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A COMPARATIVE STUDY ON A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: Analytical and numerical research is done to determine how carbon nanotubes (CNT's) nanofluid affects natural convection close to an endless vertically heated plate. The corresponding system of non-linear partial differential equations is solved analytically. The Kamal transform is used to compute the explicit approximate analytical solutions for characteristics of nanofluid flow, such as the velocity and the temperature profile. The analytical expressions for the velocity and the temperature distributions are given in explicit form. The outcomes are next compared with the numerical solution with the help of numerical inversion formula, which demonstrates a good agreement. The average absolute error percentage is calculated for both the velocity and the temperature profile to show the effectiveness of this present approach. Also, the 3D view of non-dimensional temperature and velocity are plotted. The approximate analytical expression for the Nusselt number is consequently derived. Graphical representations are given for the numerous effects of significant physical parameters.

Keywords and Phrases: Fourier's law, Carbon nanotube (CNT), Natural convection flow, The Kamal transform, Parameter perturbation method (PPM), Numerical inversion method (Stehfest's formula).

2020 Mathematics Subject Classification: 35A22, 35G31, 35G61, 34E05, 34E10.

1. Introduction

Currently, researchers are fascinated by the important role of nanofluids in many applications. There are many applications for nanofluids, such as fuel cells, hybrid-motors, domestic refrigerators, heat exchangers, electronics, biomedical and food products. One application is described in Elnaqeeb et al. [17], who examined the behavior of carbon nanotubes (CNTs) nanofluids with Prabhakar-like heat transport on natural convection, close to an endless vertical heated plate. Eastman et al. [16] highlight several characteristics of nanofluid behavior in their article [16]. According to Lee et al. [26], particle size is also thought to have a significant role in enhancing the nanofluids thermal conductivity.

Recent empirical studies have revealed that improvements in the heat transmission of conventional fluids was received a lot of interest. Common approaches to improving heat transfer using solid nanoparticles dispersed in a liquid have great potential for improving heat transfer from the base fluid. A fluid with suspended solid particles can transmit heat more effectively than a fluid without them and a range of materials might be utilized to create nanofluids [9, 14, 27, 34, 35, 38, 39]. Xie et al. [36] spoke about the specific components regarding nanoparticles in addition to the thermal conductivity for fluids. Das et al. [12] reported that improved thermal conductivity at high temperatures for nanofluid. Murshed et al. [28] found that a 5 percent volume addition of TiO_2 nanoparticles to water increased the maximum percentage of thermal conductivity.

Kim et al. [25] reported on the thermoelectric power and thermal efficiency of a single CNT. Choi et al.'s [11] preparation of oil-based nanofluids, including CNTs, revealed that the conductivity of thermal measurement was non-linear when nanotubes were loaded. Cu nanoparticles directly distributed over ethylene glycol have used to create nanofluids, and Eastman et al. [15] evaluated the energy conductivity of these fluids. An aqueous solution of multi-walled carbon nanotube passing through a horizontal tube was the subject of a study by Ding et al. [13] to determine its heat transfer characteristics. According to Hwang et al. [23], the properties of the conventional fluid and the suspended nanoparticles have an effect on the sustainability of the nanofluid.

Fractional calculus is being used in various scientific fields such as biophysics, biology, electrical engineering and mechanics (Ahmed et al. [7]). The Prabhakar derivative, the Atangana-Baleanu derivative, the Hadamard derivative, and several other fractional derivative operators are regarded as generalizations of classical derivatives [5, 6, 10, 19, 20, 21, 29].

To discover numerical approximations of the solutions to the differential equations, Stehfest [31, 32] introduced a numerical inversion approach to the Laplace transform (Jacquot et al. [24]). Abdeliah et al. [1, 2] discovered a new transformation, the Kamal transformation and the use of the Kamal transformation to solve partial differential equations. Aggarwal et al. [4] present the Kamal transformation error function and some applications in their study [3]. Furthermore, Fadhil [18] explains the convolution of the Kamal transform.

For the intended purpose for resolving ordinary and even partial differential equations that are both linear and non-linear, numerous analytical approximation techniques have been developed recently. He [22] put out a number of workable techniques, including the Variational iteration technique and Roshan et al. [8]'s Parameter Perturbation Method (PPM). See Sweilam et al. [33], Xu et al. [37], and Shou et al. [30] for further details and uses of PPM to tackle non-linear boundary value problems arises in various fields.

In this article, the main objective is to estimate the approximate analytical expression of the non-linear partial differential equation for carbon nanotube (CNT) nanofluids influence on natural convection near an infinite vertical heated plate. With the use of the Kamal transform and the parameter perturbation approach, the model problem is analytically resolved. The results are then compared with the numerical solution using numerical inversion formula. The outcomes are displayed graphically to interpret the impacts of the physical parameters including Prandtl number and Grashof number. Also, the physical quantities of interest like Nusselt number is calculated and portrayed graphically.

2. Mathematical Formulation of the Problem

Let us consider a laminar flow across an endless vertical plate in the boundary layer of an incompressible viscous fluid with CNT's. The wall is maintained at a temperature $T_\infty + (T_W - T_\infty)f(t), T_\infty$ being the ambient temperature and $f(t)$ is a piecewise continuous function of the exponential order to infinity with $f(0) = 0$; the fluid is controlled by the thermal buoyancy force.

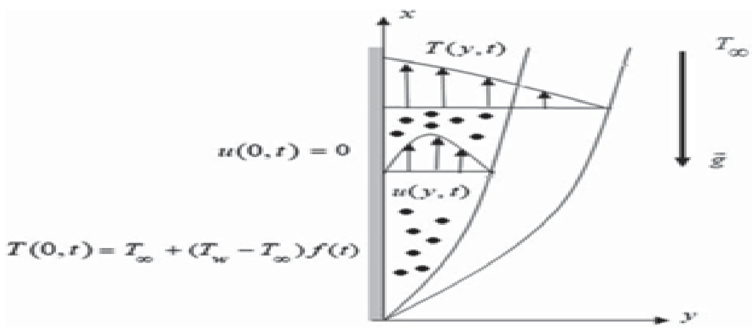

Fig.1 : A schematic illustration of the problem

In this study, a system based on Cartesian coordinates is used, as in Fig.1, where the $x-$axis is parallel to that same plate and the $y-$axis is taken normal to the plate. We simply assume that the velocity as well as the temperature fields is functions of y and t alone; so we find for fluid velocity $\bar{v}(y,t) = u(y,t)\bar{i}$ where$\bar{i}$ is the unite vector along $x-$ direction. Under consideration of the absence ofa pressure gradient depending on the path of the flow and Boussinesq's approximation, the governing set of partial differential equations reported in Awan et al. [10] and Hajizadeh et al. [19] is as follows:
The momentum equation has the form:

$$\rho_{nf}\frac{\partial U(y,t)}{\partial y} = \mu_{nf}\frac{\partial^2 U(y,t)}{\partial y^2} + g(\rho\beta)_{nf}[T(y,t) - T_\infty] \tag{1}$$

The energy balance equation is given by

$$(\rho c_p)_{nf}\frac{\partial T(y,t)}{\partial t} = -\frac{\partial q(y,t)}{\partial y} \tag{2}$$

