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WEAK PERIODIC SOLUTIONS FOR A CLASS OF REACTION-DIFFUSION PROBLEMS

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Abstract: The focus of our research paper is on exploring a quasilinear parabolic reaction-diffusion problem that includes a nonlinearity in gradient and nonlinear boundary conditions. This problem is relevant to the study of diffusion in different scientific fields. Our approach involves utilizing functional analysis techniques, specifically Schauder's fixed point theorem, to establish the existence of weak periodic solutions.

Keywords and Phrases: Reaction-diffusion equation, fixed point theorem, weak periodic solutions.

2020 Mathematics Subject Classification: 35K57, 35B10, 47H10.

1. Introduction

Reaction-diffusion equations can be used to quantitatively represent many periodic phenomena in the physical, environmental, and biological domains. An example of such systems is the one introduced by Badii [3], which models the movement of water and salt in a porous medium subject to water extraction by mangrove roots. One prominent example is also the Belousov-Zhabotinsky reaction, which involves the periodic oscillation of chemical concentrations. This reaction has been used to model phenomena such as heartbeats and circadian rhythms. Other related models can be found in [10]. For a selection of significant recent findings on this topic, refer to [1, 5, 8, 6, 11, 12, 13, 15, 16], and the associated references. This work aims to study weak periodic solutions for a class of reaction-diffusion models. To achieve this, we will use a methodology based on Schauder's fixed point theorem. We are then interested in the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - d\Delta u + a\left|\nabla u\right|^{p} = h\left(t,x\right) & \text{in } Q_T, \\ u\left(0,.\right) = u\left(T,.\right) & \text{in } \Omega, \\ -\dfrac{\partial u}{\partial \eta} = \beta\left(x,t\right)u + g\left(t,x,u\right) & \text{on } \Sigma_T, \end{cases} \tag{1.1}$$

where $u = u\left(t,x\right)$, Ω is an open regular bounded subset of $\mathbb{R}^N$, $N \geq 1$, with the smooth boundary $\partial\Omega$, $T > 0$ is the period, $Q_T = \left]0,T\right[\times \Omega$, $\Sigma_T = \left]0,T\right[\times \partial\Omega$, $p \geq 1$, $d > 0$, $a > 0$ and η denote the unit normal vector to the boundary $\partial\Omega$. h, g and β are prescribed functions satisfying the conditions of hypotheses $(H_1) - (H_3)$, which we will mention in the next section.

Our research primarily concentrates on investigating periodic solutions within a suitable space of T-periodic functions. To accomplish this objective, we will utilize a particular theorem concerning maximal monotone mappings, along with an appropriate fixed point argument.

Theorem 1.1. (for maximal monotone mappings) *Let L be a linear closed, densely defined operator from the reflexive space $\mathcal{V}$ to its dual $\mathcal{V}^*$, L maximal monotone and let A be a bounded hemicontinuous monotone mapping from $\mathcal{V}$ to $\mathcal{V}^*$, then $L + A$ is maximal monotone in $\mathcal{V} \times \mathcal{V}^*$. Moreover, if $L + A$ is coercive, then Range$(L + A) = \mathcal{V}^*$.*

See [4] and [7] for the proof of this theorem as well as various applications.

The remainder of this paper is structured as follows: The next section outlines the assumptions necessary for our problem and states our main result. In the third section, we prove the existence and uniqueness of a periodic solution for an abstract problem using maximal monotone mappings. The final section focuses on proving the main result.

2. Statement of the main result

Throughout this work, we consider the following assumptions:

(H_1) β is a periodic positive continuous and bounded function such that

$$0 < \beta_1 \leq \beta\left(t,x\right) \leq \beta_2,\ \forall \left(t,x\right) \in \Sigma_T.$$

(H_2) $g : \Sigma_T \times \mathbb{R}^+ \to \mathbb{R}$ is a periodic Carathéodory function in time, $s \mapsto g\left(t,x,s\right)$

is nondecreasing with respect to s for a.e., $(t,x) \in \Sigma_T$ and

$$s.g(t,x,s) \geq 0 \quad \text{and} \quad |g(t,x,s)| \leq \xi(t,x) + |s| \quad \text{where } \xi \in L^2(\Sigma_T).$$

(H_3) $h \in L^2(Q_T)$.

