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VALUE DISTRIBUTION OF L-FUNCTIONS AND MEROMORPHIC FUNCTIONS SHARING FINITE SETS

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Abstract: Let f be a non-constant meromorphic function with finitely many poles, and let L be an L-function in the Selberg class. In ([13]) the authors showed the existence of subsets $S, T \subset \mathbb{C}$ of 10 elements such that the condition $L^{-1}(S) = f^{-1}(T)$ implies $f = hL$ for a non-zero constant h. In this paper, we present a class of such subsets $S, T \subset \mathbb{C}$ of 9 elements. As an application of this result, we obtain a class of subsets $S \subset \mathbb{C}$ of 9 elements such that the condition $L^{-1}(S) = f^{-1}(S)$ implies $f = L$. This result improves ([22], Theorem 7).

Keywords and Phrases: L-function, Selberg class, value distribution, uniqueness, meromorphic functions.

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1. Introduction. Main results

L-functions in the Selberg class, with the Riemann zeta function as a prototype, are important objects in number theory. In this paper, an L-function always means a non-constant L-function in the Selberg class $\mathcal{S}$, with the normalized condition $a(1) = 1$, which is defined to be a Dirichlet series

$$L(s) = \sum_{i=0}^{\infty} \frac{a(n)}{n^s}$$

satisfying hypotheses:

(i) Ramanujan hypothesis; (ii) Analytic continuation; (iii) Functional equation; (iv) Euler product hypothesis (see ([23]).

In recent years the problem of determining L-functions by preimages of subsets caused increasing attentions.

On the other hand, an L-function can be analytically continued as a meromorphic function in the complex plane $\mathbb{C}$. Therefore, for the problem of determining L-functions by preimages of subsets one of the main tools is the Nevanlinna theory on the value distribution of meromorphic functions. Let us first recall some basic notions.

Let f be a non-constant meromorphic function in $\mathbb{C}$, and $a \in \mathbb{C} \cup \{\infty\}$. We assume that the reader is familiar with the notations of Nevanlinna theory (see, for example ([6]), ([8])): $T(r, f)$, $N(r, f)$, $m(r, f)$,....

Denote by $E_f(a)$ the set of all $a-$ points of f where an $a-$ point is counted with its multiplicity, and by $\overline{E}_f(a)$ where an $a-$ point is counted only one time (i.e., $\overline{E}_f(a) = f^{-1}(a)$). Let m be a positive integer. Denote by $E_{f,m)}(a)$ the set of $a-$points of f with multiplicities $\leq m$, each $a-$ point counted as many times as its multiplicity. For a non-empty subset $S \subset \mathbb{C} \cup \{\infty\}$, define $E_f(S) = \cup_{a \in S} E_f(a)$, and $\overline{E}_f(S) = f^{-1}(S)$, and $E_{f,m)}(S) = \cup_{a \in S} E_{f,m)}(a)$. Let $\mathcal{F}$ be a non - empty subset of $\mathcal{M}(\mathbb{C})$. Two non-constant meromorphic functions f, g of $\mathcal{F}$ are said to *share* S, *counting multiplicity*, (share S CM), if $E_f(S) = E_g(S)$, and to *share* S, *ignoring multiplicity*, (share S IM), if $\overline{E}_f(S) = \overline{E}_g(S)$. If the condition $E_f(S) = E_g(S)$ (resp. $\overline{E}_f(S) = \overline{E}_g(S)$) implies $f = g$ for any two non-constant meromorphic (entire) functions f, g of $\mathcal{F}$, then S is called a unique range set for meromorphic (entire) functions of $\mathcal{F}$ counting multiplicity (resp. ignoring multiplicity).

In 1976 Gross [7] proved that there exist three finite sets S_j $(j = 1, 2, 3)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$, $j = 1, 2, 3$ must be identical. In the same paper Gross [7] posed the following question:

Question A. *Can one find two (or possible even one) finite set* S_j $(j = 1, 2)$ *such that any two entire functions* f *and* g *satisfying* $E_f(S_j) = E_g(S_j)$ $(j = 1, 2)$ *must be identical?*

Many results have been obtained for this and related topics (see ([4]), ([5]), ([12]), ([13]), ([14]), ([15], [16]), ([26])).

The study of unique range sets primarily focuses on two topics:

a) Finding unique range sets with the smallest possible number of elements.

b) Characterizing the properties of unique range sets.

For Topic a), in 1998 (resp. 2000) Frank and Reinders ([4]) (resp. Fujimoto ([5])) exhibited a unique range set for meromorphic functions on $\mathbb{C}$ counting multiplicity (resp. ignoring multiplicity) with 11 (resp. 17) elements.

In recent years many results have been obtained for the problem of determining L-functions by preimages of subsets (see [16], ([17]), ([19]), ([9]), ([23]), ([27])).

Steuding ([23]), Hu and Li ([9]) showed that an L-function is uniquely defined by the preimage of a single point $c \neq 1, \infty$, counting multiplicity. Hu and Wu ([24]), Yuan, Li, and Yi ([27]) obtained uniqueness theorems for L-functions sharing values in a finite subset of $\mathbb{C}$, counting multiplicities.