The Fourier's law of thermal flux is given by

$$q(y,t) = -k_{nf}\frac{\partial T(y,t)}{\partial t} \tag{3}$$

where g is the gravitational acceleration and q is the thermal flux of the nanofluid. The thermo physical parameters of the considered nanofluid (Haq et al. [20, 21]) are given by ,

$$\rho_{nf} = (1-\phi)\rho_f + \phi\rho_{CNT}, \;\; (\rho c_p)_{nf} = (1-\phi)(\rho c_p)_f + \phi(\rho c_p)_{CNT}, \;\; \mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}}$$

$$\rho\beta_{nf} = (1-\phi)(\rho\beta_p)_f + \phi(\rho\beta)_{CNT}, \;\; \frac{k_{nf}}{k_f} = \frac{1-\phi+2\phi\left(\frac{k}{k_{nf}-k_f}\right)In\left(\frac{k_{CNT}+k_f}{2k_f}\right)}{1-\phi+2\phi\left(\frac{k_f}{k_{CNT}-k_f}\right)In\left(\frac{k_{CNT}+k_f}{2k_f}\right)} \tag{4}$$

where ρ_{nf}, μ_{nf} and $(c_p)_{nf}$ are the density, viscosity and the specific heat of the nanofluid. Also, the coefficient of nanofluid's thermal expansion at constant pressure and the nanofluid's thermal conductivity are expressed by the parameters β_{nf} and k_{nf} respectively and ϕ is a parameter of the volume fraction.
We consider here about the following initial and boundary conditions that are suitable for eqns. (1)-(3):

$$U(y,0) = 0, \;\; T(y,0) = T_\infty, \;\; y \geq 0 \tag{5}$$

$$U(0,t) = 0, \;\; T(0,t) = T_\infty + [T_W - T_\infty]f(t), \;\; t \geq 0 \tag{6}$$

$$U(y,t) \to 0, \;\; T(y,t) \to T_\infty \;\; as \;\; y \to \infty \tag{7}$$

We exhibit the successive dimensionless variables to create the non-dimensional problem:

$$y^* = \frac{v_0 y}{v_{nf}},\ U_{nf} = \frac{\mu_{nf}}{\rho_{nf}},\ t^* = \frac{v_0 t}{v_{nf}},\ U^* = \frac{U}{v_0},\ T^* = \frac{T - T_\infty}{T_W - T_\infty},\ q^* = \frac{q}{q_0},$$
$$q_0 = \frac{k_{nf}(T_W - T_\infty)v_0}{v_{nf}},\ Pr = \frac{(\mu c_p)_{nf}}{k_{nf}},\ Gr = \frac{g(v\beta)_{nf}(T_W - T_\infty)}{v_0^3} \quad (8)$$

In the preceding equations, v_{nf} denotes the kinematic viscosity, Pr is the Prandtl number, Gr is a Grashof number and $v_0 > 0$ is a characteristic velocity.
Substitute eqn. (8) into the eqns. (1)-(7); we obtain the dimensionless problem by neglecting the star notations.

$$\frac{\partial U(y,t)}{\partial t} = \frac{\partial^2 U(y,t)}{\partial t^2} + Gr\ T(y,t) \quad (9)$$
$$Pr\frac{\partial T(y,t)}{\partial t} = -\frac{\partial q(y,t)}{\partial y} \quad (10)$$
$$q(y,t) = -\frac{\partial T(y,t)}{\partial y} \quad (11)$$
$$U(y,0) = 0,\ T(y,0) = T_\infty,\ y \geq 0 \quad (12)$$
$$U(0,t) = 0,\ T(0,t) = f(t),\ t \geq 0 \quad (13)$$
$$U(y,t) \to 0,\ T(y,t) \to T_\infty\ \ as\ \ y \to \infty \quad (14)$$

3. Approximate Analytical Solution of a System of Partial Differential Equation using the Kamal Transform and PPM

Abdelilah Kamal et al. [1] created a new integral transform known as the Kamal transform. For the purpose of evaluating partial and ordinary differential equations in the time domain, it is developed from the traditional Fourier integral. One can use it to resolve both linear as well as non-linear differential equations. The application of the Kamal transformation is clearly described in Aggarwal et al. [3]. The basic definition and properties of Kamal transform are provided in Appendix-A.

In order to resolve simultaneous differential equations, the parameter perturbation method was initially proposed in 1999 by He et al. [22]. It is a sort of effective technique for resolving non-linear problems that can converge to a rough solution for a smooth simultaneous system. According to Roshan et al. [8], PPM may be easily applied to various non-linear systems and is used to simulate the non-linear elastic deformation of a beam. It has many applications in engineering and other scientific disciplines. The basic concept of parameter perturbation method is provided in Appendix-B.

The approximate analytical solutions for both the temperature and velocity profiles obtained using the Kamal transform and PPM are described below. The detailed derivations of approximate analytical solution for temperature and velocity are provided in Appendix-C.

3.1. Solution of Temperature Distribution

We get the transformed problem of the temperature profile by simply applying the Kamal transform technique to the eqns. (10), (11), $(13)_2$, $(14)_2$ and use the initial condition in eqn. $(12)_2$:

$$\frac{Pr}{v}\bar{T}(y,v) = -\frac{\partial \bar{q}(y,v)}{\partial y} \tag{15}$$

$$\bar{q}(y,v) = -\frac{\partial \bar{T}(y,v)}{\partial y} \tag{16}$$

$$\bar{T}(0,v) = G(v), \;\; \bar{T}(y,v) \to 0 \;\; as \;\; y \to \infty \tag{17}$$

By substituting eqn. (16) into eqn. (17) and rearrangement, we obtain the following (ordinary thermal transport) differential equation:

$$\frac{\partial^2 \bar{T}(y,v)}{\partial y^2} - \frac{Pr}{v}\bar{T}(y,v) = 0 \tag{18}$$

The solution of eqn. (18) under the conditions of eqn. (17) using PPM is given by

$$\bar{T}(y,v) = G(v)\; e^{-y\sqrt{\frac{Pr}{v}}} \tag{19}$$

Applying the inverse Kamal transform to the eqn. (19), we get the transformed temperature is as follows:

$$T(y,t) = f'(t) * erfc\left[\frac{y\sqrt{Pr}}{2\sqrt{t}}\right] \tag{20}$$

where $f'(t) = \frac{df(t)}{dt}$ and $erfc(\cdot)$ is the complementary Gauss error function.

3.2. Solution of Fluid Velocity

We get the transformed problem of the velocity profile by simply applying the Kamal transform technique to the eqns. (9), $(13)_1, (14)_1$ and use the initial condition in eqn. $(12)_1$:

$$\frac{1}{v}\bar{U}(y,v) = \frac{\partial^2 \bar{U}(y,v)}{\partial y * 2} + Gr\;\bar{T}(y,t) \tag{21}$$

$$\bar{U}(y,v) \to 0 \;\; as \;\; y \to \infty \tag{22}$$

Substitute eqn. (19) into eqn. (21) and rearranging, we get the following (ordinary thermal transport) differential equation:

$$\frac{\partial^2 \bar{U}(y,v)}{\partial y*2} - \frac{1}{v}\bar{U}(y,v) = -Gr\left[G(v)\ e^{-y\sqrt{\frac{Pr}{v}}}\right] \tag{23}$$

The solution of eqn. (21) under the conditions of eqn. (22) using PPM is given by the following equation:

$$\bar{U}(y,v) = \frac{v}{1-Pr}Gr.G(v)\left[e^{-y\sqrt{\frac{Pr}{v}}} - e^{-y\sqrt{\frac{1}{v}}}\right] \tag{24}$$

Applying the inverse Kamal transform to the eqn. (24), we obtain the transformed velocity.

$$U(y,t) = \frac{Gr}{1-Pr}f(t)*\left(erfc\left[\frac{y\sqrt{Pr}}{2\sqrt{t}}\right] - erfc\left[\frac{y}{2\sqrt{t}}\right]\right)\ :\ Pr \neq 1 \tag{25}$$

3.3. Particular case

In the particular case $Pr = 1$, the temperature in the Kamal domain is

$$\bar{T}(y,v) = G(v)\ e^{-y\sqrt{\frac{1}{v}}}\ = \left(\frac{1}{v}\right)G(v)\frac{e^{-y\sqrt{\frac{1}{v}}}}{\left(\frac{1}{v}\right)} \tag{26}$$

The effective inverse Kamal transform is calculated by

$$T(y,t) = f'(t)*erfc\left[\frac{y}{2\sqrt{t}}\right] \tag{27}$$

In this instance, the modified velocity has a more straightforward form.

$$\bar{U}(y,v) = \frac{y}{2}Gr.G(v)\left[\frac{e^{-y\sqrt{\frac{1}{v}}}}{\left(\frac{1}{v}\right)}\right] \tag{28}$$

With the effective inverse Kamal transform is given by

$$U(y,t) = \frac{y\ Gr}{2}f(t)*\left(\frac{e^{(-\frac{y}{4t})}}{\sqrt{\pi t}}\right) \tag{29}$$

3.4. Nusselt number

The following is the mathematical expression of the heat transfer rate:

$$Nu = \left.\frac{\partial T(y,t)}{\partial y}\right|_{y=0} = -\left\{\left.\frac{\partial \bar{T}(y,v)}{\partial y}\right|_{y=0}\right\} = K^{-1}\left\{G(v)\sqrt{\frac{Pr}{v}}\right\} \tag{30}$$

With the effective inverse Kamal transform is given by

$$Nu = f'(t) * \sqrt{\frac{Pr}{\pi t}} \tag{31}$$

4. Numerical Inversion Formula

For the confirmation of our work, we use the Stehfest's formula presented by Stehfest et al. [31, 32] and Jacquot et al. [24] regarding numerical inversion associated with the Laplace transform method to find the numerical solutions of temperature and velocity fields.

