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A NOTE ON IDENTITIES INVOLVING RIEMANN ZETA FUNCTIONS AND BELL POLYNOMIALS

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Abstract: Our basic aim is to provide some identities and a recurrence relation involving Multiple Zeta Riemann functions (MZVs). We do this by the use of complete and partial Bell polynomials.

Keywords and Phrases: Riemann Zeta Function, Multiple Riemann Zeta Function, Integer Compositions, Bell polynomials.

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1. Introduction and Background

In this paper we concern in writing sums of Multiple Zeta Riemann functions (MZVs) as a rational linear combination of products of single Riemann zeta functions. The results found here presents certain similarity to the recent results of Kim et al in [9-12].

We recall that the multiple zeta function is given by

$$\zeta(s_1, s_2, \ldots, s_k) = \sum_{n_1>n_2>\ldots>n_k\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}.$$

The k-tuple having all entries equal to s is denoted by $\{s\}^k$. Thus,

$$\zeta(\{s\}^k) = \sum_{n_1>n_2>\ldots>n_k\geq 1} \frac{1}{(n_1 n_2 \cdots n_k)^s}.$$

For convenience we let $\zeta(\{n\}^0) = 1$. A multi-index $s = (s_1, s_2, \ldots, s_k)$ is said to be admissible if the series $\zeta(s)$ converges (see Hoffman [8] and Spanier [18]).

In this sense Sitaramachandrarao and Subbarao [15] found the next equation:

$$\zeta(\{a\}^3) = \frac{1}{5}\zeta^3(a) - \frac{1}{2}\zeta(a)\zeta(2a) + \frac{1}{3}\zeta(3a),$$

where $\zeta(\{a\}^n) = \zeta(\overbrace{a, a, \ldots, a}^{n})$ and $a > 1$, is a positive integer. Generalizing the previous result, Hoffman in [8], obtained the following result:

Theorem 1. (Hoffman [8], Theorem 2.2)

$$\sum_{\sigma\in\Sigma_k} A(i_{\sigma(1)} \ldots, i_{\sigma(k)}) = \sum_{partitions\ \Pi\ of\ \{1,2,\ldots,k\}} \tilde{c}(\Pi)\zeta(i, \Pi) \tag{1}$$

where

- $$A(i_1 \ldots, i_k) = \sum_{n_1>n_2>\ldots>n_k\geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \ldots n_k^{i_k}};$$
- Σ_k *is the group of k- permutations;*
- $\Pi = \{P_1, \ldots, P_l\}$ *is a partition of the set* $\{1,2,\ldots,k\}$*;*
- $\zeta(i, \Pi) = \prod_{s=1}^{l} \zeta(\sum_{j\in P_s} i_j)$*;*
- $\tilde{c} = (-1)^{k-l}(cardP_1 - 1)!(cardP_2 - 1)! \cdots (cardP_l - 1)!$

Recently, Alegri, Bonilla, and Reddy [1] extended Hofmann's result, as outlined below.

Theorem 2. [Alegri, Bonilla and Reddy [1]] *For all complex numbers* $a_1, a_2, \ldots, a_r$ *with* $\Re\{a_i\} > 1$, $1 \le i \le r$, *we have*

$$\sum_{p_n(a_1,\ldots,a_r)\in PR_n(a_1,\ldots,a_r)} \zeta(p_n(a_1,\ldots,a_r)) =$$

$$\sum_{m=1}^{n} \sum_{\substack{w_1+\cdots+w_m\in C(n)\\ 1\le m\le n}} \frac{(-1)^{n+m}}{m! w_1 w_2\cdots w_m} \left[\sum_{k_1+\ldots+k_r\in C^*(w_1)} \binom{w_1}{k_1,\ldots,k_{r-1}} \zeta(k_1a_1+\ldots+k_ra_r)\right]$$

$$\times \left[\sum_{k_1+\ldots+k_r\in C^*(w_2)} \binom{w_2}{k_1,\ldots,k_{r-1}} \zeta(k_1a_1+\ldots+k_ra_r)\right] \times$$

$$\cdots \times \left[\sum_{k_1+\ldots+k_r\in C^*(w_m)} \binom{w_m}{k_1,\ldots,k_{r-1}} \zeta(k_1a_1+\ldots+k_ra_r)\right].$$

We denote by $PR_n(a_1, a_2, \ldots, a_r)$ the set of all n-permutations with repetition of the set $\{a_1, a_2, \ldots, a_r\}$, or equivalently the n-tuples of elements of $\{a_1, a_2, \ldots, a_r\}$. For example,

$$PR_3(a_1, a_2) = \{(a_1, a_1, a_1), (a_1, a_1, a_2),$$
$$(a_1, a_2, a_1), (a_2, a_1, a_1), (a_2, a_2, a_1), (a_2, a_1, a_2), (a_1, a_2, a_2), (a_2, a_2, a_2)\}.$$

