eta\| \leq D$.*
*It can be proved easily that,*

$$\langle S_{F\eta}x, x|\eta\rangle = \sum_{i\in I,j\in[k]} |\langle x, f_{ij}|\eta\rangle|^2,\ \ \forall x\in X.$$

**Theorem 4.2.** *The 2-woven frame operator $S_{F\eta} : X \to X$ is self adjoint, positive and invertible.*

**Proof.** Clearly $S_{F\eta} = T_{F\eta}T^*_{F\eta}$ is self adjoint.
$F$ is 2-woven frame, therefore

$$C\|x,\eta\|^2 \leq \langle S_{F\eta}x, x|\eta\rangle \leq D\|x,\eta\|^2, \ \ \forall x \in X$$

$$C\mathcal{I} \leq S_{F\eta} \leq D\mathcal{I}.$$

Hence $S_{F\eta}$ is a positive operator.

Furthermore, by Proposition 2.1,

$$\begin{aligned}\|\mathcal{I} - D^{-1}S_{F\eta}, \eta\| &= \underset{\|x,\eta\|=1}{sup} |\langle(\mathcal{I} - D^{-1}S_{F\eta})x, x|\eta\rangle| \\ &\leq \frac{D-C}{D} < 1.\end{aligned}$$

Hence $S_{F\eta}$ is invertible operator.

**Remark 4.** *Every $x \in X$ can be represented as*

$$\begin{aligned}x = S_{F\eta}S^{-1}_{F\eta}x &= \sum_{i\in I, j\in[k]} \langle S^{-1}_{F\eta}x, f_{ij}|\eta\rangle f_{ij} \\ &= \sum_{i\in I, j\in[k]} \langle x, f_{ij}|\eta\rangle S^{-1}_{F\eta}f_{ij}.\end{aligned}$$

**Theorem 4.3.** *Let $F = \{f_{ij}\}_{i\in I, j\in[k]}$ be a finite family of 2-Bessel sequences in $X$ associated to $\eta$, then the following conditions are equivalent:*

*(i) $F$ is 2-woven frame associated to $\eta$ with universal bounds $C$ and $D$.*

*(ii) For the operator $S_{F\eta}x = \sum_{i\in I, j\in[k]}\langle x, f_{ij}|\eta\rangle f_{ij}$, we have*

$$C\mathcal{I} \leq S_{F\eta} \leq D\mathcal{I},$$

*where $\mathcal{I}$ is the identity operator.*

**Proof.** $(i) \Rightarrow (ii)$
Since for every $x \in X$,

$$\langle S_{F\eta}x, x|\eta\rangle = \sum_{i\in I, j\in[k]} |\langle x, f_{ij}|\eta\rangle|^2.$$

So,

$$C\|x,\eta\|^2 \le \langle S_{F\eta}x, x|\eta\rangle \le D\|x,\eta\|^2, \ \ \forall x \in X$$

Hence,

$$C\mathcal{I} \le S_{F\eta} \le D\mathcal{I}.$$

$(ii) \Rightarrow (i)$

Let $T_{F\eta}$ be a 2-woven pre frame operator associated to $\eta$, then we have

$$\begin{aligned}\sum_{i\in I, j\in[k]} |\langle x, f_{ij}|\eta\rangle|^2 &= \langle S_{F\eta}x, x|\eta\rangle = \langle T^*_{F\eta}x, T^*_{F\eta}x|\eta\rangle \\ &= \|T^*_{F\eta}x, \eta\|^2 \\ &\le \|T^*_{F\eta}, \eta\|^2\|x,\eta\|^2 = \|S_{F\eta},\eta\|^2\|x,\eta\|^2 \\ &\le D\|x,\eta\|^2.\end{aligned}$$

Again, we have

$$\begin{aligned}\sum_{i\in I, j\in[k]} |\langle x, f_{ij}|\eta\rangle|^2 &= \|S^{\frac{1}{2}}_{F\eta}x, \eta\|^2 \\ &\ge C\|x,\eta\|^2, \ \ \forall x \in X.\end{aligned}$$

Hence, $F$ is a 2-woven frame.