Concerning Question A it is natural to ask the following question:

Question B. *Under what conditions on subsets $S, T \subset \mathbb{C} \cup \{\infty\}$ and non-constant meromorphic functions f and g the relation holds: either $E_f(S) = E_g(T)$ or $\overline{E}_f(S) = \overline{E}_g(T)$?*

Many results have been obtained for this and related topics (see ([25]), ([20]), ([3]), ([21]), ([13])).

In response to Question B, the authors ([13]) showed the existence of subsets $S, T \subset \mathbb{C}$ of 10 elements such that for a non-constant meromorphic function f with finitely many poles, and a L-function L, the condition $L^{-1}(S) = f^{-1}(T)$ implies $f = hL$ for a non-zero constant h.

Noting that the identity relationship between a L-function and a meromorphic functions is a specific instance of a linear dependency between the same functions.

Regarding this and Topic a), we may ask: What are the smallest cardinalities for such finite sets S, T such that the condition: $\overline{E}_f(S) = \overline{E}_g(T)$ implies $f = hL$ for a non-zero constant h?

In this direction, in this paper, we present a class of subsets $S, T \subset \mathbb{C}$ of 9 elements such that for a non-constant meromorphic function f with finitely many poles, and an non-constant L-function L, the condition $L^{-1}(S) = f^{-1}(T)$ and $E_{L',1)}(0) = E_{f',1)}(0)$ implies $f = hL$ for a non-zero constant h.

Now let us describe main results of the paper. Consider polynomials $P(z), Q(z) \in \mathbb{C}[z]$ of degree n of the form:

$$P(z) = az^n + bz^{n-m} + c, \text{ where } a, b, c \neq 0;$$

$$Q(z) = uz^n + vz^{n-m} + t; \text{ where } u, v, t \neq 0. \tag{1.1}$$

Assume that:

$$\frac{a^{n-m}c^m}{b^n} \neq \frac{(-1)^n(n-m)^{n-m}m^m}{n^n}, \tag{1.2}$$

$$\frac{u^{n-m}t^m}{v^n} \neq \frac{(-1)^n(n-m)^{n-m}m^m}{n^n}. \tag{1.3}$$

Note that polynomial $P(z)$ (*resp.* $Q(z)$) has n distinct simple zeros if and only if the condition (1.2) (resp. (1.3)) is satisfied (see ([18], Lemma 2.7)).

We shall prove the following theorem.

Theorem 1. *Let m, n be positive integers, $n \geq 2m+7$, let $P(z), Q(z)$ be polynomials of the form (1.1) with conditions (1.2) and (1.3), and let S, T be the zero sets of P, Q, respectively. Let f be a non-constant meromorphic function with finitely many poles in the complex plane and L be a $L-$ function. Then we have:*

1. $L^{-1}(S) = f^{-1}(T)$, and $E_{L',1)}(0) = E_{f',1)}(0)$ if only if $f = hL$ and $hS = T$, where h is a non-zero constant satisfying $h^n = \dfrac{at}{cu}$ and $h^m = \dfrac{av}{ub}$.

2. In particular, $L^{-1}(S) = f^{-1}(T)$, and f is a L-function, and $E_{L',1)}(0) = E_{f',1)}(0)$ if only if $f = L$ and $\dfrac{a}{u} = \dfrac{b}{v} = \dfrac{c}{t}$.

Applications. We discuss some applications of Theorem 1.

Theorem 2. *Let m, n be positive integers such that $(n, m) = 1$ and $n \geq 2m+7$, let $P(z)$ be polynomial of the form (1.1) with conditions (1.2), and let S be the zero set of P. Suppose that $L^{-1}(S) = f^{-1}(S)$ and $E_{L',1)}(0) = E_{f',1)}(0)$ for a non-constant meromorphic function with finitely many poles f and a L-function L. Then we have: $f = L$.*

Indeed, applying Theorem 1 with $Q(z) = P(z), T = S$, we get: $f = hL$ and $hS = S$, where h is a non-zero constant satisfying $h^n = 1$ and $h^m = 1$. By $(n, m) = 1$ we obtain $h = 1$. So $f = L$.

Examples.

Let f be a non-constant meromorphic function with finitely many poles and let L be a non-constant $L-$function, S, T are the zero set of the polynomial $P(z)$ and $Q(z)$, respectively.

Example 1. Let

$$P(z) = z^{11} - \frac{11}{9}z^9 + 1,\ S = \{a_1, ..., a_{11}\},\ Q(z) = z^{11} - \frac{11}{9}2^2 z^9 + 2^{11},\ T = \{b_1, ..., b_{11}\}.$$

Then

$$L^{-1}(S) = f^{-1}(T),$$

$$E_{L',1)}(0) = E_{f',1)}(0) \ \text{ if only if } \ f = hL, hS = T, \ \text{ where } h^{11} = 2^{11}, h^2 = 2^2.$$

Now we show the necessary condition. We investigate conditions (1.2), (1.3). We have

$$a = u = 1,\ b = -\frac{11}{9},\ c = 1,\ v = -\frac{11}{9}2^2,\ t = 2^{11},\ \frac{9^{11}}{11^{11}} \neq \frac{9^9 2^2}{11^{11}},\ \ 11 = 2.2 + 7.$$

Then applying Theorem 1 with $n = 11,\ m = 2,\ a = u = 1,\ b = -\frac{11}{9},\ c = 1,$ $v = -\frac{11}{9}2^2,\ t = 2^{11}$, we obtain:

$$f = hL,\ hS = T,\ \text{ where }\ h^{11} = \frac{at}{cu} = 2^{11}, h^2 = \frac{av}{ub} = 2^2.$$

Noticing that $h = 2$ satisfying condition above.

