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NOVEL CLASS OF FINITE INTEGRALS INVOLVING GENERALISED HYPERGEOMETRIC FUNCTION

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Abstract: The classical summation theorems like Gauss theorem, Gauss second theorem, Bailey's theorem, Kummer's theorem, Watson theorem, Dixon theorem, Whipple's theorem and Saalshütz theorem respectively for the hypergeometric series ${}_2F_1$ and ${}_3F_2$ play a key role in the theory of hypergeometric and generalized hypergeometric series and are widely used in many fields. In this research paper, we wish to evaluate a new class of integrals consisting of twenty five results related to generalized hypergeometric function. These twenty five results are expressed in the single integral in the form :

$$\int_0^1 x^{d-1}(1-x)^{d+j-1}{}_4F_3\left[\begin{matrix} a,\ b,\ c,\ 2d+j \\ \frac{1}{2}(a+b+i+1),\ 2c+j,\ d \end{matrix};x\right]dx$$

for i & $j = 0, \pm 1$ & ± 2.
The results are obtained by employing generalizations of classical Watson's summation theorem previously derived by Lavoie et al. together with an Euler's beta integral. Finally, we will be obtaining fifty new integrals and few more integrals expressed in two general integrals consisting of twenty five each as special cases.

Keywords and Phrases: Generalized hypergeometric function, Watsons theorem, definite integral, Euler's Beta integral.

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1. Introduction and Preliminaries

The generalized hypergeometric function with p numerator parameters and q denominator parameters, denoted by ${}_pF_q$, for $p, q \in \mathbb{N}_0$ [1,5] is given by

$$
{}_pF_q\left[\begin{matrix} e_1, & \ldots, & e_p \\ f_1, & \ldots, & f_q \end{matrix} ; z\right] = \sum_{n=0}^{\infty} \frac{(e_1)_n \ldots (e_p)_n}{(f_1)_n \ldots (f_q)_n} \frac{z^n}{n!} \tag{1}
$$

where $(e)_n$ is a commonly used Pochhammer symbol, defined for any complex $e \in \mathbb{C}$ through Gamma function by

$$
\begin{aligned}
(e)_n &= \frac{\Gamma(e+n)}{\Gamma(e)}, \quad \left(e \in \mathbb{C} \backslash \mathbb{Z}_0^-\right) \\
&= \begin{cases} e(e+1)\ldots(e+n-1), & (n \in \mathbb{N}) \\ 1, & (n = 0) \end{cases}
\end{aligned} \tag{2}
$$

The classical summation theorems for the series ${}_2F_1$, namely defined as Gauss theorem, second Gauss theorem, Kummer theorem, Bailey's theorem, Watson theorem, Dixon theorem, Whipple theorem and Saalschütz theorem for the series ${}_3F_2$ are vital in the theory of hypergeometric and generalized hypergeometric functions [1, 5]

Later, Lavoie et al. [2, 3, 4] generalized the aforementioned classical summation theorems. But in this research, we will focus in the Watson's summation theorem [1] as given below:

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, b, c \\ \frac{1}{2}(1+a+b), 2c \end{matrix} ; 1\right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}
\end{aligned} \tag{3}
$$

such that $\Re(2c-a-b) > -1$.
The generalization of (3) due to Lavoie et al. [3] is given by

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\, b,\, c \\ \frac{1}{2}(1+i+a+b),\, 2c+j \end{matrix}\; ;\; 1\right] \\
&= \mathcal{A}_{i,j}\, \frac{2^{a+b+i-2}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\,\Gamma\left(c+\left[\frac{j}{2}\right]+\frac{1}{2}\right)\,\Gamma\left(c-\frac{1}{2}(a+b+|i+j|-j-1)\right)}{\Gamma\left(\frac{1}{2}\right)\,\Gamma\left(a\right)\,\Gamma\left(b\right)} \\
&\times \left\{ \mathcal{B}_{i,j}\, \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{4}\left(1-(-1)^i\right)\right)\,\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}+\left[\frac{j}{2}\right]-\frac{1}{4}(-1)^j\,\left(1-(-1)^i\right)\right)\,\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}+\left[\frac{j}{2}\right]\right)} \right. \\
&\left. +\mathcal{C}_{i,j}\, \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)\,\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+\left[\frac{j+1}{2}\right]+\frac{1}{4}(-1)^j\,\left(1-(-1)^i\right)\right)\,\Gamma\left(c-\frac{1}{2}b+\left[\frac{j+1}{2}\right]\right)} \right\} \\
&= \Omega \quad \text{(let)} \qquad (4)
\end{aligned}
$$

