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UNIQUENESS OF $\mathcal{L}$-FUNCTION WITH CERTAIN POLYNOMIAL OF MEROMORPHIC FUNCTION SHARING A SMALL FUNCTION

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Abstract: This study investigates the uniqueness properties of $\mathcal{L}$-functions associated with meromorphic functions that share a small function of finite weight. Based on the Value Distribution Theory of Nevanlinna and the uniqueness properties of $\mathcal{L}$-functions, we establish several theorems that demonstrate the conditions under which two $\mathcal{L}$-functions can be considered equivalent, particularly in the context of their shared values and the behavior of associated polynomials. Our results extend, generalize, and improve those of Mandal and Datta [10]. We also provide an example that supports our results and poses open questions regarding the relaxation of conditions in uniqueness theorems.

Keywords and Phrases: Uniqueness, Meromorphic function, difference-differential polynomial, $\mathcal{L}$-function, Set sharing, Small function.

2020 Mathematics Subject Classification: 30D35.

1. Introduction and Main Results

For a long time a lot of attention have been given by many scholars on the Riemann hypothesis. At the outset, we assume that by an $\mathcal{L}$-function we always mean an $\mathcal{L}$-function $\mathfrak{L}$ in the Selberg class which includes the Riemann zeta function

$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and essentially those Dirichlet series where one might expect a Riemann hypothesis. Such an $\mathcal{L}$-function is defined [13, 14] to be a Dirichelet series

$$\mathfrak{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^{-s}} \tag{1.1}$$

satisfying the following axioms:

1. Ramanujan hypothesis : $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$;

2. Analytic continuation : There is a nonnegative integer m such that $(s-1)^m \mathfrak{L}(s)$ is an entire function of finite order;

3. Functional equation: $\mathfrak{L}$ satisfies a functional equation of type

$$\Lambda_{\mathfrak{L}}(s) = w\overline{\Lambda_{\mathfrak{L}}(1-\overline{s})}, \tag{1.2}$$

where

$$\Lambda_{\mathfrak{L}}(s) = \mathfrak{L}(s) Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \nu_j), \tag{1.3}$$

with positive real numbers Q, λ_j and complex numbers ν_j, w with $Re(\nu_j) \geq 0$ and $|w| = 1$;

4. Euler product hypothesis: $\log \mathfrak{L}(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

Also, throughout the paper, f and g are two meromorphic functions in the complex plane $\mathbb{C}$. To prove the main results, we will apply Nevanlinna's theory and adopt the standard notations, which are explained in [3, 8, 11, 15, 20]. For any $b \in \mathbb{C} \cup \{\infty\}$, the preimage of b under f is defined by $f^{-1}(b) = \{b \in \mathbb{C} : f(s) - b = 0\}$. The functions f and g are said to share the value b ignoring multiplicities (IM) if $f^{-1}(b) = g^{-1}(b)$ as two sets in $\mathbb{C}$. Furthermore, f and g are said to share b counting multiplicities (CM) if they share b IM and if each root of the equation $f(s) = b$ has exactly the same multiplicities as the root of the equation $f(s) = b$. Next, we set $E_{f(S)}$ for a set $S(\subset \mathbb{C} \cup \{\infty\})$ and a function f as

$$E_f(S) = \bigcup_{b \in S} \{s \in \mathbb{C} : f(s) - b = 0\},$$

where each zero of $f - b$ is counted according to their multiplicities, that is, $E_{f(S)}$ is a multiset. Also, we denote by $\overline{E}_f(S)$ the set of distinct elements in $E_f(S)$. We say that f and g share the set S CM if $E_f(S) = E_g(S)$ and that they share the set S IM if $\overline{E}_f(S) = \overline{E}_g(S)$. This paper deals with the uniqueness problems of value sharing and set sharing related to $\mathcal{L}$-functions and an arbitrary meromorphic function in $\mathbb{C}$.

Undoubtedly, $\mathcal{L}$-functions are crucial in number theory, and one can analytically continue an $\mathcal{L}$-function to a meromorphic function in $\mathbb{C}$. Hence, like the value distribution of meromorphic functions, the value distribution of $\mathcal{L}$-functions is a natural consequence. In this respect, a number of researchers have studied the distribution of zeros of the $\mathcal{L}$-function in great detail during the past few years [5, 16, 9, 13, 14, 21]. The aim of the work has eventually shifted to the uniqueness determination of an $\mathcal{L}$-function using shared values or sets. Hence, let us recall these basic definitions of value and set sharing.