Stehfest's formula is expressed as:

$$T(y,t) \approx \frac{In(2)}{t}\sum_{j=1}^{N} k_j \bar{T}\left(\epsilon, j\frac{In(2)}{t}\right) \tag{32}$$

$$U(y,t) \approx \frac{In(2)}{t}\sum_{j=1}^{N} k_j \bar{U}\left(r, j\frac{In(2)}{t}\right) \tag{33}$$

where $k_j = (-1)^{j+\frac{N}{2}} \sum_{i=[\frac{j+1}{2}]}^{min\left(j,\frac{N}{2}\right)} \frac{i^{\frac{N}{2}}(2n)!}{\left(\frac{N}{2}-i\right)!i!(i-1)!(2i-j)!}$, N is even and $[r]$ stands for the integer value function, also known as the bracket function. Here we take $N = 2m =$ an even number. The results are computed numerically using MAPLE software.

4. Results and Discussion

In this portion, the effects of physical insights on temperature distribution and velocity field were addressed. For the analytical results, we consider the function$f(t) = t,\ t > 0$, for the heated wall. Fig. 1 shows a schematic diagram of the issue.

For temperature: Figs. 2, 3, and 4 represent the assessment of the non-dimensional temperature distribution with the dimensionless distance. Further from Fig. 2, it shows that the temperature level at $y = 2.4$ becomes steady state. In Fig. 3, it is displayed that raising the values of time t, causes the temperature to increase for some fixed value of Pr and the change of the temperature distribution to be exhibited. As seen in Fig. 4, by increasing the values of Pr, the temperature is raised for some fixed value of t and a slight variation of the temperature is displayed.

For velocity: Figs. 5, 6, 7 and 8 represent the dimensionless distance versus the dimensionless fluid velocity. From Fig. 6, it can be seen that for some fixed values of Pr and Gr, the velocity profile increases as the time value rises. According to Fig. 7, increasing the values of Pr results in a drop in the velocity profile for some fixed values of t and Gr. From Fig. 6, it is evident that for some fixed values of t and Pr, the velocity profile is decreased by decreasing the value of Gr.

For Nusselt number: Figs. 9 and 10 depict the Nusselt number Nu using eqn. (31). Fig. 9 illustrates that by increasing the amount of Prandtl number, Nu raises. It is clear from Fig. 10 that the Nusselt number increases by raising t. From these figures, it is evident that time and Prandtl number has a great impact on Nu.

For 3D plots: Figs. 11 to 13 portray the 3D view of dimensionless temperature and velocity against time t and distance y respectively. Fig. 11 is a 3D plot of non-dimensional temperature using eqn. (20) for distinct amounts of Pr. Fig. 12 is a 3D graph of dimensionless velocity using eqn. (25) for different amounts of Gr. Fig. 13 is a 3D view of non-dimensional velocity using eqn. (25) for various amounts of Pr.

For table: From Table 1 the average error percentage was 0.03 for the temperature profile and 0.18 for the Velocity profile. It is obvious that our findings reach a remarkable agreement as compared to numerical results.

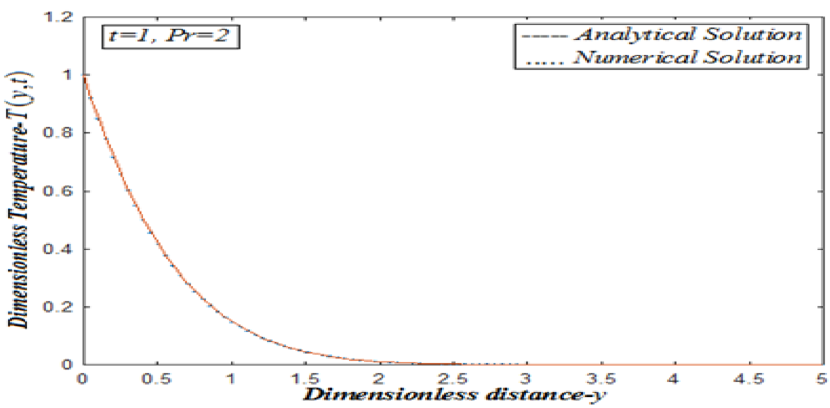

Fig. 2: Dimensionless fluid temperature $T(y,t)$ with dimensionless distance y. For certain specified values of the non-dimensional parameter t and Pr , the curves are constructed using the eqn. (20).

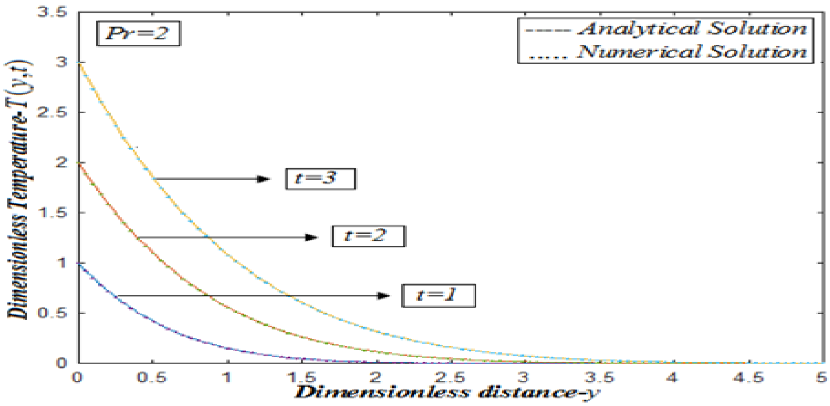

Fig. 3: Dimensionless fluid temperature $T(y,t)$ with dimensionless distance y. The curves are drawn using the eqn. (20) for varying values of the non-dimensional parameter t and in certain specified value of the non-dimensional parameter Pr.

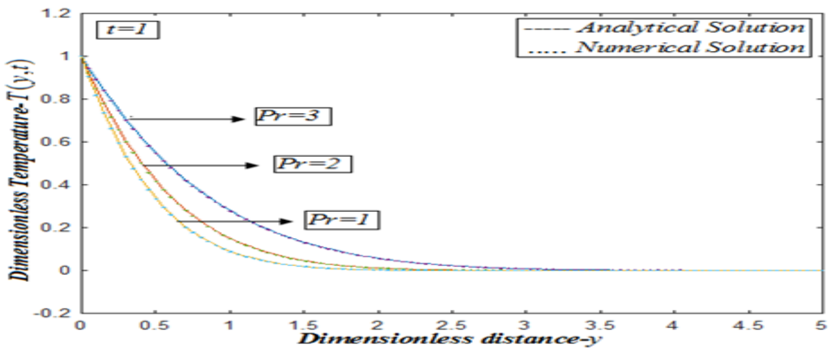

Fig. 4: Dimensionless fluid temperature $T(y,t)$ with dimensionless distance y. The curves are drawn using the eqn. (20) for varying values of the non-dimensional parameter Pr and in certain specified value of the non-dimensional parameter t.