for $i,\, j = 0,\, \pm 1,\, \pm 2$.
If $i = j = 0$, the result (4) reduces to classical Watson's summation theorem (3). In (4), $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denoted the modulus of x. Further, the coefficients $\mathcal{A}_{i,j}$, $\mathcal{B}_{i,j}$ and $\mathcal{C}_{i,j}$ are given respectively in the Table 1, Table 2 and Table 3, at the end of this paper.

In this research work, we will be evaluating twenty five integrals on generalized hypergeometric function in a single integral in the form

$$\int_0^1 x^{d-1}(1-x)^{d+j-1}{}_4F_3\left[\begin{matrix} a,\; b,\; c,\; 2d+j \\ \frac{1}{2}(a+b+i+1),\; 2c+j,\; d \end{matrix}\; ; x\right] dx$$

for $i, j = 0, \pm 1, \pm 2$.
The outcomes are evaluated by using Watson's summation theorem (generalized) on the sum of a ${}_3F_2$ together with the following Beta integral viz.

$$\int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad (5)$$

provided $Re(\alpha) > 0$ and $Re(\beta) > 0$.
Fifty results and few more integrals in the form of two general integrals (twenty five each) will also be evaluated as special cases of our main result.

2. Main Integrals

The twenty five integrals deduced from a single integral is given by the following theorem.

Theorem 2.1. *For $\Re(c) > 0$ and $\Re(1+i+2j+2c-a-b) > 0$; $i, j = 0, \pm1, \pm2$, the following formula for holds true.*

$$\int_0^1 x^{d-1}(1-x)^{d+j-1} {}_4F_3\left[\begin{matrix} a,\ b,\ c,\ 2d+j \\ \frac{1}{2}(a+b+i+1),\ 2c+j,\ d \end{matrix};x\right]dx = \frac{\Gamma(d)\ \Gamma(d+j)}{\Gamma(2d+j)}\Omega \tag{6}$$

where Ω is the same as given in (4).

Proof. The proof of the theorem is very clear. For this, let I denotes the left hand side of (6) then we have

$$\int_0^1 x^{d-1}(1-x)^{d+j-1} {}_4F_3\left[\begin{matrix} a,\ b,\ c,\ 2d+j \\ \frac{1}{2}(a+b+i+1),\ 2c+j,\ d \end{matrix};x\right]dx$$

Now expressing ${}_3F_2$ as a series and since it is uniformly convergent in the interval [0,1], changing the order of integration and summation, we have

$$I = \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n\ (c)_n(2d+j)_n}{\left(\frac{1}{2}(1+i+a+b)\right)_n\ (2c+j)_n\ (d)_n\ n!} \int_0^1 x^{d+n-1}(1-x)^{d+j-1}dx$$

Evaluating the integral by (5), using the Pochhammer symbol (2) and after some simplification, we get

$$I = \frac{\Gamma(d)\ \Gamma(d+j)}{\Gamma(2d+j)} \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n\ (c)_n}{\left(\frac{1}{2}(1+i+a+b)\right)_n\ (2c+j)_n\quad n!}$$

Now on summing the series, we have

$$I = \frac{\Gamma(d)\ \Gamma(d+j)}{\Gamma(2d+j)} {}_3F_2\left[\begin{matrix} a,\ b,\ c \\ \frac{1}{2}(1+i+a+b),\ 2c+j \end{matrix};1\right] \tag{7}$$

Now the resulting ${}_3F_2$ appearing in (7) can be evaluated by using result (4) then we can obtain the right hand side of (6). With this, the proof of the theorem is now completed.

3. Special Cases

In this section, we wish to enlist more than fifty interesting and special cases, which are the parts of our main findings and are given as follows.

It we take b is replaced by $-2n$ and a is replaced by $a+2n$ or if $b = -2n-1$ and a is replaced by $a+2n+1$ in (6), then one of the two terms appearing on the right-hand side of (6) will vanish, In each case we get twenty five special cases.

These special cases are expressed in the following two corollaries.