Definition 1.1. [7] *Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_k(a, f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, then we say that f and g share the value a with weight k.*

The definition implies that if f and g share a value a with weight k, then z_0 is an a-points of f with multiplicity $m \leq k$ if and only if it is an a-ponts of f with multiplicity $m \leq k$, and z_0 is an a-points of f with multiplicity $m > k$ if and only if it is an a-ponts of f with multiplicity $n > k$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f and g share the value a with weight k. It is clear that if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p \leq k$. Also, note that f, g share the value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

Definition 1.2. [22] *For $q \in \mathbb{N}$ and $b \in \mathbb{C} \cup \{\infty\}$, we have*

$$\Theta(b, f) = 1 - \varlimsup_{r \to \infty} \frac{\overline{N}(r, b; f)}{T(r, f)} \quad \textit{and} \quad \delta_q(b, f) = 1 - \varlimsup_{r \to \infty} \frac{N_q(r, b; f)}{T(r, f)},$$

where $N_q(r, b; f) = \overline{N}(r, b; f) + \overline{N}_{(2}(r, b; f) + \ldots + \overline{N}_{(q}(r, b; f)$ and hence

$$0 \leq \delta(b, f) \leq \delta_q(b, f) \leq \delta_{q-1}(b, f) \leq \cdots \leq \delta_2(b, f) \leq \delta_1(b, f) = \Theta(b, f) \leq 1.$$

Definition 1.3. [2] *For $q, n \in \mathbb{N}$, we define*

$$\sigma_q = min\{q, n\} \quad \textit{and} \quad \sigma_q^* = q + 1 - \sigma_q.$$

Clearly, $N_q(r,0;f) \le \sigma_q N_{\sigma_q^*}(r,0;f)$.

Now, we recall the first result due to Wu and Hu [16].

Theorem 1.1. *Let* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$ *be two* $\mathcal{L}$*-functions, and* $\alpha_1, \alpha_2 \in \mathbb{C}$ *be two distinct values. Let* k_1, k_2 *are two positive integers satisfying* $k_1k_2 > 1$. *If* $E_{k_1}(\alpha_1, \mathfrak{L}_1) = E_{k_1}(\alpha_1, \mathfrak{L}_2)$ *and* $E_{k_2}(\alpha_2, \mathfrak{L}_1) = E_{k_2}(\alpha_2, \mathfrak{L}_2)$, *then* $\mathfrak{L}_1 \equiv \mathfrak{L}_2$.

Relating the uniqueness of a $\mathcal{L}$-function with an arbitrary meromorphic function, in 2018, Hao and Chen [4] investigated the following result:

Theorem 1.2. *Let* $\mathfrak{L}_1$ *be a* $\mathcal{L}$*-function and* F *be a meromorphic function defined in the complex plane* $\mathbb{C}$ *with finitely many poles. Let* $\alpha_1, \alpha_2 \in \mathbb{C}$ *be distinct and* $k_1, k_2 \in \mathbb{N}$ *satisfying* $k_1k_2 > 1$. *If* $E_{k_i}(\alpha_i, F) = E_{k_i}(\alpha_i, \mathfrak{L}_1)$, *for i= 1, 2, then* $\mathfrak{L}_1 \equiv F$

In 2020, Mandal and Datta [10], who studied the uniqueness result, shared a small function by considering an $\mathcal{L}$-function and a differential monomial that improves and extends the results of Hao and Chen [4] as follows:

Theorem 1.3. *Let* $\mathfrak{L}_1$ *be a non-constant* $\mathcal{L}$*-function and* $\eta(z)$ *be a small function of* $\mathfrak{L}_1$ *such that* $\eta \not\equiv 0, \infty$. *If* $\overline{E}_{4)}(\eta; \mathfrak{L}_1) = \overline{E}_{4)}(\eta, (\mathfrak{L}_1^m)^{(k)}), E_{2)}(\eta; \mathfrak{L}_1) = E_{2)}(\eta, (\mathfrak{L}_1^m)^{(k)})$ *and*

$$2N_{k+2}(r,0;\mathfrak{L}_1^m) \le (\sigma + O(1))T(r,\mathfrak{L}_1),$$

where $k \ge 1, m \ge 1 \in \mathbb{N}$, *and* $0 < \sigma < 1$, *then* $\mathfrak{L}_1 \equiv (\mathfrak{L}_1^m)^{(k)}$.

In 2023, Raj and Waghamore [12] introduced new definition on the difference-differential polynomial, which motivates us to write this paper.

Definition 1.4. [12] *Let* n_{ij}, m_{ij} *with* $(i = 0, 1, ..., k)$ *and* $(j = 1, 2, ..., t)$ *be non-negative integers and* $f(z)$ *be a non-constant meromorphic function. We shall define a general difference-differential monomial as follows*

$$M_j[f] = \prod_{i=0}^{k} [f^{(i)}(z)]^{n_{ij}} [f^{(i)}(z + c_i)]^{m_{ij}},$$

where $c_i's (i = 0, 1, ..., k)$ *are complex constants. Let* $d_{M_j} = \sum_{i=0}^{k} n_{ij} + m_{ij}$ *denote the degree of* $M_j[f]$ *and* $W_{M_j} = \sum_{i=0}^{k} (i+1)(n_{ij} + m_{ij})$ *denote the weight of* $M_j[f]$. *Then the expression*

$$P[f] = \sum_{j=1}^{t} a_j M_j[f], \tag{1.4}$$

where $T(r, a_j) = S(r, f)$ *for* $j = 1, 2, ..., t$ *is called the difference-differential polynomial generated by* f *of upper degree* $U_d(P) = max_{1\le j\le t}\{d_{M_j}\}$, *lower degree* $L_d(P) =$