Now we show the sufficient condition. Assume that

$$f = hL,\ \text{where}\ h^{11} = \frac{at}{cu} = 2^{11}, h^2 = \frac{av}{ub} = 2^2.$$

Then $E_{L',1)}(0) = E_{f',1)}(0)$ and $h^9 = 2^9$, and by

$$Q(f) = f^{11} - \frac{11}{9}2^2 f^9 + 2^{11} = 2^{11}(L^{11} - \frac{11}{9}L^9 + 1) = 2^{11}P(L),$$

we get

$$2^{11}(L - a_1)\cdots(L - a_{11}) = (f - b_1)\cdots(f - b_{11}).$$

From this it follows that $L^{-1}(S) = f^{-1}(T)$.

Example 2. Let $P(z) = z^9 - \frac{9}{8}z^8 + 1$. Then

$$L^{-1}(S) = f^{-1}(S),\ E_{L',1)}(0) = E_{f',1)}(0)\ \text{ if only if }\ f = L.$$

Now we show the necessary condition. We investigate conditions (1.2). We have We have

$$\frac{8^9}{9^9} \neq \frac{8^8}{9^9},\ 9 = 2.1 + 7.$$

Then applying Theorem 2 with $n = 9,\ m = 1$, and noticing that $(9, 1) = 1$, we obtain: $f = L$.

Now we show the sufficient condition. Assume that $f = L$. Then $E_{L',1)}(0) = E_{f',1)}(0)$. By

$$P(f) = f^9 - \frac{9}{8}f^8 + 1 = L^9 - \frac{9}{8}L^8 + 1 = P(L),\ \text{ we get}$$

$$(L - a_1)\cdots(L - a_9) = (f - b_1)\cdots(f - b_9).$$

From this it follows that $L^{-1}(S) = f^{-1}(S)$.

Remark. *i) Theorem 1 improves a recent result in* $([13], Theorem 1.1)$.

ii) In ([22], *Theorem*7) *the authors showed the existence of subsets* $S_P \subset \mathbb{C}$ *of* 14 *elements such that for a non-constant meromorphic function* f *with finitely many poles, and a L-function* L*, the condition* $L^{-1}(S_P) = f^{-1}(S_P)$ *and* $L^{-1}(c) = f^{-1}(c)$ *and* $c \notin S_P$*, implies* $f = L$*. So Theorem 2 improves* ([22], *Theorem*7).

2. Preliminary results

We have other forms of two Fundamental Theorems of the Nevanlinna theory:

As an immediate consequence of the Nevanlinna's First fundamental theorem (([8], Theorem 1.2, p.5)) we have

Another form of the First Fundamental Theorem (see [26]**, Theorem 1.2, p.8).** *Let* $f(z)$ *be a non-constant meromorphic function and let* $a \in \mathbb{C}$*. Then*

$$T(r, \frac{1}{f-a}) = T(r, f) + O(1),$$

where $O(1)$ *is a bounded quantity depending on* a*.*

A another form of the Second Fundamental Theorem (see [26]**, Theorem 1.6', p.22).** *Let* f *be a non-constant meromorphic function on* $\mathbb{C}$ *and let* $a_1, a_2, ...,$ a_q *be distinct points of* $\mathbb{C}$*. Then*

$$(q-1)T(r, f) \leq \overline{N}(r, f) + \sum_{i=1}^{q} \overline{N}(r, \frac{1}{f-a_i}) - N_0(r, \frac{1}{f'}) + S(r, f),$$

where $N_0(r, \frac{1}{f'})$ *is the counting function of those zeros of* f'*, which are not zeros of the function* $(f-a_1)...(f-a_q)$*, and* $S(r, f) = o(T(r, f))$ *for all* r*, except for a set of finite Lebesgue measure.*

We need some lemmas.

Lemma 2.1. ([6]) *For any non-constant meromorphic function* f*, we have*

i) $T(r, f^{(k)}) \leq (k+1)T(r, f) + S(r, f)$*;*

ii) $S(r, f^{(k)}) = S(r, f)$*.*

([28]**)** *For any non-constant meromorphic function* f*,*

$$N(r, \frac{1}{f'}) \leq N(r, \frac{1}{f}) + \overline{N}(r, f) + S(r, f).$$

Definition. *Let* f *be a non-constant meromorphic function, and* k *be a positive integer. We denote by* $\overline{N}_{(k}(r, f)$ *the counting function of the poles of order* $\geq k$ *of* f*, where each pole is counted only once. If* z *is a zero of* f*, denote by* $\nu_f(z)$ *its multiplicity. We denote by* $\overline{N}(r, \frac{1}{f'}; f \neq 0)$ *the counting function of the zeros* z *of* f' *satisfying* $f(z) \neq 0$*, where each zero is counted only once. Denote by*