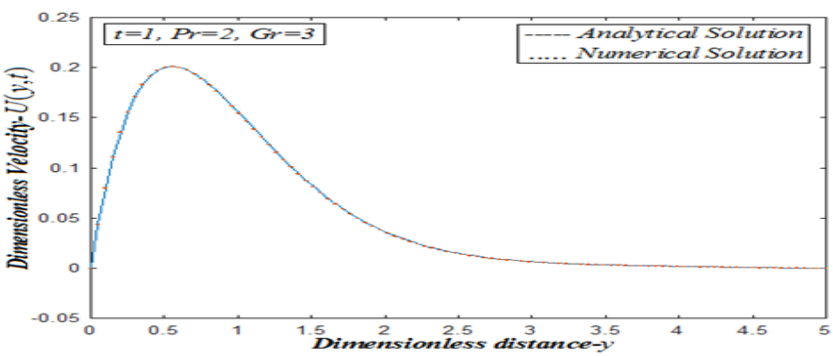

Fig. 5: Dimensionless fluid velocity $U(y,t)$ with dimensionless distance y. The curves are sketched using the eqn. (25) for certain specified values of the non-dimensional parameter t,Pr and Gr.

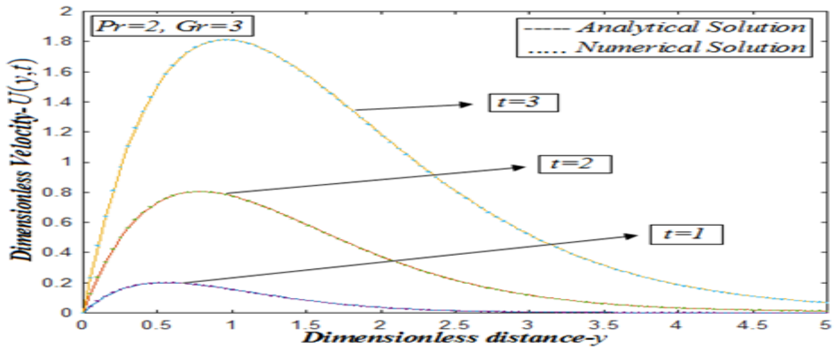

Fig. 6: Dimensionless fluid velocity $U(y,t)$ with dimensionless distance y. The curves are sketched using the eqn. (25) for varying values of the non-dimensional parameter t and in certain specified values of the non-dimensional parameter Pr and Gr.

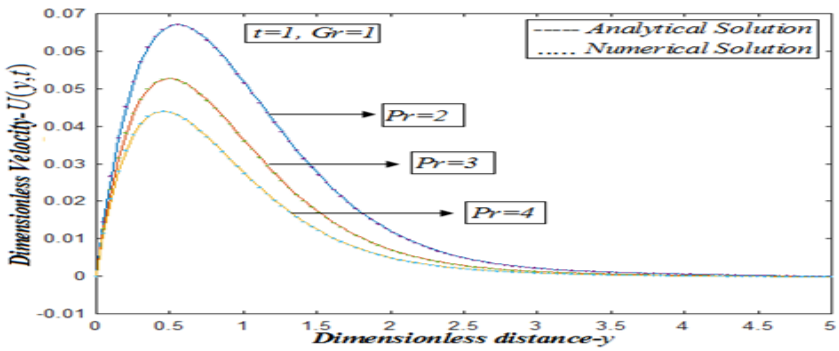

Fig. 7: Dimensionless fluid velocity $U(y,t)$ with dimensionless distance y. The lines are drawn using the eqn. (25) for varying values of the non-dimensional parameter Pr and in certain specified values of the non-dimensional parameter t and Gr.

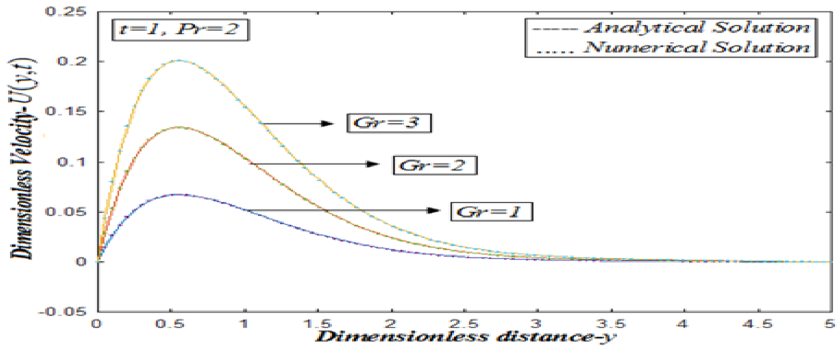

Fig. 8: Dimensionless fluid velocity $U(y, t)$ with dimensionless distance$y$. The plots are sketched using the eqn. (25) for varying values of the non-dimensional parameter Gr and in certain specified values of the non-dimensional parameter t and Pr.

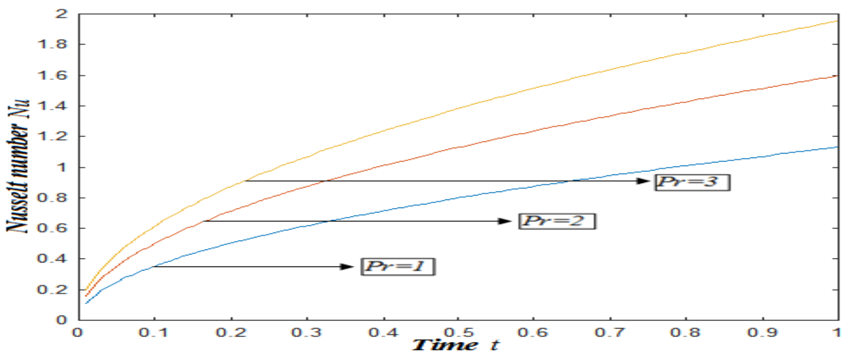

Fig. 9: Nusselt number Nu with time t. The curves are sketched using eqn. (31) for numerous amounts of the Prandtl number Pr.

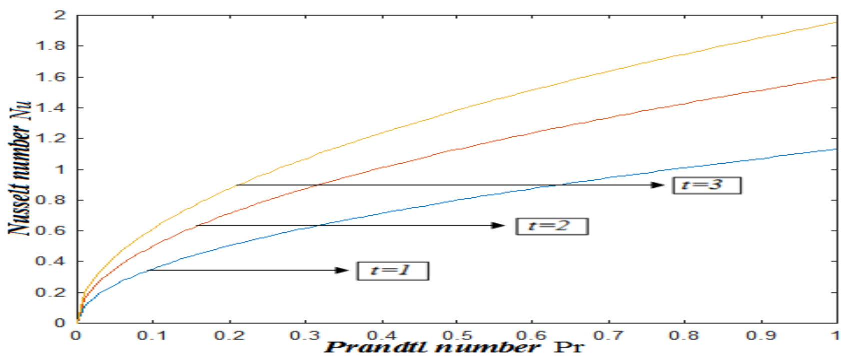

Fig. 10: Nusselt number Nu with Prandtl number Pr. The lines are sketched using the eqn. (31) for varying amounts of time t.

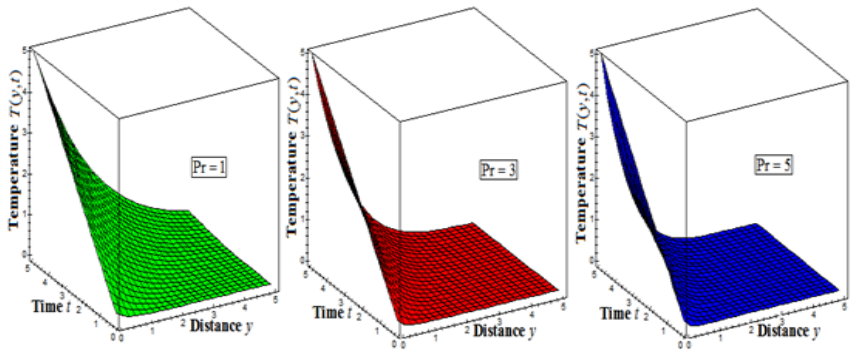

Fig. 11: 3D plot of dimensionless temperature for numerous amount of Pr.