Now, we present our functional framework for the periodic solutions of our problem. We define $\mathbf{v} = \{\psi \in D(\Omega), \ \mathbf{div}\ \psi = 0\}$, and we will denote V to the adherence of $\mathbf{v}$ in $H^1(\Omega)$, V' to the topological dual space of V, H to the adherence of $\mathbf{v}$ in $L^2(\Omega)$ and X' to the topological dual space of H.

We have $V \subset H \subset V'$ with continuous and dense injection, see Lions ([7], p156).

$$\begin{aligned}\mathcal{V} &= L^2\left(0,T;H^1(\Omega)\right) \cap L^\infty\left(0,T;H \cap W^{1,q}(\Omega)\right),\\ \mathcal{V}^* &= L^2\left(0,T;\left(H^1(\Omega)\right)^*\right) + L^1\left(0,T;X'\right),\end{aligned}$$

with $q = 2p$, and we denote by $\left(H^1(\Omega)\right)^*$ the topological dual space of $H^1(\Omega)$ and $\langle\cdot,\cdot\rangle$ present the duality pairing between $\mathcal{V}$ and $\mathcal{V}^*$. For more details and information, see Lions ([7], pp 68–69). The standard norm of $L^2(0,T;H^1(\Omega))$ is defined by

$$\|u\|_{L^2(0,T;H^1(\Omega))} := \left(\int_{Q_T} |\nabla u(t,x)|^2\, dtdx + \int_{Q_T} |u(t,x)|^2\, dtdx\right)^{\frac{1}{2}}.$$

Throughout this paper, we equipped $\mathcal{V}$ with the norm

$$\|u\|_{\mathcal{V}} := \left(\int_{Q_T} |\nabla u(t,x)|^2\, dtdx + \int_{\Sigma_T} \beta(t,x)\, |\tilde{u}(t,x)|^2\, dtd\sigma\right)^{\frac{1}{2}}$$

which is equivalent to the standard norm of $L^2(0,T;H^1(\Omega))$. We denote by $\tilde{u}$ the trace of u on Σ_T. Let us define the set

$$\mathcal{W}(0,T) := \left\{u \in \mathcal{V} \mid \frac{\partial u}{\partial t} \in \mathcal{V}^* \text{ and } u(0) = u(T)\right\}$$

equipped with the norm $\|u\|_{\mathcal{W}(0,T)} := \|u\|_{\mathcal{V}} + \left\|\frac{\partial u}{\partial t}\right\|_{\mathcal{V}^*}$.

It is obvious that $\mathcal{W}(0,T)$ is dense in $\mathcal{V}$ due to the density of $C^\infty\left(\overline{Q}\right) \subset \mathcal{W}(0,T)$ in $\mathcal{V}$. To obtain further information on this matter, the reader may consult [7].

We introduce the concept of a weak periodic solution to our problem.

Definition 2.1. *A function u is said to be a weak periodic solution of problem* (1.1)*, if $u \in \mathcal{V}$ and for all $\varphi \in \mathcal{W}(0,T) \cap L^{\infty}(Q_T)$, we have*

$$-\left\langle u, \frac{\partial \varphi}{\partial t}\right\rangle + d\int_{Q_T} \nabla u \nabla \varphi \; dxdt + a\int_{Q_T} |\nabla u|^p \varphi \; dxdt + \int_{\Sigma_T} \beta(t,x)\, \tilde{u}\tilde{\varphi}\; d\sigma dt + \int_{\Sigma_T} g(t,x,\tilde{u})\, \tilde{\varphi} d\sigma dt = \int_{Q_T} h(t,x)\, \varphi \; dxdt.$$

We are now able to state the main result of this paper, which is the following theorem.

Theorem 2.2. *Under hypotheses* $(H_1)-(H_3)$*, problem* (1.1) *admits a weak periodic solution $u \in \mathcal{W}(0,T)$.*

3. Abstract problem

Using Theorem 1.1, we will prove the existence and uniqueness of a periodic solution for an abstract problem formulated by means of maximal monotone mappings.