$\overline{E}_f(a, \nu_{f-a} = k)$ *the set of* $a-$*points of* f *with multiplicities* k*, where an* $a-$ *point is counted only one time.*

Let be given two non-constant meromorphic functions f and g. For simplicity, denote by $\nu_1(z) = \nu_f(z)$ (resp. $\nu_2(z) = \nu_g(z)$), if z is a zero of f (resp. g). Let $f^{-1}(0) = g^{-1}(0)$. We denote by $N(r, \frac{1}{f}; \nu_1 = \nu_2 = 1)$ (resp. $\overline{N}(r, \frac{1}{f}; \nu_1 > \nu_2 \geq 1)$) the counting function of the common zeros z, satisfying $\nu_1(z) = \nu_2(z) = 1$ (resp. $\nu_1(z) > \nu_2(z) \geq 1$, where each zero is counted only once), and by $N(r, \frac{1}{f}; \nu_1 \geq 2)$ the counting function of the zeros z of f, satisfying $\nu_1(z) \geq 2$. Similarly, we define the counting functions $\overline{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1)$ and $N(r, \frac{1}{g}; \nu_2 \geq 2)$.

Lemma 2.2. *Let* f, g *be two nonconstant meromorphic functions such that* $f^{-1}(0) = g^{-1}(0)$*. Set*

$$F = \frac{1}{f},\ G = \frac{1}{g},\ H = \frac{F''}{F'} - \frac{G''}{G'}.$$

Suppose that $H \not\equiv 0$.

1) We have

$$([\mathbf{13}], \mathbf{Lemma 2.4})\quad N(r, H) \leq \overline{N}_{(2}(r, f) + \overline{N}_{(2}(r, g) + \overline{N}(r, \frac{1}{f}; \nu_1 > \nu_2 \geq 1) + \overline{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1) + \overline{N}(r, \frac{1}{f'}; f \neq 0) + \overline{N}(r, \frac{1}{g'}; g \neq 0).$$

Moreover, if a *is a common simple zero of* f *and* g*, then* $H(a) = 0$.

2) Assume additionally that:

$$\overline{E}_f(0, \nu_1 = 1) \cap \overline{E}_g(0, \nu_2 = 2) = \emptyset \quad and \quad \overline{E}_f(0, \nu_1 = 2) \cap \overline{E}_g(0; \nu_2 = 1) = \emptyset.$$

We have

$$\overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + 2\overline{N}(r, \frac{1}{f}; \nu_1 \geq 2) + 2\overline{N}(r, \frac{1}{g}; \nu_2 \geq 2) \leq N(r, H) + \frac{1}{2}(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) + N(r, \frac{1}{f}; \nu_1 \geq 2) + N(r, \frac{1}{g}; \nu_2 \geq 2) + S(r, f) + S(r, g).$$

Proof. Applying ([6], p. 14) we get:

$$N(r, \frac{1}{f}) = \sum_{0 < |a_i| < r} \log \frac{r}{|a_i|} + n(0, \frac{1}{f}) \log r,$$

where the a_i's are zeros of f, counting multiplicity, and

$$\overline{N}(r, \frac{1}{f}) = \sum_{0<|a_i|<r} \log \frac{r}{|a_i|} + \overline{n}(0, \frac{1}{f}) \log r,$$

where the a_i's are zeros of f, ignoring multiplicity.

Set

$$M = \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) + 2\overline{N}(r, \frac{1}{f}; \nu_1 \geq 2) + 2\overline{N}(r, \frac{1}{g}; \nu_2 \geq 2),$$

$$T = N(r, \frac{1}{f}; \nu_1 = \nu_2 = 1) + \frac{1}{2}(N(r, \frac{1}{f}) + N(r, \frac{1}{g}))+$$

$$N(r, \frac{1}{f}; \nu_1 \geq 2) + N(r, \frac{1}{g}; \nu_2 \geq 2).$$

We first prove that $M \leq T$.

Let a be a zero of f with multiplicity p. From $\overline{E}_f(0) = \overline{E}_g(0)$, it follows that a is a zero of g with multiplicity q. We consider the following cases:

Case 1. Assume that $p = q$.

If $p = q = 1$, then a is counted with $1 + 1 + 0 + 0 = 2$ times in M and it is counted with $1 + \frac{1}{2}(1 + 1) = 2$ times in T.

If $p = q \geq 2$, then a is counted with $1 + 1 + 2 + 2 = 6$ times in M and it is counted with $0 + \frac{1}{2}(p + p) + p + p = 3p \geq 6$ times in T.

Case 2. Assume that $p > q$.

If $p > q$ and $q = 1$, then $p \geq 3$ from $\overline{E}_f(0, \nu_1 = 2) \cap \overline{E}_g(0, \nu_2 = 1) = \emptyset$, and then a is counted with $1 + 1 + 2 + 0 = 4$ times in M and by $p \geq 3$ we see that a is counted with $0 + \frac{1}{2}(p + 1) + p + 0 = p + \frac{p+1}{2} \geq 5 > 4$ times in T.