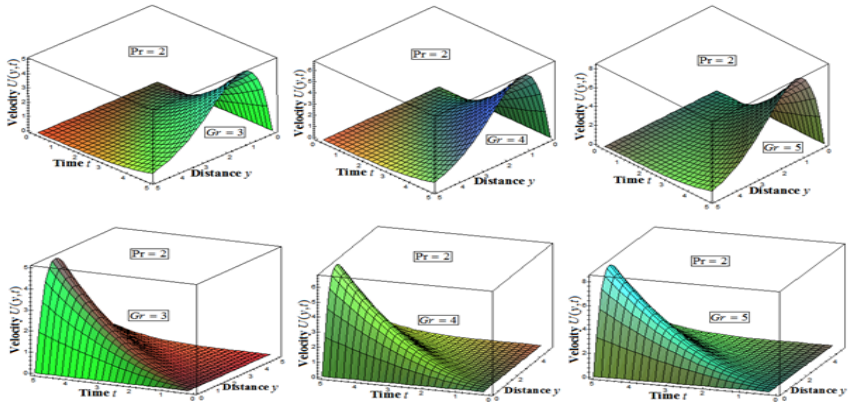

Fig. 12: 3D view of non-dimensional velocity for numerous amounts of Gr.

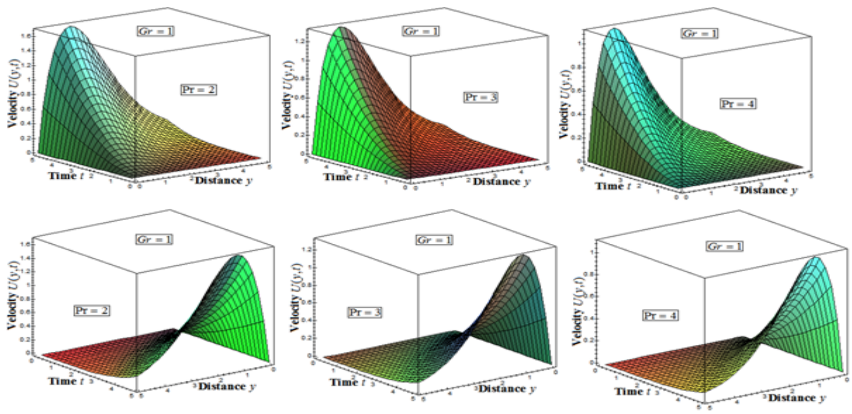

Fig. 13: 3D illustration of non-dimensional velocity for distinct amounts of Pr.

Table 1: Comparison regarding the numerical solution using the eqns. (32) and (33) and the analytical solution using the eqns. (20) and (25) for $t = 1$, $Pr = 2$ *and* $Gr = 3$.

ρ	Dimensionless distributions					
	Temperature- $T(y,t)$			Velocity- $U(y,t)$		
	Semi-analytical solution	Numerical solution	Error %	Semi-analytical solution	Numerical solution	Error %
0	1	1.00015	0.015	0	0	0
0.1	0.85015	0.85030	0.01764	0.07947	0.07955	0.10067
0.2	0.71872	0.71887	0.02087	0.13463	0.13478	0.11142
0.3	0.60412	0.60426	0.02317	0.17041	0.17063	0.12910
0.4	0.50480	0.50494	0.02773	0.19100	0.19128	0.14659
0.5	0.41927	0.41941	0.03339	0.19994	0.20028	0.17005
0.6	0.34609	0.34623	0.04045	0.20015	0.20054	0.19485
0.7	0.28389	0.28401	0.04226	0.19404	0.19449	0.23191
0.8	0.23137	0.23148	0.04754	0.18356	0.18406	0.27239
0.9	0.18734	0.18743	0.04804	0.17027	0.17081	0.31714
1	0.15067	0.15074	0.04645	0.15538	0.15597	0.37971
Average absolute error %			**0.03296**			**0.18671**

6. Conclusions

In this paper, we have looked at an analytical representation of the effects of carbon nanotubes (CNT's) nanofluid under natural convection close to an infinite vertical heating plate. The characteristics of nanofluid flow are derived by solving the governing partial differential equations analytically with the Kamal transform and PPM. The particular cases are also discussed. The graphical results are interlined to analyze the effects of different thermo physical factors on temperature and velocity profiles. The findings lead to the following conclusions:

- Temperature increases as time and Prandtl number increases.
- Velocity rises with increasing values of time.
- Velocity falls with increasing values of the Prandtl number.
- As the values of the Grashof number decrease, the velocity decreases.
- Nusselt number enhances as the amount of time and Prandtl number increase.

Appendices

Appendix A: Basic definition and properties of the Kamal transform

The Kamal Transform was first presented by Abdeliah Kamal. It is a mathematical technique used to solve time-domain partial and ordinary differential equations. We take into account the Kamal transform for exponential-order functions in the collection A by

$$A = \left\{ f(t) : \exists M,\ \lambda_1,\ \lambda_2 > 0,\ \Big| f(t) \Big| < M e^{\frac{|t|}{\lambda_i}},\ \ if\ t \in (-1)^i \times [0, \infty),\ i = 1, 2 \right\} \tag{A.1}$$

where M is a constant but finite number,$\lambda_1,\ \lambda_2$ are finite or infinite, the Kamal transform of $f(t)$ may be described as an integral equation:

$$K[f(t)] = G(v) = \int_0^\infty f(t) e^{\frac{-t}{v}} dt,\ \ t \geq 0,\ \ \lambda_1 \leq v \leq \lambda_2 \tag{A.2}$$

In the argument of the function f the variable t is factored by the variable v in the Kamal Transform and $K(\cdot)$ denotes the Kamal Transform operator.

Kamal transform of elementary functions:

Let $f(t)$ be any piecewise continuous function with exponential order for $t \geq 0$.

$i)\ \ f(t) = 1 \implies K[1] = v$

$ii)\ f(t) = t \implies K[t] = v^2$

$iii)\ f(t) = t^n \implies K[t^n] = n! v^{n+1},\ \ n \geq 0$

$iv)\ f(t) = e^{at} \implies K[e^{at}] = \frac{v}{1-av}$

$v)\ f(t) = e^{-at} \implies K[e^{at}] = \frac{v}{1+av}$

Kamal transform of derivatives:

Theorem-1:

Let $G(v)$ be the Kamal transform of $f(t)$ then

$i)\ \ K[f'(t)] = \frac{1}{v} G(v) - f(0)$

$ii)\ \ K[f''(t)] = \frac{1}{v^2} G(v) - \frac{1}{v} f(0) - f'(0)$

$iii)\ \ K[f^n(t)] = \frac{1}{v^n} G(v) - \sum_{k=0}^{n-1} v_{k-n+1} f^{(k)}(0)$

Kamal transform of partial derivatives:

$i)\ \ K\left[\frac{\partial f(x,t)}{\partial t}\right] = \frac{1}{v} G(x, v) - f(x, 0)$

$ii)\ K\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right] = \frac{1}{v^2} G(x, v) - \frac{1}{v} f(x, 0) - \frac{\partial f(x,0)}{\partial t}$

$iii)\ K\left[\frac{\partial f(x,t)}{\partial x}\right] = \frac{d}{dx}(G(x, v))$

$iv)\ K\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}(G(x, v))$

$v)\ \ K\left[\frac{\partial^n f(x,t)}{\partial x^n}\right] = \frac{d^n}{dx^n}(G(x, v))$