Having fixed $w \in \mathcal{V}$, we consider the problem

$$-\left\langle u, \frac{\partial \varphi}{\partial t}\right\rangle + d\int_{Q_T} \nabla u \nabla \varphi dxdt + a\int_{Q_T} |\nabla w|^p \varphi \; dxdt + \int_{\Sigma_T} \beta(t,x)\, \tilde{u}\tilde{\varphi} d\sigma dt + \int_{\Sigma_T} g(t,x,\tilde{u})\, \tilde{\varphi} d\sigma dt = \int_{Q_T} h(t,x)\, \varphi \; dxdt. \tag{3.1}$$

In order to use Theorem 1.1, we must define two mappings L and A:
L is the linear operator defined by

$$L : \mathcal{W}(0,T) \to \mathcal{V}^* \quad \text{with} \quad \langle Lu, \varphi\rangle = \int_{Q_T} u_t \varphi dtdx, \; \forall \varphi \in \mathcal{W}(0,T).$$

This operator is closed, skew-adjoint (i.e., $L = -L^*$) and maximal monotone (see Lemma 1.1, p 313 and Section 2.2 of Chapter 3 in Lions [7]). As for the operator A, it is defined as follows: $A : \mathcal{V} \to \mathcal{V}^*$,

$$\langle Au, \varphi\rangle = d\int_{Q_T} \nabla u \nabla \varphi \; dxdt + \int_{\Sigma_T} \beta(t,x)\, \tilde{u}\tilde{\varphi}\; d\sigma dt + \int_{\Sigma_T} g(t,x,\tilde{u})\, \tilde{\varphi}\; d\sigma dt, \; \forall \varphi \in \mathcal{W}(0,T).$$

Proposition 3.1. *If the assumptions* $(H1)$ *and* $(H2)$ *are fulfilled, then the mapping A is (i) hemicontinuous, (ii) monotone and (iii) coercive.*
Proof. (i) The hemicontinuity follows from the Hölder inequality. In fact,

$$|\langle Au, \varphi\rangle| \leq \left| d\int_{Q_T} \nabla u \nabla \varphi \; dxdt\right| + \left|\int_{\Sigma_T} \beta(t,x)\, \tilde{u}\tilde{\varphi}\; d\sigma dt\right| + \left|\int_{\Sigma_T} g(t,x,\tilde{u})\, \tilde{\varphi}\; d\sigma dt\right|.$$

We have

$$\left| d\int_{Q_T} \nabla u \nabla \varphi \; dxdt\right| \leq d\left(\int_{Q_T} |\nabla u|^2 \; dxdt\right)^{\frac{1}{2}} \left(\int_{Q_T} |\nabla \varphi|^2 \; dxdt\right)^{\frac{1}{2}} \leq d\, \|u\|_{\mathcal{V}} \|\varphi\|_{\mathcal{V}},$$

$$\left|\int_{\Sigma_T} \beta(t,x)\,\tilde{u}\tilde{\varphi}\ d\sigma dt\right| \leq \left(\int_{\Sigma_T} \beta(t,x)\,|\tilde{u}|^2\ d\sigma dt\right)^{\frac{1}{2}} \left(\int_{\Sigma_T} \beta(t,x)\,|\tilde{\varphi}|^2\ d\sigma dt\right)^{\frac{1}{2}} \leq \|u\|_{\mathcal{V}}\,\|\varphi\|_{\mathcal{V}},$$

$$\left(\int_{\Sigma_T} |\tilde{u}|^2\ d\sigma dt\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\beta_1}} \left(\int_{Q_T} |\nabla u|^2\ dxdt + \int_{\Sigma_T} \beta\,|\tilde{u}|^2\ d\sigma dt\right)^{\frac{1}{2}} = \frac{1}{\sqrt{\beta_1}}\,\|u\|_{\mathcal{V}},$$

$$\left(\int_{\Sigma_T} |\tilde{\varphi}|^2\ d\sigma dt\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{\beta_1}} \left(\int_{Q_T} |\nabla \varphi|^2\ dxdt + \int_{\Sigma_T} \beta\,|\tilde{\varphi}|^2\ d\sigma dt\right)^{\frac{1}{2}} = \frac{1}{\sqrt{\beta_1}}\,\|\varphi\|_{\mathcal{V}},$$

$$\left|\int_{\Sigma_T} g(t,x,\tilde{u})\,\tilde{\varphi}\ d\sigma dt\right| \leq \frac{1}{\sqrt{\beta_1}}\,\|\xi\|_{L^2(\Sigma_T)}\,\|\varphi\|_{\mathcal{V}} + \frac{1}{\beta_1}\,\|u\|_{\mathcal{V}}\,\|\varphi\|_{\mathcal{V}}.$$

