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CERTAIN RESULTS INVOLVING GENERALIZED MODULAR IDENTITIES

Jitendra Prasad

+2 High School Azad Nagar Chhattargachh,
Pothia, Kishanganj - 855117, Bihar, INDIA

E-mail : prasadjitendra10@gmail.com

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Abstract: In this paper, making use of certain modular relations due to Ramanujan and G. N. Watson, an attempt has been made to establish some results involving modular relations and continued fractions.

Keywords and Phrases: Modular relation, identity, continued fraction, basic hypergeometric series.

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1. Introduction, Notations and Definitions

Here and throughout the paper, we adopt the standard q-series notation as given in [2]. Let q be a complex number such that $|q| < 1$. For positive integer n, we define

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)...(1-aq^{n-1}),$$

$$(a;q)_0 = 1$$

and

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r).$$

Some times we use the compressed notation;

$$(a_1, a_2, ..., a_m; q)_n = (a_1;q)_n(a_2;q)_n...(a_m;q)_n,$$

$n \in N \cup \{\infty\}$ and $m \geq 1$.
Let

$$ {}_r\Phi_s \left[\begin{array}{c} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_s; q)_n} \left\{(-1)^n q^{n(n-1)/2}\right\}^{1+s-r}. $$

Ramanujan's generalized theta function is defined as

$$ f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, $$

which by an appeal of Jacobi's triple product identity [2; App. II (II.28)] gives for $|ab| < 1$,

$$ f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} = (-a, -b, ab; ab)_\infty. \tag{1.1} $$

Deduction from (1.1) are the product representations of the classical theta functions,

$$ \Phi(q) = f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty, \tag{1.2} $$

$$ \Phi(-q) = (q; q^2)_\infty^2 (q^2; q^2)_\infty = \frac{(q; q)_\infty}{(-q; q)_\infty}, \tag{1.3} $$

$$ \Psi(q) = f(q, q^3) = \sum_{n=-\infty}^{\infty} q^{2n^2} q^{-n} = (-q, -q^3, q^4; q^4)_\infty = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{1.4} $$

$$ f(-q) = f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty. \tag{1.5} $$

Rogers-Ramanujan functions [1; page 150] are defined as,

$$ G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_\infty} \tag{1.6} $$

and

$$ H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}. \tag{1.7} $$

Further, Ramanujan showed that

$$\frac{H(q)}{G(q)} = \frac{\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}} = \frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty} = \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+...}. \tag{1.8}$$

2. Main Results

In this section, taking modular relations due to Ramanujan and G. N. Watson certain new modular relation has been established. Certain continued fractions have also been established in this section.

Ramanujan in his 'lost' notebook asserted that

$$G(q)G(q^4) + qH(q)H(q^4) = \frac{\Phi(q)}{(q^2;q^2)_\infty} = (-q;q^2)^2_\infty \tag{2.1}$$

G. N. Watson [3] proved (2.1) and also established in [4] following modular relations

$$G(-q)\Phi(q) - G(q)\Phi(-q) = 2qH(q^4)\Psi(q^2) \tag{2.2}$$

and

$$H(-q)\Phi(q) + H(q)\Phi(-q) = 2G(q^4)\Psi(q^2) \tag{2.3}$$

Multiplying (2.2) by $H(q)$ and (2.3) by $G(q)$ then adding those new modular relations we get,

$$\{G(-q)H(q) + G(q)H(-q)\}\Phi(q) = 2\{qH(q)H(q^4) + G(q)G(q^4)\}\Psi(q^2). \tag{2.4}$$

Now, making use of (2.1) in (2.4) we get,

$$G(-q)H(q) + G(q)H(-q) = 2\frac{\Psi(q^2)}{(q^2;q^2)_\infty}. \tag{2.5}$$

(2.5) is assumed to be a new modular relation.

Again taking $-q$ for q in (2.1) we have a modular relation,

$$G(-q)G(q^4) - qH(-q)H(q^4) = \frac{\Phi(-q)}{(q^2;q^2)_\infty}. \tag{2.6}$$

Putting the value of $\Phi(-q)$ (given in (1.3)) in (2.6) we have

$$G(-q)G(q^4) - qH(-q)H(q^4) = \frac{1}{(-q;q)^2_\infty} = (q;q^2)^2_\infty. \tag{2.7}$$

Taking the ratio of (2.1) and (2.7) we find,

$$\frac{G(q)G(q^4)+qH(q)H(q^4)}{G(-q)G(q^4)-qH(-q)H(q^4)} = \frac{(-q;q^2)_\infty^2}{(q;q^2)_\infty^2}. \tag{2.8}$$

Taking the ratio of (2.2) and (2.3) we get,

$$\frac{G(-q)\Phi(q)-G(q)\Phi(-q)}{H(-q)\Phi(q)+H(q)\Phi(-q)} = q\frac{H(q^4)}{G(q^4)}. \tag{2.9}$$

Now, making use of (1.8), (2.9) yields

$$\frac{G(-q)\Phi(q)-G(q)\Phi(-q)}{H(-q)\Phi(q)+H(q)\Phi(-q)} = \frac{q}{1+}\frac{q^4}{1+}\frac{q^8}{1+}\frac{q^{12}}{1+...}. \tag{2.10}$$

References

[1] Andrews, G. E. and Berndt, B. C., Ramanujan's Lost Notebook, Part II, Springer 2009.

[2] Gasper, G. and Rahman, M., Basic Hypergeometric Series, Second edition, Cambridge 2004.

[3] Watson, G. N., Proof of certain identities in combinatory analysis, J. Indian Math. Soc., 20 (1933), 57-60.

[4] Watson, G. N., the mock theta functions (2), Proc. London Math. Soc., 42 (1937), 274-304.