$min_{1\leq j\leq t}\{d_{M_j}\}$, weight $W_P = max_{1\leq j\leq t}\{W_{M_j}\}$ and the order k (where k is the highest order of the derivative of f in $P[f]$). Let ϑ denote $max_{1\leq j\leq t}\{W_{M_j} - d_{M_j}\}$, i.e.,

$$\begin{aligned}\vartheta &= max_{1\leq j\leq t}\sum_{i=0}^{k}[(i+1)-1](n_{ij}+m_{ij})\\ &= max_{1\leq j\leq t}(n_{1j}+m_{1j}+2n_{2j}+2m_{2j}+\ldots+kn_{kj}+km_{kj}).\end{aligned}$$

Definition 1.5. *Let*

$$\begin{aligned}\mathcal{Q}(z) &= b_{m+n}z^{m+n}+\ldots+b_nz^n+\ldots+b_0\\ &= b_{m+n}\prod_{j=1}^{r}(z-z_{q_j})^{q_j},\end{aligned}$$

where $b_i(i=0,1,2,...,n+m-1), b_{n+m}\neq 0$ and $z_{q_j}(j=1,2,...,r)$ are distinct finite complex numbers and $n+m\geq r\geq 2$ and $q_1,q_2,...,q_r, r\geq 2, n, m$ and k are all positive integers with $\sum_{j=1}^{r} q_j = n+m$. Also let $q > max_{q\neq q_j, j=1,2,...,s}\{q_j\}, s = r-1$, where r and s are two positive integers.

Let

$$\begin{aligned}\mathcal{P}(z_1) &= b_{n+m}\prod_{j=1}^{r-1}(z_1+z_q-z_{q_j})^{q_j}\\ &= a_pz_1^p+a_{p-1}z_1^{p-1}+\ldots+a_0,\end{aligned}$$

where $b_{n+m}=a_p, z_1=z-z_q, p=n+m-q$. Therefore, $\mathcal{Q}(z)=z_1^q\mathcal{P}(z_1)$.
Next we assume

$$\mathcal{P}(z_1)=a_p\prod_{j=1}^{s}(z_1-\beta_j)^{q_j}$$

where $\beta_j = z_{q_j}-z_q, (j=1,2,...,s)$, be distinct zeros of $\mathcal{P}(z_1)$.

It is important to recognize that no research has been undertaken on difference-differential polynomials in a broader context. As a result, in order to broaden the applicability of Theorem 1.3, some unavoidable problems arise. Given the data reported by Mandal and Datta before, it is natural to ask the following question, which provides the inspiration behind this paper.

Question: Can an analogous uniqueness result be obtained by incorporating $\mathfrak{L}_1$ with $\mathscr{P}_n(\mathfrak{L}_1)$, $\mathcal{Q}(\mathfrak{L}_1)$, and $(\mathfrak{L}_1^m)^{(k)}$ with $P[\mathfrak{L}_1]$?

We demonstrate the following theorems with respect to the preceding question:

Theorem 1.4. *Let $\mathfrak{L}_1$ be a non-constant $\mathcal{L}$-function, $\mathscr{P}_n(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z$, be a non-zero polynomial, where $a_j \in \mathbb{C}$ for $j = 1, 2, \cdots, n$ with $a_n \neq 0$ and m be the number of distinct zeros of $\mathscr{P}_n(z)$. Let $\varphi(z)(\neq 0, \infty)$ be a small function of $\mathfrak{L}_1, k$ be a positive integer, and let $P[\mathfrak{L}_1]$ be a difference-differential polynomial of $\mathfrak{L}_1$. If $n \geq m+1$, then $\mathscr{P}_n(\mathfrak{L}_1)P[\mathfrak{L}_1] - \varphi$ has infinitely many zeros.*

Theorem 1.5. *Let $\mathfrak{L}_1$ be a non-constant $\mathcal{L}$-function. Let $\varphi(z)(\neq 0, \infty)$ be a small function of $\mathfrak{L}_1, k$ be a positive integer, and let $P[\mathfrak{L}_1]$ be a difference-differential polynomial of $\mathfrak{L}_1$. Suppose $\overline{E}_{4)}(\varphi; \mathfrak{L}_1) = \overline{E}_{4)}(\varphi; P[\mathfrak{L}_1]), E_{2)}(\varphi; \mathfrak{L}_1) = E_{2)}(\varphi; P[\mathfrak{L}_1])$ and*

$$\left(\frac{(k+3)U_d(P) + 4}{1 + U_d(P)}\right) N(r, 0; \mathfrak{L}_1) < (\lambda + O(1))T(r, \mathfrak{L}_1),$$

where $0 < \lambda < 1$, then $\mathfrak{L}_1 \equiv P[\mathfrak{L}_1]$.

Example 1.1. Let $\mathfrak{L}_1 = \sum\limits_{n=1}^{\infty} \frac{1}{n^z}$, $P[\mathfrak{L}_1] = \sum\limits_{n=1}^{\infty} \frac{1}{n^z} + \sum\limits_{n=1}^{\infty} \frac{1}{(n+c)^z}$, where c is a non-zero constant $k = 1$, and $U_d(P) = 2$. Then we can see that $\mathfrak{L}_1$ and $P[\mathfrak{L}_1]$ share 1 and ∞ CM, but the conclusion of Theorem 1.5 does not hold, which shows that the conditions given in the theorem are necessary but not sufficient.