Corollary 3.1. *The following twenty five results hold true for* $i, j = 0, \pm 1, \pm 2,$.

$$\int_0^1 x^{d-1}(1-x)^{d+j-1}{}_4F_3\left[\begin{array}{c} -2n,\ a+2n,\ c,\ 2d+j \\ \frac{1}{2}(a+i+1),\ 2c+j,\ d \end{array};x\right]dx$$
$$= D_{i,j}\ \frac{\Gamma(d)\ \Gamma(d+j)}{\Gamma(2d+j)}\ \frac{\left(\frac{1}{2}\right)_n\ \left(\frac{1}{2}a-c+\frac{3}{4}-\frac{(-1)^i}{4}-\left[\frac{1}{2}j+\frac{1}{4}\left(1+(-1)^i\right)\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j}{2}\right]\right)_n\ \left(\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)_n} \quad (8)$$

The coefficients $D_{i,j}$ *are given in the Table 4.*

Corollary 3.2. *The following twenty five results hold true for* $i, j = 0, \pm 1, \pm 2,$.

$$\int_0^1 x^{d-1}(1-x)^{d+j-1}{}_4F_3\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c,\ 2d+j \\ \frac{1}{2}(a+i+1),\ 2c+j,\ d \end{array};x\right]dx$$
$$= E_{i,j}\ \frac{\Gamma(d)\ \Gamma(d+j)}{\Gamma(2d+j)}\ \frac{\left(\frac{3}{2}\right)_n\ \left(\frac{1}{2}a-c+\frac{5}{4}+\frac{(-1)^i}{4}-\left[\frac{1}{2}j+\frac{1}{4}\left(1+(-1)^i\right)\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j+1}{2}\right]\right)_n\ \left(\frac{1}{2}a+\frac{1}{4}\left(3-(-1)^i\right)\right)_n} \quad (9)$$

The coefficients $E_{i,j}$ *are given in the Table 5.*

If $i = j = 0$, in (7), we get the following interesting result:

$$\int_0^1 x^{d-1}(1-x)^{d-1}{}_4F_3\left[\begin{array}{c} -2n,\ a+2n,\ c,\ 2d \\ \frac{1}{2}(a+1),\ 2c,\ d \end{array};x\right]dx = \frac{\Gamma(d)\ \Gamma(d)}{\Gamma(2d)}\ \frac{\left(\frac{1}{2}\right)_n\ \left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(c+\frac{1}{2}\right)_n\ \left(\frac{1}{2}a+\frac{1}{2}\right)_n} \quad (10)$$

For $d = c$, it reduces to

$$\int_0^1 x^{c-1}(1-x)^{c-1}{}_2F_1\left[\begin{array}{c} -2n,\ a+2n \\ \frac{1}{2}(a+1) \end{array};x\right]dx = \frac{\Gamma(c)\ \Gamma(c)}{\Gamma(2c)}\ \frac{\left(\frac{1}{2}\right)_n\ \left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(c+\frac{1}{2}\right)_n\ \left(\frac{1}{2}a+\frac{1}{2}\right)_n}$$

Similarly, if we take $i = j = 0$ in (8), we get the following elegant result:

$$\int_0^1 x^{d-1}(1-x)^{d-1}{}_4F_3\left[\begin{array}{c} -2n-1,\ a+2n+1,\ c,\ 2d \\ \frac{1}{2}(a+1),\ 2c,\ d \end{array};x\right]dx = 0 \quad (11)$$

For $d = c$, it reduces to

$$\int_0^1 x^{c-1}(1-x)^{c-1}{}_2F_1\left[\begin{array}{c} -2n-1,\ a+2n+1 \\ \frac{1}{2}(a+1) \end{array};x\right]dx = 0$$

We can obtain other results in the same manner. The details are left as an exercise to the interested readers.

4. Concluding Remarks

In this paper, we have evaluated twenty -five new and interesting integrals in the form of a single integral. These results are obtained by employing generalization of classical Watson's summation theorem earlier discovered by Lavoie et al. together with the well-known Euler's Beta integral. We have also deduced more than fifty interesting integrals in the form of two integrals (twenty five each) as the special cases of our main findings.