As in Charalambides [4], the number of n-permutations of r with unrestricted repetitions, denoted by $U(r, n)$, is $U(r, n) = r^n$.
We denote by $C(n; k_1, k_2, \ldots, k_r)$ the multinomial coefficient of $x_1^{k_1} x_2^{k_2} \ldots x_r^{k_r}$ in the expansion of $(x_1 + x_2 + \cdots + x_r)^n$, where $n = k_1 + k_2 + \ldots + k_r$. The number $C(n; k_1, k_2, \ldots, k_r)$ is equal to

$$\frac{n!}{k_1! k_2! \cdots k_r!},$$

as seen in Charalambides [4] and Comtet [6]. An integer composition of a positive integer n is a way of writing n as the sum of a sequence of strictly positive integers. The set of compositions of n is denoted by $C(n)$. For example,

$$C(4) = \{4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1\}.$$

A weak composition of n is an integer composition of n in which the number zero is allowed. The set of weak compositions of n is denoted by $C^*(n)$. More information

about compositions and weak compositions can be found in Alegri [2], Heubach and Mansour [7] and Sills [16].
For $a = a_1 = a_2 = \ldots = a_r$, with $\Re\{a\} > 1$, it is possible to obtain an expression for $\zeta(\{a\}^n)$ as combinations of ordinary Riemann zeta functions.

Corollary 1. [Alegri, Bonilla and Reddy [1]]

$$\zeta(\{a\}^n) = \frac{1}{n^n} \sum_{\substack{w_1+\cdots+w_m \in C(n) \\ 1\le m\le n}} \frac{(-1)^{n+m}}{m! w_1 w_2 \cdots w_m} \left[\sum_{k_1+\ldots+k_r \in C^*(w_1)} C(w_1; k_1, \ldots, k_r)\zeta((k_1+\ldots+k_r)a) \right]$$

$$\times \left[\sum_{k_1+\ldots+k_r \in C^*(w_2)} C(w_2; k_1, \ldots, k_r)\zeta((k_1+\ldots+k_r)a) \right] \times$$

$$\cdots \times \left[\sum_{k_1+\ldots+k_r \in C^*(w_m)} C(w_m; k_1, \ldots, k_r)\zeta((k_1+\ldots+k_r)a) \right].$$

Similar to the previous theorem, the authors [1] find another way to represent the MZV $\zeta(\{a\}^n)$, as given next.

Theorem 3. [Alegri, Bonilla and Reddy [1]] *For all complex numbers a, wherein $\Re\{a\} > 1$, we have*

$$\zeta(\{a\}^n) = \frac{(-1)^{n+1}}{n}\zeta(na) + \frac{1}{n!}\zeta^n(a) + \sum_{x_1+\ldots+x_m \,\in\, C(n)} \frac{(-1)^{ne(x_1+\ldots+x_m)}}{m! x_1 x_2 \cdots x_m}\zeta(x_1 a)\zeta(x_2 a)\ldots\zeta(x_m a),$$

where $ne(x_1+\ldots+x_m)$ is equals to the number of x_i even in the integer composition $x_1+\ldots+x_m \in C(n)$.

In this paper, we aim to present a new expression for $\zeta(\{a\}^n)$, for $\Re\{a\} > 1$, utilizing Bell polynomials. The partial Bell polynomial and complete exponential Bell partition polynomial, as defined in Chapter 11 of Charalambides [4], Comtet [6], Sivaraman et al. [17] and Zúñiga-Segundo et al. [20], are given respectively by the sums:

$$B_{n,j}(x_1, x_2, \ldots, x_{n-j+1}) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_i\ge 0 \\ k_1+k_2+\ldots+k_n=j}} \frac{n!}{k_1(1!)^{k_1} k_2!(2!)^{k_2} \cdots k_{n-j+1}!((n-j+1)!)^{k_{n-j+1}}} x_1^{k_1} x_2^{k_2} \cdots x_{n-j+1}^{k_{n-j+1}}, \tag{2}$$

$$B_n(x_1, x_2, \ldots, x_n) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_i\ge 0}} \frac{n!}{k_1(1!)^{k_1} k_2!(2!)^{k_2} \cdots k_n!(n!)^{k_n}} x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}. \tag{3}$$

The generating function for the exponential Bell partition polynomial (3) is

$$\sum_{n=0}^{\infty} B_n(x_1, \ldots, x_n) \frac{t^n}{n!} = \exp\left(\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right).$$

In addition, we have an important and well-known recurrence for Bell polynomials:

$$B_{n+1}(x_1, x_2, \ldots, x_{n+1}) = \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(x_1, x_2, \ldots, x_{n-i}) x_{i+1}, \tag{4}$$

with the initial value $B_0 = 1$.

Bell polynomials are some of the most important special polynomials due to their various applications in different mathematical frameworks (See [9-12]).