Theorem 1.6. *Let $\mathfrak{L}_1$ be a non-constant $\mathcal{L}$-function, $\mathscr{P}_n(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z$, be a non-zero polynomial, where $a_j \in \mathbb{C}$ for $j = 1, 2, \cdots, n$ with $a_n \neq 0$ and m be the number of distinct zeros of $\mathscr{P}_n(z)$. Let $\varphi(z)(\neq 0, \infty)$ be a small function of $\mathfrak{L}_1, k$ be a positive integer, and let $P[\mathfrak{L}_1]$ be a difference-differential polynomial of $\mathfrak{L}_1$. Suppose $\overline{E}_{4)}(\varphi; \mathscr{P}_n(\mathfrak{L}_1)) = \overline{E}_{4)}(\varphi; P[\mathfrak{L}_1]), E_{2)}(\varphi; \mathscr{P}_n(\mathfrak{L}_1)) = E_{2)}(\varphi; P[\mathfrak{L}_1])$ and*

$$\left(\frac{(k+3)U_d(P) + 4m}{n + U_d(P)}\right) N(r, 0; \mathfrak{L}_1) < T(r, \mathfrak{L}_1),$$

then $\mathscr{P}_n(\mathfrak{L}_1) \equiv P[\mathfrak{L}_1]$.

Theorem 1.7. *Let $\mathfrak{L}_1$ be a non-constant $\mathcal{L}$-function. Let $\varphi(z)(\neq 0, \infty)$ be a small function of $\mathfrak{L}_1$, $\mathcal{Q}(z) = b_{m+n} z^{m+n} + \ldots + b_n z^n + \ldots + b_0, b_{m+n} \neq 0$, be a polynomial of degree $(m+n)$ such that $\mathcal{Q}(f) = f_1^q \mathcal{P}(f_1)$. Suppose $\overline{E}_{4)}(\varphi; \mathcal{Q}(\mathfrak{L}_1)) = \overline{E}_{4)}(\varphi; P[\mathfrak{L}_1]), E_{2)}(\varphi; \mathcal{Q}(\mathfrak{L}_1)) = E_{2)}(\varphi; P[\mathfrak{L}_1])$ and*

$$2\sigma_2 \delta_{\sigma_2^*}(z_q, \mathfrak{L}_1) + U_d(P)\delta(0, \mathfrak{L}_1) + (k+2)U_d(P)\Theta(0, \mathfrak{L}_1) > n + m + 2\sigma_2 + (k+2)U_d(P) - q,$$

then $\mathcal{Q}(\mathfrak{L}_1) \equiv P[\mathfrak{L}_1]$.

Remark 1.1.

1. Theorems 1.4, 1.5, 1.6, and 1.7 directly improves, extends Theorem 1.3 by extending from differential polynomial to difference-differential polynomial.

2. If we choose $m_{ij} = 0$ in Definition 1.4, the difference-differential polynomial reduces to the differential polynomial, yielding three equivalent results; rather than proving these results, we ask an open question at the end of the paper.

2. Auxiliary Lemmas

In this portion, we will introduce certain lemmas that will be utilized to prove the main results. Let $\mathscr{F}$ and $\mathscr{G}$ be two non-constant meromorphic functions. Henceforth, we shall denote by $\mathscr{H}$ the following function.

$$\mathscr{H} = \left(\frac{\mathscr{F}''}{\mathscr{F}'} - \frac{2\mathscr{F}'}{\mathscr{F}-1}\right) - \left(\frac{\mathscr{G}''}{\mathscr{G}'} - \frac{2\mathscr{G}'}{\mathscr{G}-1}\right). \tag{2.1}$$

Lemma 2.1. [1] *If* $\overline{E}_{4)}(1;\mathscr{F}) = \overline{E}_{4)}(1;\mathscr{G}), E_{2)}(1;\mathscr{F}) = E_{2)}(1;\mathscr{G})$ *and* $\mathscr{H} \not\equiv 0$, *then*

$$\begin{aligned} T(r,\mathscr{F}) + T(r,\mathscr{G}) \leq 2\{&N_2(r,0;\mathscr{F}) + N_2(r,\infty;\mathscr{F}) + N_2(r,0;\mathscr{G}) + N_2(r,\infty;\mathscr{G})\} \\ &+ S(r,\mathscr{F}) + S(r,\mathscr{G}). \end{aligned}$$

Lemma 2.2. [12] *Suppose f is a non-constant meromorphic function of finite order, and $P[f]$ is a difference-differential polynomial in f. Then*

$$m\left(r, \frac{P[f]}{f^{U_d(P)}}\right) \leq \{U_d(P) - L_d(P)\} m\left(r, \frac{1}{f}\right) + S(r,f).$$

Lemma 2.3. [12] *Suppose f is a non-constant meromorphic function of finite order, and $P[f]$ is a difference-differential polynomial in f. Then*

$$N\left(r, P[f]\right) \leq U_d(P) N(r,f) + \vartheta \overline{N}(r,f) + S(r,f).$$

Lemma 2.4. [12] *Suppose f is a non-constant meromorphic function of finite order, and $P[f]$ is a difference-differential polynomial in f. Then*

$$\begin{aligned} N\left(r, \frac{P[f]}{f^{U_d(P)}}\right) \leq \vartheta\left(\overline{N}(r,f) + \overline{N}(r,\frac{1}{f})\right) + \sum_{j=1}^{k}\sum_{i=0}^{k} m_{ij}\overline{N}(r,f) \\ + \{U_d(P) - L_d(P)\} N\left(r, \frac{1}{f}\right) + S(r,f). \end{aligned}$$