We believe that the results established in this research paper are simple, interesting, easily established, have not appeared in the literature and represent a definite contribution in the theory of generalized hypergeometric series of one or several variables. It is hoped that the results could be of potential use in the area of applied mathematics, statistics, engineering and mathematical physics.

In addition to this, it is well-known that whenever the results are expressed in terms of gamma functions, the results are very important from the applications point of view. Since our results are in terms of gamma functions, so we conclude our paper by remarking that, as an application, in our subsequent paper, we are applying these results in the evaluations of a new class of integrals involving generalized functions such as those of MacRobert's E-function, Meijer's G-function, Fox's H-function, Inayat Hussain's bar H-function, Rathie's I-function and Saxena (V.P.) I-function.

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i \ j	-2	-1	0	1	2
2	$\frac{1}{2(c-1)(a-b-1)(a-b+1)}$	$\frac{1}{2(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{8(c+1)(a-b-1)(a-b+1)}$
1	$\frac{1}{(c-1)(a-b)}$	$\frac{1}{(a-b)}$	$\frac{1}{(a-b)}$	$\frac{1}{2(a-b)}$	$\frac{1}{2(c+1)(a-b)}$
0	$\frac{1}{2(c-1)}$	1	1	1	$\frac{1}{2(c+1)}$
-1	$\frac{1}{(c-1)}$	1	2	2	$\frac{2}{(c+1)}$
-2	$\frac{1}{2(c-1)}$	1	1	2	$\frac{2}{(c+1)}$

Table 1: Table for $A_{i,j}$

i \ j	-2	-1	0	1	2
2	$c(a+b-1)-(a+1)(b+1)+2$	$a+b-1$	$a(2c-a)+b(2c-b)-2c+1$	$2c(a+b-1)-(a-b)^2+1$	$B_{2,2}$
1	$c-b-1$	1	1	$2c-a+b$	$2c(c+1)-(a-b)(c-b+1)$
0	$(c-a-1)(c-b-1)+(c-1)(c-2)$	1	1	1	$(c-a+1)(c-b+1)+c(c+1)$
-1	$2(c-1)(c-2)-(a-b)(c-b-1)$	$2c-a+b-2$	1	1	$c-b+1$
-2	$B_{-2,-2}$	$B_{-2,-1}$	$a(2c-a)+b(2c-b)-2c+1$	$a+b-1$	$c(a+b-1)-(a-1)(b-1)$

Table 2: Table for $B_{i,j}$

$$\mathcal{B}_{2,2} = 2c(c+1)\{(2c+1)(a+b-1)-a(a-1)-b(b-1)\} - (a-b-1)(a-b+1)\{(c+1)(2c-a-b+1)+ab\};$$

$$\mathcal{B}_{-2,-2} = 2(c-1)(c-2)\{(2c-1)(a+b-1)-a(a+1)-b(b+1)+2\} - (a-b-1)(a-b+1)\{(c-1)(2c-a-b-3)+ab\};$$

$$\mathcal{B}_{-2,-1} = 2(c-1)(a+b-1)-(a-b)^2+1;$$

i \ j	-2	-1	0	1	2
2	-4	$-(4c-a-b-3)$	-8	$-[8c^2-2c(a+b-1)-(a-b)^2+1]$	$-4(2c+a-b+1)(2c-a+b+1)$
1	$-(c-a-1)$	-1	-1	$-(2c+a-b)$	$-[2c(c+1)+(a-b)(c-a+1)]$
0	4	1	0	-1	-4
-1	$2(c-1)(c-2)+(a-b)(c-a-1)$	$2c+a-b-2$	1	1	$c-a+1$
-2	$4(2c-a+b-3)(2c+a-b-3)$	$C_{-2,-1}$	8	$4c-a-b+1$	4