If $p > q$ and $q \geq 2$, then $p \geq 3$ and a is counted with $1 + 1 + 2 + 2 = 6$ times in M, and by $p \geq 3$, and $q \geq 2$, we see that $p + q \geq 5$, and then a is counted with $0 + \frac{1}{2}(p + q) + p + q = 0 + \frac{3(p+q)}{2} > 6$ times in T.

Case 3. Assume that $q > p$.

The proof of Case 3 is completed by using the arguments similar to the ones in Case 2.

So $M \leq T$. Noting that if a is a common simple zero of f and g, then $H(a) = 0$, we have

$$\begin{aligned}\overline{N}(r,\frac{1}{f};\nu_1=\nu_2=1) &\le N(r,\frac{1}{H}) \le T(r,H)+O(1)\\ &= N(r,H)+m(r,H)+O(1)\\ &\le N(r,H)+m(r,\frac{F''}{F'})+m(r,\frac{G''}{G'})+O(1)\\ &\le N(r,H)+S(r,f)+S(r,g).\end{aligned}$$

Lemma 2.2 is proved.

Lemma 2.3. [19] *Suppose L is a non-constant L-function, there is no generalized Picard exceptional value of L in the complex plane.*

Lemma 2.4. [23] *Let L be a non-constant $L-$ function. Then*

i) $T(r,L)=\frac{d_L}{\pi}r\log r+O(r)$, where $d_L=2\sum_{i=1}^{K}\lambda_i$ is the degree of $L-$ function, and K,λ_i are respectively the positive integer and positive real number in the functional equation of the definition of $L-$ functions;

ii) $N(r,L)=S(r,L)$.

Lemma 2.5. [10] *Let $L_1,...,L_N$ be distinct non-constant L-functions. Then $L_1,...,L_N$ are linearly independent over $\mathbb{C}$.*

Lemma 2.6. [13] *Let $P(z)$, $Q(z)$ be two non-constant polynomials of degree n and let f be a non-constant meromorphic function with finitely many poles in the complex plane, L be a non-constant L-function. Assume that f and L satisfy*

$$\frac{1}{Q(f)}=\frac{c}{P(L)}+c_1,$$

where $c,c_1\in\mathbb{C}$ and $c\neq 0$. Then $c_1=0$.

Lemma 2.7. [13] *Let L be an L-function in the extended Selberg class, f be a meromorphic function, a,b,d,u,v,t be non-zero complex constants, n and m be positive integers, $n\ge 2m+3$. Set*

$$P(x)=ax^n+bx^{n-m}+d;\quad Q(y)=uy^n+vy^{n-m}+t.$$

1. If $P(L)=Q(f)$, then $f=hL$, where h is a constant, satisfying $h^n=a/u$, $h^{n-m}=b/v$ and $d=t$.

2. If $(n,m)=1$, $a=u$, $b=v$, and $P(L)=Q(f)$, then $f=L$ and $d=t$.

3. If L_1,L_2 are L-functions in the extended Selberg class and $P(L_1)=Q(L_2)$, then $L_1=L_2$ and $P=Q$.

3. Proof of Theorem 1

Recall that

$$P(z) = az^n + bz^{n-m} + d,\ Q(z) = uz^n + vz^{n-m} + t, a, b, d, u.v, t \neq 0.$$

Therefore, we have:

$$P(z) = a(z-a_1)...(z-a_n), P'(z) = Az^{n-m-1}(z-d_1)...(z-d_m), A \neq 0, d_i \neq d_j,$$

$$Q(z) = u(z-b_1)...(z-b_n), Q'(z) = Bz^{n-m-1}(z-t_1)...(z-t_m), B \neq 0, t_i \neq t_j, \quad (3.1)$$

$$P(L) = a(L-a_1)...(L-a_n),\ Q(f) = u(f-b_1)...(f-b_n). \quad (3.2)$$

$$[P(L)]' = AL^{n-m-1}(L-d_1)...(L-d_m)L',\ [Q(f)]' = Bf^{n-m-1}(f-t_1)...(f-t_m)f'. \quad (3.3)$$

The necessary condition.

Lemma 3.1. *We have*

1)

$$(n-1)T(r, L) + S(r, L) \leq nT(r, f) + S(r, f),$$

2)

$$(n-1)T(r, f) + S(r, f) \leq nT(r, L) + S(r, L),\ S(r, f) = S(r, L).$$

Proof. By Lemma 2.4, $T(r, L) = \frac{d_L}{\pi} r \log r + O(r)$, and therefore $\overline{N}(r, L) = S(r, L)$. On the other hand, we have

$$\overline{N}(r, f) = \sum_{0<|a_i|<r} \log \frac{r}{|a_i|} + \overline{n}(0, f) \log r,$$

where the a_i's are poles of f, ignoring multiplicity (see ([6], p. 14)). From this and f is a non-constant meromorphic function with finitely many poles it follows that $\overline{N}(r, f) = O(\log r)$, and so $\overline{N}(r, f) = S(r, L)$. Applying another form of the two Fundamental Theorems and noting that $\overline{E}_L(S) = \overline{E}_f(T)$, we obtain

$$(n-1)T(r, L) \leq \overline{N}(r, L) + \sum_{i=1}^{n} \overline{N}(r, \frac{1}{L-a_i}) + S(r, L),$$

$$\begin{aligned}(n-1)T(r, L) + S(r, L) &\leq \sum_{i=1}^{n} \overline{N}(r, \frac{1}{f-b_i}) + S(r, f)\\ &\leq nT(r, f) + S(r, f).\end{aligned}$$