Lemma 2.5. [12] *Suppose f is a non-constant meromorphic function of finite order, and $P[f]$ is a difference-differential polynomial in f. Then*

$$N\left(r,\frac{1}{P[f]}\right)\le \vartheta\overline{N}(r,f)+\{U_d(P)-L_d(P)\}m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{U_d(P)}}\right)+S(r,f).$$

Lemma 2.6. [12] *Suppose f is a non-constant meromorphic function of finite order, and $P[f]$ is a difference-differential polynomial in f. Then*

$$N_q\left(r,\frac{1}{P[f]}\right)\le U_d(P)N_{k+q}\left(r,\frac{1}{f}\right)+\vartheta\overline{N}(r,f)+S(r,f).$$

Lemma 2.7. [19] *Let $f(z)$ be a non-constant meromorphic function in the complex plane, and let*

$$\mathscr{P}_n(f(z))=a_nf^n(z)+a_{n-1}f^{n-1}(z)+...+a_1f(z)+a_0, \tag{2.2}$$

where $a_0,a_1,...,a_n$ are constants and $a_n\neq 0$. Then $T(r,\mathscr{P}_n(f))=n\,T(r,f)+O(1)$.

Lemma 2.8. [14] *Let $\mathfrak{L}$ be a $\mathcal{L}$-function with degree d. Then*

$$T(r,\mathfrak{L})=\frac{d}{\pi}r\log r+O(r).$$

Lemma 2.9. [10] *Let $\mathfrak{L}$ be a $\mathcal{L}$-function. Then $N(r,\infty;\mathfrak{L})=S(r,\mathfrak{L})$.*

Lemma 2.10. *Suppose f is a non-constant meromorphic function and $P[f]$ is a difference-differential polynomial of f. Then*

$$\begin{aligned}
&T(r,P[f])\le U_d(P)T(r,f)+\sigma\overline{N}(r,f)+S(r,f),\\
&N\left(r,\frac{1}{P[f]}\right)\le T(r,P[f])-T\left(r,\frac{1}{f^{U_d(P)}}\right)+N\left(r,\frac{1}{f^{U_d(P)}}\right)+S(r,f),\\
&N\left(r,\frac{1}{P[f]}\right)\le U_d(P)N\left(r,\frac{1}{f}\right)+\sigma\overline{N}(r,f)+S(r,f).
\end{aligned}$$

Proof. From Nevanlinna's first fundamental theorem, we have

$$N\left(r,\frac{1}{P[f]}\right)=T(r,P[f])-m\left(r,\frac{1}{P[f]}\right)+O(1). \tag{2.3}$$

Also, we have

$$m\left(r,\frac{1}{f^{U_d(P)}}\right)\le m\left(r,\frac{P[f]}{f^{U_d(P)}}\right)+m\left(r,\frac{1}{P[f]}\right), \tag{2.4}$$

which implies that

$$m\left(r, \frac{1}{f^{U_d(P)}}\right) \leq m\left(r, \frac{1}{P[f]}\right) + S(r,f), \tag{2.5}$$

which further implies that

$$-m\left(r, \frac{1}{P[f]}\right) \leq -m\left(r, \frac{1}{f^{U_d(P)}}\right) + S(r,f). \tag{2.6}$$

Using (2.6) in (2.3), we get

$$N\left(r, \frac{1}{P[f]}\right) \leq T(r, P[f]) - T\left(r, \frac{1}{f^{U_d(P)}}\right) + N\left(r, \frac{1}{f^{U_d(P)}}\right) + S(r,f). \tag{2.7}$$

Since

$$\begin{aligned} T(r, P[f]) &= m(r, P[f]) + N\left(r, P[f]\right) + O(1) \\ &\leq m\left(r, \frac{P[f]}{f^{U_d(P)}}\right) + m\left(r, f^{U_d(P)}\right) + N(r, P[f]) \\ &\leq m\left(r, f^{U_d(P)}\right) + U_d(P)N\left(r, f\right) + \sigma\overline{N}(r,f) + S(r,f) \\ &\leq U_d(P)T(r,f) + \sigma\overline{N}(r,f) + S(r,f). \end{aligned} \tag{2.8}$$

Substituting (2.8) in (2.7), we get

$$N\left(r, \frac{1}{P[f]}\right) \leq U_d(P)T(r,f) + \sigma\overline{N}(r,f) - T\left(r, \frac{1}{f^{U_d(P)}}\right) + N\left(r, \frac{1}{f^{U_d(P)}}\right) + S(r,f),$$

implies that

$$N\left(r, \frac{1}{P[f]}\right) \leq U_d(P)N(r, \frac{1}{f}) + \sigma\overline{N}(r,f) + S(r,f).$$

Lemma 2.11. [18] *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Then*

$$N\left(r, \infty; \frac{f}{g}\right) - N\left(r, \infty; \frac{g}{f}\right) = N(r,\infty;f) + N(r,0;g) - N(r,\infty;g) - N(r,0;f).$$