Convolution of two functions:
Convolution of two functions $F(t)$ and $H(t)$ is denoted by $F(t)*H(t)$ and is defined by
$F(t)*H(t)=\int_0^t F(x)H(t-x)dx=\int_0^t H(x)F(t-x)dx$
Convolution theorem for Kamal transform:
If $K[F(t)]=G(v)$ and $K[H(t)]=I(v)$ then $k[F(t)*H(t)]=k[F(t)]K[H(t)]=g(v)I(v)$
Inverse Kamal transform:
If $K[F(t)]=G(v)$ then $F(t)$ is called the inverse Kamal transform of $G(v)$ and mathematically it is defined as $F(t)=K^{-1}[G(v)]$
Kamal transform of error function and complementary error function:
The error and complementry error functions are defined as
$erf(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ and $erfc(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$
The Kamal transform of error and complementary error functions are defined as
$K[erf(\sqrt{t})]=\frac{v^{3/2}}{\sqrt{(1+v)}}$ and
$K[erfc(\sqrt{t})]=K[1-erf(\sqrt{t})]=K[1]-K[erf(\sqrt{t})]=v-\frac{v^{3/2}}{\sqrt{(1+v)}}$
Appendix B: Basic concept of parameter perturbation method (PPM)
It is a method of perturbation where the coefficients of an expression are specified in terms of the power of an artificial parameter that may be applied to non-linear systems.
Consider a general non-linear equation is as follows:

$$L(u)+N(u)=f(x),\ \ x\in R^d \tag{B.1}$$

where L is a linear portion, N is a non-linear part and $f(x)$ is a known analytical function.
Correspondingly, expanding parameters are incorporated via linear transformation:

$$u(x)=\epsilon v(x)+b \tag{B.2}$$

where ϵ is the perturbation variable; in order to eliminate the secular term from the equation, we may get the unknown constant variable b by substituting the eqn. (B.2) into the eqn.(B.1)
The solution is then extended to take the following form:

$$v=\sum_{i=0}^{n}\epsilon^i v_i \ = \ v_0+\epsilon v_1+\epsilon^2 v_2+... \tag{B.3}$$

where ϵ is an artificial book keeping parameter. We keep $v_0(0) = v(0)$and $\sum_i v_i(0) = 0$

Appendix C: Approximate analytical solution using the Kamal transform and the Parameter perturbation method (PPM)

To find the viable solution of transformed eqns. (9), (10) and (11) under the initial and boundary conditions in eqns. (12), (13) and (14), we use the new technique namely the Kamal transform.

By applying the Kamal transform to the eqns. (9)-(14), we get

$$\left.\begin{aligned}
&K\left[\frac{\partial U(y,t)}{\partial t}\right] = K\left[\frac{\partial^2 U(y,t)}{\partial y^2} + Gr\ T(y,t)\right]\\
&K\left[Pr\frac{\partial T(y,t)}{\partial t}\right] = K\left[-\frac{\partial q}{\partial y}\right]\\
&K[q(y,t)] = K\left[-\frac{\partial T(y,t)}{\partial y}\right]\\
&K[U(y,0)] = 0\ \ K[T(y,0)] = 0\ \ y \geq 0\\
&K[U(0,t)] = 0\ \ K[T(0,t)] = K[f(t)]\ \ v \geq 0\\
&K[U(y,t)] \rightarrow 0\ \ K[T(0,t)] \rightarrow 0\ \ as\ \ y \rightarrow \infty
\end{aligned}\right\} \tag{C.1}$$

$$\left.\begin{aligned}
\Longrightarrow\ &\frac{1}{v}\bar{U}(y,v) - \bar{U}(y,0) = \frac{\partial^2 \bar{U}(y,v)}{\partial y^2} + Gr\ \bar{T}(y,v)\\
&Pr\left[\frac{1}{v}\bar{T}(y,v) - \bar{T}(y,0)\right] = -\frac{\partial \bar{q}(y,v)}{\partial y}\\
&\bar{q}(y,v) = -\frac{\partial \bar{T}(y,v)}{\partial y}\\
&\bar{U}(y,0) = 0\ \ \bar{T}(y,0) = 0\ \ y \geq 0\\
&\bar{U}(0,v) = 0,\ \ \bar{T}(0,v) = G(v),\ \ v \geq 0\\
&\bar{U}(y,v) \rightarrow 0,\ \ \bar{T}(y,v) \rightarrow 0\ \ as\ \ y \rightarrow \infty
\end{aligned}\right\} \tag{C.2}$$

$$\left.\begin{aligned}
\Longrightarrow\ &\frac{1}{v}\bar{U}(y,v) = \frac{\partial^2 \bar{U}(y,v)}{\partial y^2} + Gr\ \bar{T}(y,v)\\
&\frac{Pr}{v}\bar{T}(y,v) = \frac{\partial^2 \bar{T}(y,v)}{\partial y^2}\\
&\bar{U}(0,v) = 0,\ \ \bar{T}(0,v) = G(v),\ \ v \geq 0\\
&\bar{U}(y,v) \rightarrow 0,\ \ \bar{T}(y,v) \rightarrow 0\ \ as\ \ y \rightarrow \infty
\end{aligned}\right\} \tag{C.3}$$

To solve eqn. (C.3) by means of PPM, first we rearrange the terms in eqn. (C.3):

$$\left(\frac{\partial^2 \bar{U}(y,v)}{\partial y^2}\right) - \left(\frac{1}{v}\bar{U}(y,v)\right) + (Gr\ \bar{T}(y,v)) = 0 \tag{C.4}$$

with

$$\bar{U}(0,v) = 0,\ \ v \geq 0\ \ and\ \ \bar{U}(y,v) \to 0\ \ as\ \ y \to \infty \tag{C.5}$$

$$\left(\frac{\partial^2 \bar{T}(y,v)}{\partial y^2}\right) - \left(\frac{Pr}{v}\bar{T}(y,v)\right) = 0 \tag{C.6}$$

with

$$\bar{T}(0,v) = G(v),\ \ v \geq 0\ \ and\ \ \bar{T}(y,v) \to 0\ \ as\ \ y \to \infty \tag{C.7}$$

Parameter perturbation method (PPM) to find the solution of the temperature profile:

To solve the second-order differential eqn. (C.6) with the boundary condition in eqn. (C.7) by using the parameter perturbation method.

We introduced an expanding variable via a linear transformation:

$$\bar{T}(y,v) = \epsilon \bar{S}(y,v) + b \tag{C.8}$$

where ϵ is the perturbation variable and b is the unknown steady(constant) parameter.

Substituting eqn. (C.8) into eqn. (C.6), we get

$$\left(\frac{\partial^2}{\partial y^2}(\epsilon \bar{S}(y,v) + b)\right) - \left(\frac{Pr}{v}(\epsilon \bar{S}(y,v) + b)\right) = 0$$

$$\implies \epsilon \frac{\partial^2 \bar{S}(y,v)}{\partial y^2} - \epsilon \frac{Pr}{v}\bar{S}(y,v) - \frac{Pr}{v}b = 0,$$

$$\implies \frac{\partial^2 \bar{S}(y,v)}{\partial y^2} - \frac{Pr}{v}\bar{S}(y,v) - \frac{Pr}{\epsilon v}b = 0 \tag{C.9}$$

The solution is elaborated in the following manner:

$$\bar{S} = \sum_{i=0}^{n} \epsilon^i \bar{S}_i = \bar{S}_0 + \epsilon \bar{S}_1 + \epsilon^2 \bar{S}_2 + ... \tag{C.10}$$

Substituting eqn. (C.10) into eqn. (C.9), we get

$$\frac{\partial^2}{\partial y^2}(\bar{S}_0 + \epsilon \bar{S}_1 + \epsilon^2 \bar{S}_2 + ...) - \frac{Pr}{v}(\bar{S}_0 + \epsilon \bar{S}_1 + \epsilon^2 \bar{S}_2 + ...) - \frac{Pr}{\epsilon v}b = 0 \tag{C.11}$$

Equating the powers of ϵ -terms in eqn. (C.11), we get

$$\epsilon^0 \ : \ \frac{\partial^2 \bar{S}_0}{\partial y^2} - \frac{Pr}{v}\bar{S}_0 - \frac{Pr}{\epsilon v} b = 0 \tag{C.12}$$

$$\epsilon^1 \ : \ \frac{\partial^2 \bar{S}_1}{\partial y^2} - \frac{Pr}{v}\bar{S}_1 = 0 \tag{C.13}$$

$$\vdots$$

On solving the eqn. (C.12), we obtain the solution

$$\bar{S}_0(y, v) = Ae^{y\sqrt{\frac{Pr}{v}}} + Be^{-y\sqrt{\frac{Pr}{v}}} - \frac{b}{\epsilon} \tag{C.14}$$

Substituting eqn. (C.14) into eqn. (C.8), we obtain

$$\bar{T}_0(y, v) = \epsilon \left(Ae^{y\sqrt{\frac{Pr}{v}}} + Be^{-y\sqrt{\frac{Pr}{v}}} - \frac{b}{\epsilon} \right) + b \tag{C.15}$$