Similarly

$$(n-1)T(r,f) \leq \overline{N}(r,f) + \sum_{i=1}^{n} \overline{N}(r, \frac{1}{f-b_i}) + S(r,f),$$

$$(n-1)T(r,f) + S(r,f) \leq \sum_{i=1}^{n} \overline{N}(r, \frac{1}{L-a_i}) + S(r,L),$$

$$(n-1)T(r,f) + S(r,f) \leq nT(r,L) + S(r,L).$$

Lemma 3.1 is proved.

We will prove that there is a constant $l \neq 0$ such that $P(L) = lQ(f)$. Set

$$F = \frac{1}{P(L)},\ G = \frac{1}{Q(f)}, H = \frac{F''}{F'} - \frac{G''}{G'},$$

$$S(r) = S(r,L) = S(r,f), T(r) = T(r,L) + T(r,f). \tag{3.4}$$

Then $T(r, P(L)) = nT(r,L) + O(1)$ and $T(r, Q(f)) = nT(r,f) + O(1)$, and hence $S(r, P(L)) = S(r,L)$ and $S(r, Q(f)) = S(r,f)$. We first prove that $H \equiv 0$. Suppose that $H \not\equiv 0$, on the contrary.

Claim 1. *We have*

i) $(n-1)T(r,L) \leq \overline{N}(r, \frac{1}{P(L)}) - N_o(r, \frac{1}{L'}) + S(r)$, *where* $N_o(r, \frac{1}{L'})$ *is the counting function of those zeros of* $L^{'}$, *which are not zeros of the function* $(L-a_1)(L-a_2)\cdots(L-a_n)$.

ii) $(n-1)T(r,f) \leq \overline{N}(r, \frac{1}{Q(f)}) - N_o(r, \frac{1}{f'}) + S(r)$, *where* $N_o(r, \frac{1}{f'})$ *is the counting function of those zeros of* $f^{'}$, *which are not zeros of the function* $(f-b_1)\cdots(f-b_n)$.

Proof. i) Applying another form of the two Fundamental Theorems to L and the values $a_1, a_2, ..., a_q$, and noticing that

$$N(r,L) = S(r,L),\ \sum_{i=1}^{n} \overline{N}(r, \frac{1}{L-a_i}) = \overline{N}(r, \frac{1}{P(L)}),$$

we obtain

$$(n-1)T(r,L) \leq \overline{N}(r,L) + \sum_{i=1}^{n} \overline{N}(r, \frac{1}{L-a_i}) - N_o(r, \frac{1}{L'}) + S(r,L),$$

$$(n-1)T(r,L) \leq \overline{N}(r, \frac{1}{P(L)}) - N_o(r, \frac{1}{L'}) + S(r).$$

ii) The inequality for f is proved by a similar argument.
Claim 1 is proved.

Claim 2. *We have*

$$N(r,H) \leq (m+1)T(r) + N_o(r,\frac{1}{f'}) + N_o(r,\frac{1}{L'}) + S(r).$$

Proof.
Noting that H has only simple poles, from Lemma 2.2 we obtain

$$\begin{aligned} N(r,H) \leq &\overline{N}_{(2}(r,P(f)) + \overline{N}_{(2}(r,P(L)) + \\ &\overline{N}(r,\frac{1}{P'(f)};P(f)\neq 0) + \overline{N}(r,\frac{1}{P'(L)};P(L)\neq 0) + S(r). \end{aligned}$$

On the other hand,

$$\overline{N}_{(2}(r,P(L)) = \overline{N}(r,L) = S(r), \overline{N}_{(2}(r,Q(f)) = \overline{N}(r,f) = S(r).$$

Moreover, we have

$$\begin{aligned} &\overline{N}(r,\frac{1}{[P(L)]'};P(L)\neq 0) \leq \\ &\quad \overline{N}(r,\frac{1}{L}) + \sum_{i=1}^{m}\overline{N}(r,\frac{1}{L-d_i};(L-a_1)\cdots(L-a_n)\neq 0) + N_o(r,\frac{1}{L'}) \\ &\quad \leq \overline{N}(r,\frac{1}{L}) + \sum_{i=1}^{m}\overline{N}(r,\frac{1}{L-d_i}) + N_o(r,\frac{1}{L'}) \leq (m+1)T(r,L) + N_o(r,\frac{1}{L'}) + S(r). \end{aligned}$$

Thus

$$\overline{N}(r,\frac{1}{[P(L)]'};P(L)\neq 0) \leq (m+1)T(r,L) + N_o(r,\frac{1}{L'}) + S(r).$$

Similarly,

$$\overline{N}(r,\frac{1}{[Q(f)]'};P(f)\neq 0) \leq (m+1)T(r,f) + N_o(r,\frac{1}{f'}) + S(r).$$

Claim 2 is proved.