**Theorem 4.4.** *Let $F$ be 2-woven frame for $X$ and $S_{F\eta}$ be 2-woven frame operator of $F$. If the positive square root of $S^{-1}_{F\eta}$ is denoted as $S^{-\frac{1}{2}}_{F\eta}$, then $\{S^{-\frac{1}{2}}_{F\eta} f_{ij}\}_{i\in I, j\in[k]}$ is a tight 2-woven frame.*

**Proof.** For $x \in X$,

$$S_{F\eta}x = \sum_{i\in I, j\in[k]} \langle x, f_{ij}|\eta\rangle f_{ij}.$$

Substituting $x = S^{-\frac{1}{2}}_{F\eta}x$, we get

$$S^{\frac{1}{2}}_{F\eta}x = \sum_{i\in I, j\in[k]} \langle S^{-\frac{1}{2}}_{F\eta}x, f_{ij}|\eta\rangle f_{ij}.$$

Hence,

$$\begin{aligned}x &= S^{-\frac{1}{2}}\Big(\sum_{i\in I, j\in[k]} \langle x, S^{-\frac{1}{2}}_{F\eta} f_{ij}|\eta\rangle f_{ij}\Big) \\ &= \sum_{i\in I, j\in[k]} \langle x, S^{-\frac{1}{2}}_{F\eta} f_{ij}|\eta\rangle S^{-\frac{1}{2}} f_{ij}.\end{aligned}$$

Now,

$$\begin{aligned}\|x,\eta\|^2 &= \langle x, x|\eta\rangle \\ &= \langle \sum_{i\in I, j\in[k]} \langle x, S_{F\eta}^{-\frac{1}{2}} f_{ij}|\eta\rangle S^{-\frac{1}{2}} f_{ij}, x|\eta\rangle \\ &= \sum_{i\in I, j\in[k]} \langle x, S_{F\eta}^{-\frac{1}{2}} f_{ij}|\eta\rangle \langle S_{F\eta}^{-\frac{1}{2}} f_{ij}, x, |\eta\rangle \\ &= \sum_{i\in I, j\in[k]} |\langle x, S_{F\eta}^{-\frac{1}{2}} f_{ij}|\eta\rangle|^2.\end{aligned}$$

So $\{S_{F\eta}^{-\frac{1}{2}} f_{ij}\}_{i\in I, j\in[k]}$ is tight 2-woven frame.

## References

[1] Arefijamaal A. A. and Sadeghi Ghadir, Frames in 2-inner Product Spaces, Iran. J. Math. Sci. Inform., Vol. 8, No. 2 (2013), 123-130.

[2] Bemrose T., Casazza P. G., Grochenig K., Lammers M. C., and Lynch R. G., Weaving Frames, Oper. Matrices, Vol. 10, No. 4 (2016), 1093-1116.

[3] Casazza P. G., The art of frame theory, Taiwan. J. Math., Vol. 4, No. 2 (2000), 129-201.

[4] Daubechies I., Grossmann A. and Meyer Y., Painless nonorthogonal expansions, J. Math. Phys., Vol. 27, No. 5 (1986), 1271-1283.

[5] Diminnie C., Gähler S. and White A., 2-Inner Product Spaces, Demonstr. Math., Vol. 6 (1973), 525-536.

[6] Duffin R. J. and Schaeffer A. C., A class of nonharmonic Fourier series, Trans. Am. Math. Soc., Vol. 72, No. 2 (1952), 341-366.

[7] Han D., Frame representations and Parseval duals with applications to Gabor frames, Trans. Am. Math. Soc., Vol. 360, No. 6 (2008), 3307-3326.

[8] Rahimi A., Samadzadeh Z. and Daraby B., Frames related operators for woven frames, Int. J. Wavel. Multiresolut. Inf. Process, Vol. 17, No. 3 (2019), 1950010.

[9] Sun Q. and Tang W., Nonlinear frames and sparse reconstructions in Banach spaces, J. Fourier Anal. Appl., Vol. 23 (2017), 1118-1152.