Finally, we find

$$\begin{aligned} |\langle Au,\varphi\rangle| &\leq \left[\left(d+1+\frac{1}{\beta_1}\right)\|u\|_{\mathcal{V}} + \frac{1}{\sqrt{\beta_1}}\,\|\xi\|_{L^2(Q_T)}\right]\|\varphi\|_{\mathcal{V}}, \\ \|Au\|_* &\leq \left(d+1+\frac{1}{\beta_1}\right)\|u\|_{\mathcal{V}} + \frac{1}{\sqrt{\beta_1}}\,\|\xi\|_{L^2(\Sigma_T)}. \end{aligned}$$

(ii) We have

$$\begin{aligned} \langle Au-Av,u-v\rangle &= d\int_{Q_T} \nabla(u-v)\,\nabla(u-v)\ dxdt + \int_{\Sigma_T} \beta(t,x)\,|\tilde{u}-\tilde{v}|^2\ d\sigma dt \\ &\quad + \int_{\Sigma_T} [g(t,x,\tilde{u}) - g(t,x,\tilde{u})]\,(\tilde{u}-\tilde{v})\ d\sigma dt. \end{aligned}$$

According to $(H2)$, the function $s \mapsto g(t,x,s)$ is nondecreasing with respect to s for a.e., (t,x). So,

$$\int_{\Sigma_T} [g(t,x,\tilde{u}) - g(t,x,\tilde{u})]\,(\tilde{u}-\tilde{v})\ d\sigma dt > 0,$$

then $\langle Au-Av,u-v\rangle > 0$, which shows the strict monotony of A.

(iii) We have

$$\langle Au,u\rangle = d\int_{Q_T} |\nabla u|^2\ dxdt + \int_{\Sigma_T} \beta(t,x)\,|\tilde{u}|^2\ d\sigma dt + \int_{\Sigma_T} g(t,x,\tilde{u})\,\tilde{u}\ d\sigma dt.$$

According to $(H2)$, we have $g(t,x,\tilde{u})\,\tilde{u} \geq 0$. Then

$$\langle Au,u\rangle \geq d\int_{Q_T} |\nabla u|^2\ dxdt + \int_{\Sigma_T} \beta(t,x)\,|\tilde{u}|^2\ d\sigma dt,$$

and $\langle Au, u\rangle \geq d\,\|u\|_{\mathcal{V}}^{2}$, which implies $\lim\limits_{\|u\|_{\mathcal{V}}\to+\infty}\frac{\langle Au,u\rangle}{\|u\|_{\mathcal{V}}}=+\infty$; hence the coercivity.

Besides that, let $G \in \mathcal{V}^*$ be the linear functional defined as follows.

$$\langle G, \varphi\rangle = -a\int_{Q_T}|\nabla w|^p\,\varphi\,dxdt + \int_{Q_T} h(t,x)\,\varphi\,dxdt,\ \forall\varphi\in\mathcal{W}(0,T),$$

then, problem (3.1) can be reformulated in the following abstract form

$$Lu + Au = G. \tag{3.2}$$

Proposition 3.2. *Let $w \in \mathcal{V}$ be given and assuming $(H1)$ and $(H2)$,then the problem (3.2) has a unique weak periodic solution.*

Proof. The existence of weak periodic solution descends from Theorem 1.1. Uniqueness is a consequence of the strict monotonicity. Indeed, suppose that $u_1,\ u_2$ are solutions of problem (3.2). So, $Lu_1 + A(u_1) = G$ and $Lu_2 + A(u_2) = G$, which implies $\langle Lu_1 + A(u_1) - Lu_2 - A(u_2), u_1 - u_2\rangle = 0$. It is a contradiction because of the strict monotonicity.

4. Proof of the main result

This section will focus on proving Theorem 2.2.