Claim 3. *We have*

$$\overline{N}(r, \frac{1}{P(L)}) + \overline{N}(r, \frac{1}{Q(f)}) \leq (\frac{n}{2} + m + 2)T(r) + N_o(r, \frac{1}{L'}) + N_o(r, \frac{1}{f'}) + S(r).$$

Proof. Set $\nu_{P(L)}(z) = \nu_1, \nu_{Q(f)}(z) = \nu_2$. Then,

$$\overline{E}_{P(L)}(0, \nu_1 = 1) \cap \overline{E}_{Q(f)}(0, \nu_2 = 2) = \emptyset.$$

Indeed, suppose that $\overline{E}_{P(L)}(0, \nu_1 = 1) \cap \overline{E}_{Q(f)}(0, \nu_2 = 2) \neq \emptyset$, on the contrary. From this and $P(L) = a(L - a_1)...(L - a_n)$, $Q(f) = u(f - b_1)...(f - b_n)$ it follows that there exists a_i, b_j and z_0 such that $z_0 \in \overline{E}_L(a_i, \nu_{L-a_i} = 1)$ and $z_0 \in \overline{E}_f(b_j, \nu_{f-b_j} = 2)$. Therefore, $z_0 \in E_{f',1)}(0)$. Because $E_{f',1)}(0) = E_{L',1)}(0)$ we get $z_0 \in E_{L',1)}(0)$, which is a contradiction since $\nu_{L-a_i}(z_0) = 1$ and $(L - a_i)' = L'$. So

$$\overline{E}_{P(L)}(0, \nu_1 = 1) \cap \overline{E}_{Q(f)}(0, \nu_2 = 2) = \emptyset.$$

Similarly, we have

$$\overline{E}_{P(L)}(0, \nu_1 = 2) \cap \overline{E}_{Q(f)}(0, \nu_2 = 1) = \emptyset.$$

Now applying the Lemma 2.2 to the functions $P(L), Q(f)$ and noticing that

$$\overline{N}_{(2}(r, P(L)) = \overline{N}(r, L) = S(r), \overline{N}_{(2}(r, P(f)) = \overline{N}(r, f) = S(r),$$

we obtain

$$N(r, H) \leq \overline{N}(r, \frac{1}{P(L)}; \nu_1 > \nu_2 \geq 1) + \overline{N}(r, \frac{1}{Q(f)}; \nu_2 > \nu_1 \geq 1)$$

$$+\overline{N}(r, \frac{1}{[P(L)]'}; P(L) \neq 0) + \overline{N}(r, \frac{1}{[Q(f)]'}; Q(f) \neq 0) + S(r), \tag{3.5}$$

and

$$\overline{N}(r, \frac{1}{P(L)}) + \overline{N}(r, \frac{1}{Q(f)}) + 2\overline{N}(r, \frac{1}{P(L)}; \nu_1 \geq 2)+$$

$$2\overline{N}(r, \frac{1}{Q(f)}; \nu_2 \geq 2) \leq N(r, H) + \frac{1}{2}(N(r, \frac{1}{P(L)}) + N(r, \frac{1}{Q(f)}))+$$

$$N(r, \frac{1}{P(L)}; \nu_1 \geq 2) + N(r, \frac{1}{Q(f)}; \nu_2 \geq 2) + S(r). \tag{3.6}$$

From (3.5) and (3.6) and noticing that

$$\overline{N}(r, \frac{1}{P(L)}; \nu_1 > \nu_2 \geq 1) \leq \overline{N}(r, \frac{1}{P(L)}; \nu_1 \geq 2),$$

$$\overline{N}(r, \frac{1}{Q(f)}; \nu_2 > \nu_1 \geq 1) \leq \overline{N}(r, \frac{1}{Q(f)}; \nu_2 \geq 2),$$

we get

$$\overline{N}(r, \frac{1}{P(L)}) + \overline{N}(r, \frac{1}{Q(f)}) \leq (m+1)T(r) + \frac{1}{2}(N(r, \frac{1}{P(L)})+$$

$$N(r, \frac{1}{Q(f)})) + N(r, \frac{1}{P(L)}; \nu_1 \geq 2) - \overline{N}(r, \frac{1}{P(L)}; \nu_1 \geq 2)+$$

$$N(r, \frac{1}{Q(f)}; \nu_2 \geq 2) - \overline{N}(r, \frac{1}{Q(f)}; \nu_2 \geq 2) + N_o(r, \frac{1}{L'}) + N_o(r, \frac{1}{f'}) + S(r). \quad (3.7)$$

On the other hand, from $P(L) = a(L-a_1)...(L-a_n)$ it follows that if z_0 is a zero of $P(L)$ with multiplicity $d \geq 2$, then z_0 is a zero of $L - a_i$ with multiplicity $d \geq 2$ for some $i \in \{1, 2, ..., n\}$, and therefore, it is a zero of $L^{'}$ with multiplicity $d-1$, so we have

$$N(r, \frac{1}{P(L)}; \nu_1 \geq 2) - \overline{N}(r, \frac{1}{P(L)}; \nu_1 \geq 2) \leq N(r, \frac{1}{L'}).$$