Lemma 2.12. *Let $f(z)$ be a transcendental entire function of finite order, $\mathscr{P}_n(z)$ be a non-zero polynomial as defined in Theorem 1.4, and $P[f]$ is a difference-differential polynomial of $f(z)$. Set $\mathcal{F}_1 = \mathscr{P}_n(f)P[f]$. Then, we have*

$$nT(r,f) \leq T(r, \mathcal{F}_1) - N(r, 0; P[f]) + S(r,f).$$

Proof. Note that by Lemma 2.9 and 2.11, we have

$$\begin{aligned} m(r,f^{n+1}) &= m(r,\mathscr{P}_n(f)P[f]) + m\left(r,\frac{f}{P[f]}\right) + S(r,f) \\ &\leq m(r,\mathcal{F}_1) + T\left(r,\frac{f}{P[f]}\right) - N\left(r,\infty;\frac{f}{P[f]}\right) + S(r,f) \\ &\leq m(r,\mathcal{F}_1) + T\left(r,\frac{P[f]}{f}\right) - N\left(r,\infty;\frac{f}{P[f]}\right) + S(r,f) \\ &\leq m(r,\mathcal{F}_1) + N\left(r,\infty;\frac{P[f]}{f}\right) + m\left(r,\frac{P[f]}{f}\right) - N\left(r,\infty;\frac{f}{P[f]}\right) + S(r,f) \\ &\leq m(r,\mathcal{F}_1) + N\left(r,\infty;P[f]\right) + N(r,0;f) - N(r,\infty;f) - N\left(r,0;P[f]\right) + S(r,f) \\ &\leq m(r,\mathcal{F}_1) + N(r,0;f) - N\left(r,0;P[f]\right) + S(r,f). \end{aligned}$$

By Lemma 2.7, we get

$$(n+1)T(r,f) = m(r,f^{n+1}) \leq T(r,\mathcal{F}_1) + T(r,f) - N\left(r,0;P[f]\right) + S(r,f),$$

i.e.,

$$nT(r,f) \leq T(r,\mathcal{F}_1) - N\left(r,0;P[f]\right) + S(r,f).$$

This completes the Lemma.

3. Proof of Theorems

3.1. Proof of Theorem 1.4.

Let $\mathcal{F}_1(z) = \frac{\mathscr{P}_n(\mathfrak{L}_1)P[\mathfrak{L}_1]}{\varphi(z)}$. In view of Lemma 2.12 and by the second theorem for small functions (see [17]), we get

$$\begin{aligned} nT(r,\mathfrak{L}_1) &\leq T(r,\mathcal{F}_1) - N\left(r,0;P[\mathfrak{L}_1]\right) + S(r,\mathfrak{L}_1) \\ &\leq \overline{N}(r,0;\mathcal{F}_1) - \overline{N}(r,\infty;\mathcal{F}_1) + \overline{N}(r,0;\mathcal{F}_1-\varphi) - N(r,0;P[\mathfrak{L}_1]) + S(r,\mathfrak{L}_1) \\ &\leq \overline{N}(r,0;\mathscr{P}_n(\mathfrak{L}_1)) + \overline{N}(r,0;P[\mathfrak{L}_1]) + \overline{N}(r,0;\mathcal{F}_1-\varphi) - N(r,0;P[\mathfrak{L}_1]) + S(r,\mathfrak{L}_1) \\ &\leq mT(r,\mathfrak{L}_1) + \overline{N}(r,0;\mathcal{F}_1-\varphi) + S(r,\mathfrak{L}_1), \end{aligned}$$

which is a contradiction, since $n \geq m+1$, from above one can easily say that $\mathcal{F}_1 - \varphi$ has infinitely many zeros. This completes the proof.

3.2. Proof of Theorem 1.5.

Proof. Let $\mathscr{L} = \frac{\mathfrak{L}_1(z)}{\varphi(z)}$ and $\mathscr{D} = \frac{P[\mathfrak{L}_1]}{\varphi(z)}$. Clearly, we have $\overline{E}_{4)}(1,\mathscr{L}) = \overline{E}_{4)}(1,\mathscr{D})$ and $E_{2)}(1,\mathscr{L}) = E_{2)}(1,\mathscr{D})$ except for the poles and zeros of $\varphi(z)$.

Now, we shall investigate the following two cases:
Case 1: If $\mathscr{H} \not\equiv 0$. Then from Lemma 2.1, we write

$$T(r,\mathscr{L}) + T(r,\mathscr{D}) \leq 2\{N_2(r,0;\mathscr{L}) + N_2(r,\infty;\mathscr{L}) + N_2(r,0;\mathscr{D}) + N_2(r,\infty;\mathscr{D})\} + S(r,\mathscr{L}) + S(r,\mathscr{D}).$$