Proof of Theorem 2.2. The existence of weak solutions to (1.1) will be based on the research of fixed points for the mapping $\Psi : \mathcal{V} \to \mathcal{V}$ with $w \mapsto \Psi(w) = u$, where u is the unique weak periodic solution of the following problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - d\Delta u + a\,|\nabla w|^p = h(t,x) & \text{in } Q_T,\\ u(0,.) = u(T,.) & \text{in } \Omega,\\ -\dfrac{\partial u}{\partial \nu} = \beta(x,t)\,u + g(t,x,u) & \text{on } \Sigma_T. \end{cases} \tag{4.1}$$

The existence and uniqueness of weak periodic solution $u \in \mathcal{W}(0,T) \subset \mathcal{V}$ of (4.1) is a direct consequence of Badii's theorem in Badii [2], and satisfy for all $\varphi \in \mathcal{W}(0,T)$,

$$\begin{gathered} -\left\langle u, \tfrac{\partial\varphi}{\partial t}\right\rangle + d\int_{Q_T}\nabla u\nabla\varphi\,dxdt + a\int_{Q_T}|\nabla w|^p\,\varphi\,dxdt + \\ \int_{\Sigma_T}\beta(t,x)\,\tilde{u}\tilde{\varphi}\,d\sigma dt + \int_{\Sigma_T} g(t,x,\tilde{u})\,\tilde{\varphi}\,d\sigma dt = \int_{Q_T} h(t,x)\,\varphi\,dxdt \end{gathered} \tag{4.2}$$

which shows that the application is well defined.

It only remains to prove the existence of a fixed point for Ψ. For this, it is sufficient to prove that the conditions of Schauder's fixed point theorem are satisfied. This is exactly what we will present and prove in the following three points.

(i) Continuity of Ψ : We will prove some very important estimates and convergences. Let $w_n \in \mathcal{V}$ be a sequence strongly converges to w in $\mathcal{V}$. Moreover, let u_n denote the weak periodic solution of the problem

$$\begin{array}{c} -\left\langle u_n, \dfrac{\partial \varphi}{\partial t} \right\rangle + d \int_{Q_T} \nabla u_n \nabla \varphi dxdt + a \int_{Q_T} |\nabla w_n|^p \varphi dxdt + \\ \int_{\Sigma_T} \beta(t,x) \tilde{u}_n \tilde{\varphi} dxdt + \int_{\Sigma_T} g(t,x,\tilde{u}_n) \tilde{\varphi} dxdt = \int_{Q_T} h(t,x) \varphi \; dxdt. \end{array} \tag{4.3}$$

Setting $\varphi = u_n$ as a test function in (4.3), we have

$$\begin{array}{c} -\left\langle u_n, \dfrac{\partial u_n}{\partial t} \right\rangle + d \int_{Q_T} |\nabla u_n|^2 dxdt + a \int_{Q_T} |\nabla w_n|^p u_n dxdt + \\ \int_{\Sigma_T} \beta(t,x) \tilde{u}_n^2 d\sigma dt + \int_{\Sigma_T} g(t,x,\tilde{u}_n) \tilde{u}_n d\sigma dt = \int_{Q_T} h(t,x) u_n \; dxdt. \end{array} \tag{4.4}$$

The conditions $(H_1) - (H_3)$, the periodicity and the Young's inequality give us

$$\int_{Q_T} u_n(t,x) (u_n)_t (t,x) dtdx = \int_\Omega \left[u_n^2(T,x) - u_n^2(0,x) \right] dx = \int_{Q_T} h(t,x) u_n \; dxdt.$$

$(H1) - (H3)$ and the Young's inequality give us

$$d \int_{Q_T} |\nabla u_n|^2 dxdt + \int_{\Sigma_T} \beta(t,x) |\tilde{u}_n|^2 d\sigma dt + \int_{Q_T} \int_{\Sigma_T} g(t,x,\tilde{u}_n) \tilde{u}_n dxdt$$

$$\leq \frac{a}{2\varepsilon} \int_{Q_T} (|\nabla w_n|^p)^2 dxdt + \frac{\varepsilon}{2} \int_{Q_T} |u_n|^2 dxdt + \frac{1}{2\varepsilon} \int_{Q_T} (h(t,x))^2 dxdt + \frac{\varepsilon}{2} \int_{Q_T} |u_n|^2 dxdt.$$