From this and Lemma 2.1 we obtain

$$N(r, \frac{1}{P(L)}; \nu_1 \geq 2) - \overline{N}(r, \frac{1}{P(L)}; \nu_1 \geq 2) \leq N(r, \frac{1}{L'}) \leq$$

$$N(r, \frac{1}{L}) + \overline{N}(r, L) + S(r, L) \leq T(r, L) + S(r, L).$$

Similarly, we have

$$N(r, \frac{1}{Q(f)}; \nu_2 \geq 2) - \overline{N}(r, \frac{1}{Q(f)}; \nu_2 \geq 2) \leq N(r, \frac{1}{f'}) \leq T(r, f) + S(r, f).$$

Therefore,

$$N(r, \frac{1}{P(L)}; \nu_1 \geq 2) - \overline{N}(r, \frac{1}{P(L)}; \nu_1 \geq 2)+$$

$$N(r, \frac{1}{Q(f)}; \nu_2 \geq 2) - \overline{N}(r, \frac{1}{Q(f)}; \nu_2 \geq 2) \leq T(r) + S(r). \quad (3.8)$$

Combining (3.7)-(3.8) and noticing that $N(r, \frac{1}{P(L)}) + N(r, \frac{1}{Q(f)}) \leq \frac{n}{2}T(r)$ we get

$$\overline{N}(r, \frac{1}{P(L)}) + \overline{N}(r, \frac{1}{Q(f)}) \leq (\frac{n}{2} + m + 2)T(r) + N_o(r, \frac{1}{L'}) + N_o(r, \frac{1}{f'}) + S(r).$$

Claim 3 is proved.

Now we use Claims 1, 3 to obtain a contradiction, and complete the proof of $H \equiv 0$. Claim 1 and Claim 3 give us

$$(n-1)T(r) \leq (\frac{n}{2} + m + 2)T(r) + S(r),\ (n-2m-6)T(r) \leq S(r),$$

which is a contradiction since $n \geq 2m+7$.

So $H \equiv 0$. Therefore, $\frac{1}{Q(f)} = \frac{l}{P(L)} + c_1$ for some constants $l (\neq 0)$ and c_1. By Lemma 2.6 we obtain $c_1 = 0$. Thus there is a constant $l \neq 0$ such that $P(L) = lQ(f)$. That is

$$aL^n + bL^{n-m} + c = luf^n + lvf^{n-m} + lt. \tag{3.9}$$

Now we return proof the necessary condition of Theorem 1.

Then we have (3.9). Applying Lemma 2.7 to equation (3.9) we get

$$aL^n + bL^{n-m} + c = luf^n + lvf^{n-m} + lt,\ f = hL,\ l = \frac{c}{t}, h^n = \frac{a}{lu},\ h^m = \frac{lv}{b}\frac{at}{uc}.$$

Therefore $h^n = \frac{at}{cu}$, $h^m = \frac{av}{ub}$. Because $f = hL$, we have $E_{L',1)}(0) = E_{f',1)}(0)$. Now we prove $hS = T$. Take $a_i \in S, i = 1, ..., n$. By Lemma 2.3, there exists $z_0 \in \mathbb{C}$ such that $L(z_0) - a_i = 0$. Moreover, from $P(L) = lQ(f)$ and (3.2) we obtain

$$P(L) = a(L-a_1)...(L-a_n),\ u(f-b_1)...(f-b_n) = Q(f);$$

$$a(L-a_1)...(L-a_n) = lu(f-b_1)...(f-b_n). \tag{3.10}$$

By (3.10) we see that: $L(z_0) - a_i = 0$ if and only if z_0 is a zero of $P(L)$ and $L(z_0) - a_j \neq 0$ with $i \neq j$, and therefore, there exists a unique $b_k \in \{b_1, ..., b_n\}$ such that $f(z_0) - b_k = 0$. Because $f = hL$, we have $hL(z_0) - b_k = 0$, and therefore $ha_i = b_k$. From this and cardinalities of S, T are n it follows that $hS = T$.

The sufficient condition. Then we have:

$$P(L) = aL^n + bL^{n-m} + c,\ Q(f) = uf^n + vf^{n-m} + t,$$

$$f = hL, h^n = \frac{at}{cu}, h^m = \frac{av}{ub}, Q(f) = uh^nL^n + vh^{n-m}L^{n-m} + t.$$

Therefore,

$$E_{L',1)}(0) = E_{f',1)}(0),\ tP(L) = cQ(f) \text{ and } at(L-a_1)...(L-a_n) = cu(f-b_1)...(f-b_n). \tag{3.11}$$

From this it follows that $L^{-1}(S) = f^{-1}(T)$.

The necessary condition. Then, by 1/ we get: $f = hL$, where h is a non-zero constant satisfying $h^n = \dfrac{at}{cu}$, $h^m = \dfrac{av}{ub}$. If $f \in \mathcal{S}$, then $f = L$ from Lemma 2.5, and then $h = 1$, $\dfrac{a}{u} = \dfrac{b}{v} = \dfrac{c}{t}$.

The sufficient condition. The proof is completed by using the arguments similar to the ones in the sufficient condition of 1.

Theorem 1 is proved.

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