2. Results

The authors: Arakawa and Kaneko [3], Chen et al. [5] and Ihara et al. [13] by using $x_j = (-1)^{j-1}(j-1)!\zeta(ja)$, $j = 1, 2, \ldots, n$, they found the next result:

Theorem 4. ([3, 5, 13])

$$\zeta(\{a\}^n) = \frac{1}{n!} B_n(x_1, x_2, \ldots, x_n), x_j = (-1)^{j-1}(j-1)!\zeta(ja), j = 1, 2, \ldots, n.$$

Considering the recurrence given by (4), and using Theorem 4, we are able to state the result as follows:

Theorem 5. *For n a positive integer,*

$$n\zeta(\{a\}^n) = \sum_{j=1}^{n} (-1)^{j-1} \zeta(ja) \zeta(\{a\}^{n-j}). \tag{5}$$

Proof. For $x_i = (-1)^{i-1}(i-1)!\zeta(ia)$, we have

$$\begin{aligned} B_{n+1}(x_1, x_2, \ldots, x_{n+1}) = (n+1)!\zeta(\{a\}^{n+1}) &= \sum_{i=0}^{n} \binom{n}{i} B_{n-i}(x_1, x_2, \ldots, x_{n-i}) x_{i+1} \\ &= \sum_{i=0}^{n} \binom{n}{i} \left[(n-i)!\zeta(\{a\}^{n-i})(-1)^i i! \zeta((i+1)a)\right] \\ &= \sum_{i=0}^{n} (-1)^i \zeta(\{a\}^{n-i}) \zeta((i+1)a). \end{aligned}$$

By making the appropriate changes to the variables, we obtain the relationship in (5).

Example 1. If $n = 5$ in the above Theorem, we have the following expression for $\zeta(\{a\}^5)$:

$$\zeta(\{a\}^5) = -\frac{1}{6}\zeta(3a)\zeta(2a) - \frac{1}{4}\zeta(a)\zeta(4a) + \frac{1}{8}\zeta^2(2a)\zeta(a).$$

Using the well-known initial condition: $\frac{\pi^2}{6} = \zeta(2)$ and the use of Theorem 5, allows us to obtain the Hoffman-Zagier formula, as in Lupu [14] and Xu [19]:

Corollary 2.

$$\zeta(\{2\}^n) = \frac{\pi^{2n}}{(2n+1)!}.$$

Still using Theorem 5, it is possible to show the following determinantal expression for $\zeta(\{a\}^n)$ as follows:

Corollary 3. $$\zeta(\{a\}^n) = \frac{1}{n!}\begin{vmatrix} \zeta(a) & n-1 & 0 & \dots & 0 \\ \zeta(2a) & \zeta(a) & n-2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \zeta((n-1)a) & \zeta((n-2)a) & \vdots & \dots & 1 \\ \zeta(na) & \zeta((n-1)a) & \dots & \dots & \zeta(a) \end{vmatrix}$$

Our last result leads with some kind of inversion of Theorem 4.

Theorem 6.

$$\zeta(na) = \sum_{i=1}^{n}(-1)^{n+i}(i-1)! \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_i\geq 0 \\ k_1+k_2+\ldots+k_n=j}} \frac{n\zeta(a)\zeta(\{a\}^2)\cdots\zeta(\{a\}^{n-i+1})}{k_1!(1!)^{k_1-1}k_2!(2!)^{k_2-1}\cdots k_{n-j+1}!((n-j+1)!)^{k_{n-j+1}-1}}.$$

Proof. It is a well-known fact about Bell polynomials that, for

$$y_n = \sum_{i=1}^{n} B_{n,i}(x_1, \ldots, x_{n-i+1}) = B_n(x_1, \ldots, x_n), \tag{6}$$

we have

$$x_n = \sum_{i=1}^{n}(-1)^{i-1}(i-1)!B_{n,i}(y_1, \ldots, y_{n-i+1}).$$

For $y_i = i!\zeta(\{a\}^i)$, $i = 1, 2, \ldots$, by the definition of partial Bell polynomials,

$$B_{n,i}(y_1, \ldots, y_n) = \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_i\geq 0 \\ k_1+k_2+\ldots+k_n=j}} \frac{n!\zeta(a)\zeta(\{a\}^2)\cdots\zeta(\{a\}^{n-i+1})}{k_1!(1!)^{k_1-1}k_2!(2!)^{k_2-1}\cdots k_{n-j+1}!((n-j+1)!)^{k_{n-j+1}-1}},$$

and by equation (6),

$$x_n = \sum_{i=1}^{n} (-1)^{i-1}(i-1)! \sum_{\substack{k_1+2k_2+\ldots+nk_n=n \\ k_i \geq 0 \\ k_1+k_2+\ldots+k_n=j}} \frac{n!\zeta(a)\zeta(\{a\}^2)\cdots\zeta(\{a\}^{n-i+1})}{k_1!(1!)^{k_1-1}k_2!(2!)^{k_2-1}\cdots k_{n-j+1}!((n-j+1)!)^{k_{n-j+1}-1}}.$$

Since $x_n = (-1)^{n-1}(n-1)!\zeta(na)$, and using the last equation, we have conclude the proof.

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