Using Lemmas 2.7, 2.9 and 2.10, we write

$$\begin{aligned} T(r,\mathfrak{L}_1) + T(r,P[\mathfrak{L}_1]) &\leq 2\{N_2(r,0;\mathfrak{L}_1) + N_2(r,\infty;\mathfrak{L}_1) + N_2(r,0;P[\mathfrak{L}_1]) \\ &\quad + N_2(r,\infty;P[\mathfrak{L}_1])\} + S(r,\mathfrak{L}_1) + S(r,P[\mathfrak{L}_1]) \\ &\leq 2N_2(r,0;\mathfrak{L}_1) + T(r,P[\mathfrak{L}_1]) - T\left(r,\frac{1}{\mathfrak{L}_1{}^{U_d(P)}}\right) + N\left(r,\frac{1}{\mathfrak{L}_1{}^{U_d(P)}}\right) \\ &\quad + U_d(P)N_{k+2}(r,0;\mathfrak{L}_1) + \sigma\overline{N}(r,\infty;\mathfrak{L}_1) + S(r,\mathfrak{L}_1), \end{aligned}$$

which implies that

$$\begin{aligned} (U_d(P)+1)T(r,\mathfrak{L}_1) &\leq 2N_2(r,0;\mathfrak{L}_1) + U_d(P)N(r,0;\mathfrak{L}_1) + (k+2)U_d(P)\overline{N}(r,0;\mathfrak{L}_1) \\ &\quad + S(r,\mathfrak{L}_1) \\ &\leq 4\overline{N}(r,0;\mathfrak{L}_1) + U_d(P)N(r,0;\mathfrak{L}_1) + (k+2)U_d(P)\overline{N}(r,0;\mathfrak{L}_1) \\ &\quad + S(r,\mathfrak{L}_1) \\ &\leq ((k+2)U_d(P)+4)\overline{N}(r,0;\mathfrak{L}_1) + U_d(P)N(r,0;\mathfrak{L}_1) + S(r,\mathfrak{L}_1) \\ &\leq ((k+3)U_d(P)+4)N(r,0;\mathfrak{L}_1) + S(r,\mathfrak{L}_1), \end{aligned}$$

which contradicts

$$T(r,\mathfrak{L}_1) > \left(\frac{(k+3)U_d(P)+4}{1+U_{d(P)}}\right)N(r,0;\mathfrak{L}_1). \tag{3.1}$$

Case 2: Suppose $\mathscr{H} \equiv 0$. Then we get

$$\left(\frac{\mathscr{L}''}{\mathscr{L}'} - \frac{2\mathscr{L}'}{\mathscr{L}-1}\right) \equiv \left(\frac{\mathscr{D}''}{\mathscr{D}'} - \frac{2\mathscr{D}'}{\mathscr{D}-1}\right). \tag{3.2}$$

By integrating twice the (3.2), we get

$$\frac{1}{\mathscr{L}-1} = \frac{a}{\mathscr{D}-1} + b, \tag{3.3}$$

where a and b are constants and $a \neq 0$. From (3.3), it is easy to see that $\mathscr{L}$ and $\mathscr{D}$ share 1 CM. We claim that $b = 0$. Suppose if $b \neq 0$. Then from (3.3), we get

$$\frac{1}{\mathscr{L}-1} = \frac{b(\mathscr{D}-1+\frac{a}{b})}{\mathscr{D}-1}. \tag{3.4}$$

From (3.4), we clearly have,

$$\overline{N}\left(r,0;\mathscr{D}-1+\frac{a}{b}\right)=\overline{N}(r,\infty;\mathscr{L})=S(r,\mathfrak{L}_1). \tag{3.5}$$

Suppose $a\neq b$, then from Nevanlinna's Fundamental Theorem-II, (3.5), and Lemma 2.9, we have

$$\begin{aligned} T(r,\mathscr{D}) &\leq \overline{N}\left(r,\infty;\mathscr{D}\right)+\overline{N}\left(r,0;\mathscr{D}\right)+\overline{N}\left(r,0;\mathscr{D}-1+\frac{a}{b}\right)+S(r,\mathscr{D})\\ &\leq \overline{N}\left(r,0;\mathscr{D}\right)+S(r,\mathfrak{L}_1)\\ &\leq T(r,\mathscr{D})+S(r,\mathfrak{L}_1). \end{aligned} \tag{3.6}$$

Thus from Lemma 2.6, 2.9, 2.10, and (3.6), we get

$$\begin{aligned} T(r,\mathscr{D}) &= \overline{N}\left(r,0;\mathscr{D}\right)+S(r,\mathfrak{L}_1)\\ &\leq N\left(r,0;\mathscr{D}\right)+S(r,\mathfrak{L}_1)\\ &\leq T(r,\mathscr{D})-T\left(r,\frac{1}{{\mathfrak{L}_1}^{U_d(P)}}\right)+N(r,0;{\mathfrak{L}_1}^{U_d(P)})+S(r,\mathfrak{L}_1), \end{aligned}$$

which implies $T(r,\mathfrak{L}_1)\leq N(r,0;\mathfrak{L}_1)+S(r,\mathfrak{L}_1)$, which contradicts (3.1).