Utilizing the boundary conditions in eqn. (C.7), we get the solution:

$$\bar{T}_0(y, v) = G(v)e^{-y\sqrt{\frac{Pr}{v}}}$$

The solution of the temperature field is obtained by

$$\bar{T}(y, v) = G(v)e^{-y\sqrt{\frac{Pr}{v}}} \tag{C.16}$$

Parameter perturbation approach to find the solution of the velocity field:

To solve the second-order differential eqn. (C.4) with the boundary condition in eqn. (C.5) by using the parameter perturbation method.
Substituting eqn. (C.16) into eqn. (C.4), we get

$$\left(\frac{\partial^2 \bar{U}(y, v)}{\partial y^2} \right) - \left(\frac{1}{v}\bar{U}(y, v) \right) + \left(Gr \left(G(v)e^{-y\sqrt{\frac{Pr}{v}}} \right) \right) = 0 \tag{C.17}$$

We introduced an expanding variable via a linear transformation:

$$\bar{U}(y, v) = \epsilon \bar{R}(y, v) + b \tag{C.18}$$

where ϵ is the perturbation variable and b is the unknown steady(constant) parameter.
Substituting eqn. (C.18) into eqn. (C.17), we get

$$\left(\frac{\partial^2}{\partial y^2}(\epsilon\bar{R}(y,v)+b)\right)-\left(\frac{1}{v}(\epsilon\bar{R}(y,v)+b)\right)+\left(Gr\left(G(v)e^{-y\sqrt{\frac{Pr}{v}}}\right)\right)=0 \quad (C.19)$$

$$\implies \epsilon\frac{\partial^2\bar{R}(y,v)}{\partial y^2}-\epsilon\frac{1}{v}\bar{R}(y,v)-\frac{b}{v}+Gr\left(G(v)e^{-y\sqrt{\frac{Pr}{v}}}\right)=0$$

$$\implies \frac{\partial^2\bar{R}(y,v)}{\partial y^2}-\frac{1}{v}\bar{R}(y,v)-\frac{b}{\epsilon v}+\frac{1}{\epsilon}Gr\left(G(v)e^{-y\sqrt{\frac{Pr}{v}}}\right)=0 \quad (C.20)$$

The solution is elaborated in the following manner:

$$\bar{R}=\sum_{i=0}^{n}\epsilon^i\bar{R}_i=\bar{R}_0+\epsilon\bar{R}_1+\epsilon^2\bar{R}_2+... \quad (C.21)$$

Substituting eqn. (C.21) into eqn. (C.20), we get

$$\frac{\partial^2}{\partial y^2}(\bar{R}_0+\epsilon\bar{R}_1+\epsilon^2\bar{R}_2+...)-\frac{1}{v}(\bar{R}_0+\epsilon\bar{R}_1+\epsilon^2\bar{R}_2+...)-\frac{b}{\epsilon v}+\frac{1}{\epsilon}Gr\left(G(v)e^{-y\sqrt{\frac{Pr}{v}}}\right)=0 \quad (C.22)$$

Equating the powers of ϵ-terms in eqn. (C.22), we get

$$\epsilon^0 \ : \ \frac{\partial^2\bar{R}_0}{\partial y^2}-\frac{1}{v}\bar{R}_0-\frac{b}{\epsilon v}+\frac{1}{\epsilon}Gr\left(G(v)e^{-y\sqrt{\frac{Pr}{v}}}\right)=0 \quad (C.23)$$

$$\epsilon^1 \ : \ \frac{\partial^2\bar{R}_1}{\partial y^2}-\frac{1}{v}\bar{R}_1=0 \quad (C.24)$$

$$\vdots$$

On solving the eqn. (C.23), we obtain the solution

$$\bar{R}_0(y,v)=Ae^{y\sqrt{\frac{1}{v}}}+Be^{-y\sqrt{\frac{1}{v}}}-\frac{b}{\epsilon}+\frac{v\ Gr}{(1-Pr)\epsilon}G(v)e^{-y\frac{Pr}{v}} \quad (C.25)$$

Substituting eqn. (C.25) into eqn. (C.18), we obtain

$$\bar{U_0}(y,v)=\epsilon\left(Ae^{y\sqrt{\frac{1}{v}}}+Be^{-y\sqrt{\frac{1}{v}}}-\frac{b}{\epsilon}+\frac{v\ Gr}{(1-Pr)\epsilon}G(v)e^{-y\frac{Pr}{v}}\right)+b \quad (C.26)$$

Utilizing the boundary conditions in eqn. (C.5), we get the solution:

$$\bar{U_0}(y,v) = \frac{v}{(1-Pr)} Gr.G(v) \left[e^{-y\frac{Pr}{v}} - e^{-y\frac{1}{v}} \right]$$

The solution of the velocity field is obtained by

$$\bar{U}(y,v) = \frac{v}{(1-Pr)} Gr.G(v) \left[e^{-y\frac{Pr}{v}} - e^{-y\frac{1}{v}} \right] \tag{C.27}$$

Finally, by means of parameter perturbation technique, we find the solution of eqns. (C.4) and (C.6),

$$\left.\begin{aligned} \bar{T}(y,v) &= G(v) e^{-y\sqrt{\frac{Pr}{v}}} \\ \bar{U}(y,v) &= \frac{v}{(1-Pr)} Gr.G(v) \left[e^{-y\frac{Pr}{v}} - e^{-y\frac{1}{v}} \right] \end{aligned}\right\} \tag{C.28}$$

By applying the inverse Kamal transform to the eqn. (C.28), we get

$$\left.\begin{aligned} K^{-1}\left[\bar{T}(y,v)\right] &= K^{-1}\left[G(v) e^{-y\sqrt{\frac{Pr}{v}}} \right] \\ K^{-1}\left[\bar{U}(y,v)\right] &= K^{-1}\left[\frac{v}{(1-Pr)} Gr.G(v) \left(e^{-y\frac{Pr}{v}} - e^{-y\frac{1}{v}} \right) \right] \end{aligned}\right\} \tag{C.29}$$

$$\left.\begin{aligned} \implies T(y,t) &= K^{-1}\left[G(v) \left(\frac{1}{v}\right) \frac{e^{-y\frac{Pr}{v}}}{\left(\frac{1}{v}\right)} \right] \\ U(y,t) &= K^{-1}\left[\frac{Gr}{(1-Pr)} G(v) \left(\frac{e^{-y\frac{Pr}{v}}}{\left(\frac{1}{v}\right)} - \frac{e^{-y\frac{1}{v}}}{\left(\frac{1}{v}\right)} \right) \right] \end{aligned}\right\} \tag{C.30}$$

The approximate analytical solution of the temperature and velocity fields is obtained

$$\left.\begin{aligned} T(y,t) &= f'(t) * erfc\left[\frac{y\sqrt{Pr}}{2\sqrt{t}} \right] \\ U(y,t) &= \frac{Gr}{1-Pr} f(t) * \left(erfc\left[\frac{y\sqrt{Pr}}{2\sqrt{t}} \right] - erfc\left[\frac{y}{2\sqrt{t}} \right] \right) \quad Pr \neq 1 \end{aligned}\right\} \tag{C.31}$$

Appendix D: Nomenclature

Symbol	Meaning
g	Gravitational acceleration
q	Thermal flux of the nanofluid
x *and* y	Cartesian coordinates
v_0	Characteristic velocity
ρ_{nf}	Density of the nanofluid
μ_{nf}	Viscosity of the nanofluid
p	Fluid pressure
$(c_p)_{nf}$	Specific heat of the nanofluid
β_{nf}	Thermal expansion coefficient of nanofluid at constant pressure
k_{nf}	Thermal conductivity of the nanofluid
π	Volume fraction parameter
v_{nf}	Kinematic viscosity
T	Temperature
T_∞	Ambient temperature
T_W	Temperature of the sheet
$f(t)$	Piecewise continuous function
Pr	Prandtl number
Gr	Grashof number
Nu	Nusselt number

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References

[1] Abdeliah, K., and Hassan, S., The new integral transform Kamal Transform, Advances in theoretical and applied mathematics, 11(4) (2016), 451-458.

[2] Abdeliah, K., and Hassan, S., The use of Kamal Transform for solving partial differential equations, Advances in theoretical and applied mathematics, 12(1) (2017), 7-13.