Taking into account the equivalence of the norms in $\mathcal{V}$, we have

$$\begin{array}{rl} & d \displaystyle\int_{Q_T} |\nabla u_n|^2 dxdt + \int_{\Sigma_T} \beta(t,x) |\tilde{u}_n|^2 d\sigma dt - \varepsilon c(T,\Omega) \|u_n\|_{\mathcal{V}} \\ \leq & \dfrac{a}{2\varepsilon} \displaystyle\int_{Q_T} (|\nabla w_n|^p)^2 dxdt + \dfrac{1}{2\varepsilon} \int_{Q_T} |h(t,x)|^2 \; dxdt \end{array}$$

which give

$$(\min\{1,d\} - \varepsilon c(T,\Omega)) \|u_n\|_{\mathcal{V}} \leq \frac{a}{2\varepsilon} \int_{Q_T} (|\nabla w_n|^p)^2 dxdt + \frac{1}{2\varepsilon} \int_{Q_T} |h(t,x)|^2 \; dxdt \leq c'(\varepsilon).$$

We choose ε small enough to obtain the following classical energy estimate $\|u_n\|_{\mathcal{V}} \leq c''$, where the positive real constant c'' is independent of n. From (4.3), we get that

$\left(\frac{\partial u_n}{\partial t}\right)$ is bounded in the $\mathcal{V}^*$ norm; which proves the boundedness of u_n in $\mathcal{W}(0,T)$, i.e., $\|u_n\|_{\mathcal{W}(0,T)} \leq c''$, for all $n \in \mathbb{N}$. Thus, we can extract a subsequence denoted u_n such that $u_n \rightharpoonup u$ weakly in $\mathcal{V}$ as $n \to +\infty$. By Aubin's theorem in [14], the sequence u_n is precompact in $L^2(Q_T)$. So, $u_n \to u$ in $L^2(Q_T)$ and a.e., in Q_T. Furthermore, according to the trace theorem, see Morrey ([9], Theorem 3.1.4), we have $u_n \to u$ in $L^2(\Sigma_T)$ and a.e., in Σ_T.

Now, we prove that the sequence ∇u_n strongly converges to ∇u in $L^2(Q_T)$. From (4.4), we have

$$\begin{aligned} d\int_{Q_T} |\nabla u_n|^2\, dxdt &= -\int_{\Sigma_T} \beta(t,x)\, u_n^2 dxdt - \int_{\Sigma_T} g(t,x,u_n)\, u_n dxdt \\ &\quad -a\int_{Q_T} |\nabla w_n|^p\, u_n dxdt + \int_{Q_T} h(t,x)\, u_n\, dxdt. \end{aligned}$$

Since w_n strongly converges in $\mathcal{V}$ and strongly converges in $L^2(Q_T)$, we can pass to the limit in the last inequality, we obtain

$$\begin{aligned} d\lim_{n\to+\infty} \textstyle\int_{Q_T} |\nabla u_n|^2\, dxdt = -\int_{\Sigma_T} \beta(t,x)\, u^2 d\sigma dt - \int_{\Sigma_T} g(t,x,u)\, udxdt \\ -a\textstyle\int_{Q_T} |\nabla w|^p\, udxdt + \int_{Q_T} h(t,x)\, u\, dxdt. \end{aligned} \tag{4.5}$$

Moreover, setting $\varphi = u$ as a test function in (4.3) , it comes

$$\begin{aligned} -\left\langle u_n, \frac{\partial u}{\partial t}\right\rangle + d\int_{Q_T} \nabla u_n \nabla u dxdt = -\int_{\Sigma_T} \beta(t,x)\, u_n u d\sigma dt - \int_{\Sigma_T} g(t,x,u_n)\, udxdt \\ - a\int_{Q_T} |\nabla w_n|^p\, udxdt + \int_{Q_T} h(t,x)\, u\, dxdt. \end{aligned}$$

Taking the limit as $n \to +\infty$, we have

$$\begin{aligned} d\textstyle\int_{Q_T} |\nabla u|^2\, dxdt = -\int_{\Sigma_T} \beta(t,x)\, u^2 d\sigma dt - \int_{\Sigma_T} g(t,x,u)\, udxdt \\ -a\textstyle\int_{Q_T} |\nabla w|^p\, udxdt + \int_{Q_T} h(t,x)\, u\, dxdt, \end{aligned} \tag{4.6}$$

and by comparing (4.5) and (4.6), it results $\lim_{n\to+\infty} \int_{Q_T} |\nabla u_n|^2\, dxdt = \int_{Q_T} |\nabla u|^2\, dxdt$. Consequently, the mapping Ψ is continuous.