Table 3: Table for $C_{i,j}$

$$C_{-2,-1} = 8c^2 - 2(c-1)(a+b+7) - (a-b)^2 - 7$$

i \ j	-2	-1	0	1	2
2	$\frac{(a+1)[(c-1)(a-1)+2n(a+2n)]}{(c-1)(a+4n-1)(a+4n+1)}$	$\frac{(a+1)(a-1)}{(a+4n+1)(a+4n-1)}$	$\frac{(a+1)[(a-1)(2c-a-1)-4n(a+2n)]}{(2c-a-1)(a+4n+1)(a+4n-1)}$	$\frac{(a+1)[(a-1)(2c-a-1)-8n(a+2n)]}{(2c-a-1)(a+4n+1)(a+4n-1)}$	$D_{2,2}$
1	$\frac{a(c+2n-1)}{(c-1)(a+4n)}$	$\frac{a}{a+4n}$	$\frac{a}{a+4n}$	$\frac{a(2c-a-4n)}{(2c-a)(a+4n)}$	$\frac{a[(c+1)(2c-a)-2n(2c+a+4n+2)]}{(c+1)(2c-a)(a+4n)}$
0	$1-\frac{2n(a+2n)}{(c-1)(2c-a-3)}$	1	1	1	$1-\frac{2n(a+2n)}{(c+1)(2c-a+1)}$
-1	$1-\frac{2n(2c+a+4n-2)}{(c-1)(2c-a-4)}$	$1-\frac{4n}{(2c-a-2)}$	1	1	$1+\frac{2n}{(c+1)}$
-2	$D_{-2,-2}$	$1-\frac{8n(a+2n)}{(a-1)(2c-a-3)}$	$1-\frac{4n(a+2n)}{(a-1)(2c-a-1)}$	1	$1+\frac{2n(a+2n)}{(c+1)(a-1)}$

Table 4: Table for $D_{i,j}$

$$D_{2,2}=\frac{(a+1)[(a-1)(c+1)(2c-a+1)(2c-a-1)-2an(6c+a+5)(2c-a+1)+4n^2(5a^2+4a-5-4c(3c-a+4))+64n^3(a+n)]}{(c+1)(2c-a+1)(2c-a-1)(a+4n+1)(a+4n-1)}$$

$$D_{-2,-2}=1-\frac{2an(6c+a-7)(2c-a-3)-4n^2[5a^2-4a-21-4c(3c-a-8)]-64n^3(a+n)}{(c-1)(a-1)(2c-a-3)(2c-a-5)}$$

i \ j	-2	-1	0	1	2
2	$\frac{(a+1)(2c-a-3)}{(c-1)(a+4n+1)(a+4n+3)}$	$\frac{(a+1)(4c-a-3)}{(a+4n+1)(a+4n+3)(2c-1)}$	$\frac{2(a+1)}{(a+4n+1)(a+4n+3)}$	$E_{2,1}$	$\frac{(a+1)(2c+a+4n+3)(2c-a-4n-1)}{(c+1)(2c-a-1)(a+4n+1)(a+4n+3)}$
1	$\frac{(c-a-2n-2)}{(c-1)(a+4n+2)}$	$\frac{2c-a-2}{(a+4n+2)(2c-1)}$	$\frac{1}{a+4n+2}$	$\frac{(2c+a+4n+2)}{(2c+1)(a+4n+2)}$	$\frac{(c+a+2)(2c-a)-2n(3a-2c+4n+2)}{(c+1)(2c-a)(a+4n+2)}$
0	$\frac{-1}{(c-1)}$	$\frac{-1}{(2c-1)}$	0	$\frac{1}{(2c+1)}$	$\frac{1}{(c+1)}$
-1	$E_{-1,-2}$	$\frac{-(2c+a+4n)}{a(2c-1)}$	$\frac{-1}{a}$	$\frac{-(2c-a)}{a(2c+1)}$	$\frac{-(c-a-2n)}{a(c+1)}$
-2	$\frac{-(2c+a+4n-1)(2c-a-4n-5)}{(a-1)(c-1)(2c-a-5)}$	$E_{-2,-1}$	$\frac{-2}{(a-1)}$	$\frac{-(4c-a+1)}{(a-1)(2c+1)}$	$\frac{-(2c-a+1)}{(a-1)(c+1)}$

Table 5: Table for $E_{i,j}$

$$E_{2,1} = \frac{(a+1)[(4c+a+3)(2c-a-1)-8n(a+2n+2)]}{(a+4n+1)(a+4n+3)(2c+1)(2c-a-1)} \ ;$$

$$E_{-2,-1} = -\ \frac{[(4c+a-1)(2c-a-3)-8n(a+2n+2)]}{(a-1)(2c-1)(2c-a-3)}$$

$$E_{-1,-2} = -\ \frac{[(c+a)(2c-a-4)-2n(3a-2c+4n+6)]}{a(c-1)(2c-a-4)}$$