[3] Aggarwal, S., and Sharma, S. D., Application of Kamal transform for solving Abel's integral equation, Global Journal of Engineering Science and Researches, 6(3) (2019), 82-90.

[4] Aggarwal, S., and Singh, G. P., Kamal Transform of Error Function, Journal of Applied Sciences and Computations, 6(5) (2019), 2223-2235.

[5] Ahmed, N., Shah, N. A., Ahmad, B., Shah, S. I. A. S., Ulhaq and Gorji,M. R., Transient MHD convective flow of fractional nanofluid between vertical plates, J. Appl. Comput. Mech., 5(4) (2019), 592-602.

[6] Ahmed, N., Shah, N. A., and Vieru, D., Natural convection with damped thermal flux in a vertical circular cylinder, Chin. J. Physiol., 56(2) (2018), 630-644.

[7] Ahmed, N., Vieru, D., Fetecau, C., and Shah, N. A., Convective flows of generalized time-nonlocal nanofluids through a vertical rectangular channel, Phys. Fluids., 30(5) (2018).

[8] Alireza Roshan, Masoumeh Ahmadi and Davood Domiri Ganji, Variational Iteration Method (VIM) and Parameter Perturbation Method (PPM) for the solution of nonlinear differential equation of beam elastic deformation, Southest Asian Bulletin of Mathematics, 4(6) (2022), 35-44.

[9] Assael, M. J., Chen, C. F., MetaxaI, I., and Wakeham, W. A., Thermal conductivity of suspensions of carbon nanotubes in water, Int. J. Thermophys., (25) (2007), 971-985.

[10] Awan, A. U., Shah, N. A., Ahmed, N., Ali, Q., and Riaz, S., Analysis of free convection flow of viscous fluid with damped thermal and mass fluxes, Chin. J. Physiol., (60) (2019), 98-106.

[11] Choi, S. U. S., Zhang, Z. G., Yu, W., Lockwood, F. E., and Grulke, E. A., Anomalous thermal conductivity enhancement in nanotube suspensions, Appl. Phys. Lett., 79(14) (2001), 2252-2254.

[12] Das, S. K., Putra, N., Thiesen, P., and Roetzel, W., Temperature dependence of thermal conductivity enhancement for nanofluids, J. Heat. Transfer., 125(4) (2003), 567-574.

[13] Ding, Y., Alias, H., Wen, D., and Williams, R. A., Heat transfer of aqueous suspensions of carbon nanotubes (CNT nanofluids), Int. J. Heat. Mass. Tran., 49(1-2) (2006), 240-250.

[14] Eastman, J. A., Choi, S. U. S., Li, S., Thompson, L. J., and Lee, S., Enhanced thermal conductivity through the development of nanofluids, Materials Research Society - Fall Meeting, Boston, MA, (1996).

[15] Eastman, J. A., Choi, S. U. S., Li, S., Yu, W., and Thompson, L. J., Anomalously increased effective thermal conductivities of ethylene glycol-based nanofluids containing copper nanoparticles, Appl. Phys. Lett., 78(6) (2001), 718-720.

[16] Eastman, J. A., Phillpot, S. R., Choi, S. U. S., and Keblinski, P., Thermal transport in nanofluids, Annu. Rev. Mat. Res., 34(1) (2004), 219-246.

[17] Elnaqeeb, T., Shah, N. A. and Mirza, I. A., Natural convection flows of carbon nanotubes nanofluids with Prabhakar-like thermal transport, Mathematical Methods in the Applied Sciences, 1(14) (2020).

[18] Fadhil, R. A., Convolution of Kamal and Mahgoub transforms, Bulletin of Mathematics and Statistics Research, 5(4) (2017), 11-16.

[19] Hajizadeh, A., Shah, N. A., Shah, S. I. A., Animasaun, I. L., Gorji, M. R., and Alarifi, I. M., Transient free convection flow of nanofluids between two vertical parallel plates with damped thermal flux, J. Mol. Liq., 289(2019), 289:110964.

[20] Haq, R. U., Kazmi, S. N., and Mekkaoui, T., Thermal management of water based SWCNTs enclosed in a partially heated trapezoidal cavity via FEM, Int. J. Heat. Mass. Transf., 112(2017), 972-982.

[21] Haq, R. U., Soomro, F. A., Hammouch, Z., and Rehman, S. U., Heat exchange within the partially heated C-shape cavity filled with the water based SWCNTs, Int. J. Heat. Mass. Transf., 127 (2018), 506-514.

[22] He, J. H., Some new approaches to duffing equation with strongly and high order nonlinearity (ii) parameterized perturbation technique, Commun. Nonlinear Sci. Numer. Simul., 4(1) (1999) 81-83.

[23] Hwang, Y., Lee, J. K., and Lee, C. H., Stability and thermal conductivity characteristics of nanofluids, Thermochim Acta., 455(1-2) (2007), 70-74.

[24] Jacquot, R. G., Steadman, J. W., and Rhodine, C. N., The Gaver- Stehfest algorithm for approximate inversion of Laplace transforms, 5(1) (1983) 4-8.

[25] Kim, P., Shi, L., Majumdar, A., and McEuen P. L., Thermal transport measurements of individual multiwalled nanotubes, Phys. Rev. Lett., 87(21) (2001), 215-502.

[26] Lee,S., Choi, S. U. S., Li, S., and Eastman, J. A., Measuring thermal conductivity of fluids containing oxide nanoparticles, J. Heat Transfer., 121(2) (1999), 280-289.

[27] Masuda, H., Ebata, A., Teramae, M., and Hishinuma, N., Alteration of thermal conductivity and viscosity of liquid by dispersing ultra-fine particles(dispersion of γ-Al2O3, SiO2 and TiO2 ultra-fine particles), Netsu Bussei., 4(4) (1993), 227-233.

[28] Murshed, S. M. S., Leong, K. C., and Yang, C., Enhanced thermal conductivity of TiO2—water based nanofluids, Int. J. Therm. Sci., 44(4) (2005), 367-373.

[29] Shah, N. A., Ahmed, N., Elnaqeeb, T., and Rashidi, M. M., Magnetohydrodynamic free convection flows with thermal memory over a moving vertical plate in porous medium, J. Appl. Comput. Mech., 5(1) (2019), 150-161.

[30] Shou, D. H., and He, J.H., Application of parameter-expanding method to strongly nonlinear oscillators, Int. J. Nonlin. Sci. Num., 8(1) (2007), 121-124.

[31] Stehfest, H., Algorithm 368: numerical inversion of Laplace transforms, Commun. ACM., 13(1) (1970), 47-49.

[32] Stehfest, H., Remark on algorithm 368: numerical inversion of Laplace transforms, Commun. ACM., 13(10) (1970), 624.

[33] Sweilam, N. H., and Khader, M. M., Application of He's parameter-expansion method for the nonlinear differential equations, Int. J. Nonlin. Sci. Num., 10(2) (2009), 265-272.

[34] Wang, X., Xu, X., and Choi, S. U. S., Thermal conductivity of nanoparticle–fluid mixture, J. Thermophys. Heat Tran., 13(4) (1999), 474-480.

[35] Xie, H., Lee, H., You, W., and Choi, M., Nanofluids containing multiwalled carbon nanotubes and their enhanced thermal properties, J. Appl. Phys., 94(8) (2003), 4967-4971.

[36] Xie, H., Wang, J., Xi, T., Liu, Y., Ai, F., and Wu, Q., Thermal conductivity enhancement of suspensions containing man-sized alumina particles, J. Appl. Phys., 91(7) (2002), 4568-4572.

[37] Xu, L., Application of He's parameter-expansion method to an oscillation of a mass attached to a stretched elastic wire, Phys. Lett. A., 368(3-4) (2007), 259-262.

[38] Xuan, Y., and Li, Q., Investigation on convective heat transfer and flow features of nanofluids, J. Heat Transfer., 125(1) (2003), 151-155.

[39] Xuan, Y. M., and Li, Q., Heat transfer enhancement of nanofluids, Int. J. Heat Fluid Flow., 21(1) (2000), 58-64.