(ii) Compactness of Ψ : Let (w_n) be a bounded sequence in $\mathcal{V}$ and we denote $u_n = \Psi(w_n)$. As in the previous step (up to a subsequence), we have

$$\begin{aligned} w_n &\rightharpoonup w \text{ weakly in } \mathcal{V},\ u_n \rightharpoonup u \text{ weakly in } \mathcal{V},\ \frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ weakly in } \mathcal{V}^*, \\ u_n &\to u \text{ strongly in } L^2(Q_T) \text{ and a.e., in } Q_T, \\ u_n &\to u \text{ strongly in } L^2(\Sigma_T) \text{ and a.e., in } \Sigma_T. \end{aligned}$$

It suffices to prove the strong convergence of (∇u_n) in $L^2(Q_T)$. Note that the absence of almost everywhere convergence of (∇w_n) in Q_T poses a difficulty, but we can overcome it. It is obvious that

$$\int_{Q_T} |\nabla u_n - \nabla u|^2 \, dxdt = \int_{Q_T} \nabla u_n (\nabla u_n - \nabla u) \, dxdt - \int_{Q_T} \nabla u (\nabla u_n - \nabla u) \, dxdt.$$

Thanks to the weak convergence of (u_n) in $\mathcal{V}$, we have

$$\lim_{n\to+\infty} \int_{Q_T} \nabla u (\nabla u_n - \nabla u) \, dxdt = 0.$$

Now, let (w_n) be a bounded sequence in $\mathcal{V}$ and we denote $u_n = \Psi(w_n)$. By the same reasoning of the first step, we have

$$\begin{array}{rcl} w_n & \rightharpoonup & w \text{ weakly in } \mathcal{V},\ u_n \rightharpoonup u \text{ weakly in } \mathcal{V},\ \dfrac{\partial u_n}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} \text{ weakly in } \mathcal{V}^*, \\ u_n & \longrightarrow & u \text{ strongly in } L^2(Q_T) \text{ and a.e., in } Q_T, \\ u_n & \longrightarrow & u \text{ strongly in } L^2(\Sigma_T) \text{ and a.e., in } \Sigma_T. \end{array}$$

To get the compactness of Ψ, it suffices to prove the strong convergence of (∇u_n) in $L^2(Q_T)$, noting that the difficulty is presented by the absence of the almost everywhere convergence of (∇w_n) in Q_T , but we can overcome this problem by observing that

$$\int_{Q_T} |\nabla u_n - \nabla u|^2 \, dxdt = \int_{Q_T} \nabla u_n (\nabla u_n - \nabla u) \, dxdt - \int_{Q_T} \nabla u (\nabla u_n - \nabla u) \, dxdt.$$

Thanks to the weak convergence of (u_n) in $\mathcal{V}$, we get

$$\lim_{n\to+\infty} \int_{Q_T} \nabla u (\nabla u_n - \nabla u) \, dxdt = 0.$$

On the other hand, setting $\varphi = u$ as a test function in (4.3), we obtain

$$\begin{aligned} -\left\langle u_n, \frac{\partial u}{\partial t} \right\rangle + \int_{Q_T} \nabla u_n \nabla u dxdt &= -\int_{Q_T} |\nabla w_n|^p \, u dxdt - \int_{\Sigma_T} \beta(t,x)\, u_n u d\sigma dt \\ &\quad - \int_{\Sigma_T} g(t,x,u_n)\, u dxdt + \int_{Q_T} h(t,x)\, u \; dxdt, \end{aligned}$$

and setting $\varphi = u_n$ as a test function in (4.3), we obtain

$$\begin{gathered} -\left\langle u_n, \tfrac{\partial u_n}{\partial t} \right\rangle + \textstyle\int_{Q_T} |\nabla u_n|^2 \, dxdt + \int_{Q_T} |\nabla w_n|^p \, u_n dxdt \\ + \textstyle\int_{\Sigma_T} \beta(t,x)\, u_n^2 d\sigma dt + \int_{\Sigma_T} g(t,x,u_n)\, u_n dxdt = \int_{Q_T} h(t,x)\, u_n \; dxdt, \end{gathered}$$