Hence, if $a=b$, then from (3.3), we can write

$$\frac{-\varphi^2(z)}{\mathfrak{L}_1(b\mathfrak{L}_1-b\varphi-\varphi)}\equiv\frac{P[\mathfrak{L}_1]}{\mathfrak{L}_1},$$

Thus, by (3.3), Lemma 2.7 and Lemma 2.9, we obtain

$$\frac{P[\mathfrak{L}_1]}{{\mathfrak{L}_1}^{U_d(P)}}\equiv\frac{-\varphi^2(z)}{{\mathfrak{L}_1}^{U_d(P)}(b\mathfrak{L}_1-b\varphi-\varphi)}. \tag{3.7}$$

Thus, by (3.7), Lemmas 2.2, 2.4, 2.7, and 2.9, we obtain

$$\begin{aligned} (U_d(P)+1)T(r,\mathfrak{L}_1) &= T\left(r,\frac{P[\mathfrak{L}_1]}{{\mathfrak{L}_1}^{U_d(P)}}\right)+S(r,\mathfrak{L}_1)\\ &\leq N\left(r,\frac{P[\mathfrak{L}_1]}{{\mathfrak{L}_1}^{U_d(P)}}\right)+m\left(r,\frac{P[\mathfrak{L}_1]}{{\mathfrak{L}_1}^{U_d(P)}}\right)+S(r,\mathfrak{L}_1)\\ &\leq \vartheta\left(\overline{N}(r,\mathfrak{L}_1)+\overline{N}(r,\frac{1}{\mathfrak{L}_1})\right)+\sum_{j=1}^{k}\sum_{i=0}^{k}m_{ij}\overline{N}(r,\mathfrak{L}_1)\\ &\quad+\{U_d(P)-L_d(P)\}N\left(r,\frac{1}{\mathfrak{L}_1}\right)+\{U_d(P)-L_d(P)\}m\left(r,\frac{1}{\mathfrak{L}_1}\right) \end{aligned}$$

$$
\begin{aligned}
&+ S(r,\mathfrak{L}_1)\\
&\leq \{U_d(P)-L_d(P)\}T(r,\mathfrak{L}_1)+\vartheta N\left(r,\frac{1}{\mathfrak{L}_1}\right)+S(r,\mathfrak{L}_1),
\end{aligned}
$$

which implies that

$$
T(r,\mathfrak{L}_1)\leq \frac{\vartheta}{(L_d(P)+1)}N\left(r,\frac{1}{\mathfrak{L}_1}\right)+S(r,\mathfrak{L}_1),
$$

which is impossible. Hence $b=0$ and so from (3.3), we get

$$
\frac{\mathscr{D}-1}{\mathscr{L}-1}\equiv a. \tag{3.8}
$$

Suppose $a\neq 1$, then from (3.8), we get

$$
\overline{N}(r,0;\mathscr{D}+a-1)=\overline{N}(r,0;\mathscr{L}). \tag{3.9}
$$

Now, from Nevanlinna's Second Fundamental Theorem-II, Lemma 2.9, 2.10, and (3.9), we obtain

$$
\begin{aligned}
T(r,\mathscr{D})&\leq \overline{N}\left(r,\infty;\mathscr{D}\right)+\overline{N}\left(r,0;\mathscr{D}\right)+\overline{N}\left(r,0;\mathscr{D}-1+a\right)+S(r,\mathscr{D})\\
&\leq N\left(r,0;P[\mathfrak{L}_1]\right)+\overline{N}\left(r,0;\mathfrak{L}_1\right)+S(r,\mathfrak{L}_1)\\
&\leq T(r,P[\mathfrak{L}_1])-T\left(r,\frac{1}{{\mathfrak{L}_1}^{U_d(P)}}\right)+N\left(r,\frac{1}{{\mathfrak{L}_1}^{U_d(P)}}\right)+\overline{N}\left(r,0;\mathfrak{L}_1\right)+S(r,\mathfrak{L}_1),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
U_d(P)T(r,\mathfrak{L}_1)&\leq U_d(P)N\left(r,0;\mathfrak{L}_1\right)+\overline{N}\left(r,0;\mathfrak{L}_1\right)+S(r,\mathfrak{L}_1)\\
&\leq (U_d(P)+1)N\left(r,0;\mathfrak{L}_1\right),
\end{aligned}
$$

which contradicts (3.1). Therefore, $a=1$ and hence from (3.8), we get $\mathfrak{L}_1\equiv P[\mathfrak{L}_1]$. Which completes the proof.

3.3. Proof of Theorems 1.6 and 1.7.

Proof. Similar to the proof of Theorem 1.5, Theorems 1.6 and 1.7 can be demonstrated.

4. Conclusion

We studied the uniqueness problems of an $\mathcal{L}$-function sharing a small function with a finite weight with an intention to determine whether the analogues uniqueness results of Mandal and Datta [10] be obtained by incorporating $\mathfrak{L}_1$ with $\mathscr{P}_n(\mathfrak{L}_1)$,

$\mathcal{Q}(\mathfrak{L}_1)$, and $(\mathfrak{L}_1^m)^{(k)}$ with $P[\mathfrak{L}_1]$. The main results of this study are Theorems 1.4, 1.5, 1.6, and 1.7, which mostly focus on the uniqueness properties of $\mathcal{L}$- functions with some difference-differential polynomials generated by a non-constant $\mathcal{L}$- function. The results improve upon previous results in terms of generalizing the existing result. To study further, we can pose the following open questions:

Open Questions:

1. Is it possible to further relax the conditions in all the theorems by examining them in relation to differential polynomials, as demonstrated in [6] ?

2. Can we study all the results for difference-differential monomial

$$M_j[f] = \prod_{i=0}^{k} [f^{(i)}(z)]^{n_{ij}} [\Delta_{c_i} f^{(i)}(z)]^{m_{ij}}$$

   or a difference-differential polynomial ?

3. Can we study all the results for the homogeneous difference-differential polynomial as in Definition 1.4 ?

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