and we get

$$\int_{Q_T} \nabla u_n \left(\nabla u_n - \nabla u\right) dxdt = -\left\langle \frac{\partial u_n}{\partial t}, u_n - u \right\rangle - \int_{\Sigma_T} \beta\left(t,x\right) u_n \left(u_n - u\right) d\sigma dt$$
$$- \int_{\Sigma_T} g\left(t,x,u_n\right)\left(u_n - u\right) dxdt - \int_{Q_T} \left|\nabla w_n\right|^p \left(u_n - u\right) dxdt + \int_{Q_T} h\left(t,x\right)\left(u_n - u\right) \ dxdt, \tag{4.7}$$

then

$$a \int_{Q_T} \left|\left|\nabla w_n\right|^p \left(u_n - u\right)\right| dxdt \leq a \left(\int_{Q_T} \left(\left|\nabla w_n\right|^p\right)^2 dxdt\right)^{\frac{1}{2}} \left\|u_n - u\right\|_{L^2(Q_T)},$$

$$\lim_{n\to+\infty} \int_{Q_T} \left|\left|\nabla w_n\right|^p \left(u_n - u\right)\right| = 0.$$

Note that $\left(\int_{Q_T} \left(\left|\nabla w_n\right|^p\right)^2\right)^{\frac{1}{2}}$ is convergent because $w_n \in \mathcal{V}$, and

$$\int_{\Sigma_T} \beta\left(t,x\right) u_n \left(u_n - u\right) d\sigma dt \leq \beta_2 \left(\int_{\Sigma_T} \left(\left|u_n\right|\right)^2 dxdt\right)^{\frac{1}{2}} \left\|u_n - u\right\|_{L^2(\Sigma_T)},$$

$$\lim_{n\to+\infty} \int_{\Sigma_T} \beta\left(t,x\right) u_n \left(u_n - u\right) d\sigma dt = 0.$$

Also

$$\int_{\Sigma_T} g\left(t,x,u_n\right)\left(u_n - u\right) dxdt \leq \left(\left(\int_{\Sigma_T} \left(\xi\left(t,x\right)\right)^2\right)^{\frac{1}{2}} + \left(\int_{\Sigma_T} \left(\left|u_n\right|\right)^2\right)^{\frac{1}{2}}\right) \left\|u_n - u\right\|_{L^2(\Sigma_T)},$$

$$\lim_{n\to+\infty} \int_{\Sigma_T} g\left(t,x,u_n\right)\left(u_n - u\right) dxdt = 0.$$

Also note that the periodicity and the weak convergence of $\left(\dfrac{\partial u_n}{\partial t}\right)$ in $\mathcal{V}^*$ yields

$$\lim_{n\to+\infty} \left\langle \frac{\partial u_n}{\partial t}, u_n - u \right\rangle = -\lim_{n\to+\infty} \left\langle \frac{\partial u_n}{\partial t}, u_n \right\rangle - \lim_{n\to+\infty} \left\langle \frac{\partial u_n}{\partial t}, u \right\rangle = 0.$$

Now, we pass to the limit in (4.7), it results

$$\lim_{n\to+\infty} \int_{Q_T} \nabla u_n \left(\nabla u_n - \nabla u\right) dxdt = 0$$

which ensures the compactness of Ψ.

(iii) Ψ send the ball of $\mathcal{V}$ of R radius to itself. Indeed, we get the existence of a constant $R > 0$ such that $\Psi\left(B\left(0, R\right)\right) \subset B\left(0, R\right)$ where $B\left(0, R\right)$ is the ball of $\mathcal{V}$ with radius R. Let $w \in \mathcal{V}$ and $u = \Psi\left(w\right)$, by taking u as test function in the equation satisfied by u, we easily find this estimate which completes the proof of our theorem.

$$\|u\|_{\mathcal{V}} \leq \left(\int_{Q_T} \left(|\nabla w_n|^p\right)^2\right)^{\frac{1}{2}} := R.$$

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