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ON THE DIFFERENCE OF INITIAL LOGARITHMIC COEFFICIENTS FOR THE CLASS OF UNIVALENT FUNCTIONS

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Abstract: In this paper we give estimates of the differences $|\gamma_3|-|\gamma_2|$ and $|\gamma_4|-|\gamma_3|$ for the class of functions f univalent in the unit disc and normalized by $f(0) = f'(0) - 1 = 0$. Here, γ_2, γ_3 and γ_4 are the initial logarithmic coefficients of the function f.

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1. Introduction and Preliminaries

As usual, let $\mathcal{A}$ be the class of functions f that are analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ of the form

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots , \tag{1}$$

and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D}$.

One of the problems in the theory of univalent functions is finding sharp estimates of logarithmic coefficient, γ_n, of a univalent function $f \in \mathcal{S}$, defined by

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n. \tag{2}$$

From the relations (1) and (2), after equating the coefficients we receive the next initial logarithmic coefficients:

$$\gamma_1 = \frac{a_2}{2}, \quad \gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \quad \text{and} \quad \gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \tag{3}$$

Relatively little exact information is known about the logarithmic coefficients. The natural conjecture $|\gamma_n| \le 1/n$, inspired by the Koebe function (whose logarithmic coefficients are $1/n$) is false even in order of magnitude (see Duren [3, Section 8.1]), and true only for the class of starlike functions ([11]). For the class $\mathcal{S}$ the sharp estimates of single logarithmic coefficients are known only for γ_1 and γ_2, namely,

$$|\gamma_1| \le 1 \quad \text{and} \quad |\gamma_2| \le \frac{1}{2} + \frac{1}{e} = 0.635\ldots.$$

In their papers [7] and [8], the authors gave estimates $|\gamma_3| \le 0.5566178\ldots$ and $|\gamma_4| \le 0.51059\ldots$ for the class $\mathcal{S}$. Another reference on this topic is [10] with variety of valuable results presented within.

In this paper we will give the estimates for the differences $|\gamma_3|-|\gamma_2|$ and $|\gamma_4|-|\gamma_3|$ in the class $\mathcal{S}$. The sharp estimates of $|\gamma_2| - |\gamma_1|$ were established in [6], with a simple proof given in [9] using different technique. The difference of the modulus of two consecutive logarithmic coefficients gives a picture of their distribution and is further useful for understanding their nature and estimating individual logarithmic coefficients.

In this paper our main tool will be a method based on Grunsky's coefficients, used previously in [7] and [8]. This method is different than the method used in [6], or in [9].

We will use mainly the notations and results given in the book of N. A. Lebedev ([4]).

Let $f \in \mathcal{S}$ and let

$$\log \frac{f(t) - f(z)}{t - z} = \sum_{p,q=0}^{\infty} \omega_{p,q} t^p z^q,$$

where $\omega_{p,q}$ are so-called Grunsky's coefficients with property $\omega_{p,q} = \omega_{q,p}$. For those coefficients we have the next Grunsky's inequality ([3, 4]):

$$\sum_{q=1}^{\infty} q \left| \sum_{p=1}^{\infty} \omega_{p,q} x_p \right|^2 \leq \sum_{p=1}^{\infty} \frac{|x_p|^2}{p}, \tag{4}$$

where x_p are arbitrary complex numbers such that last series converges.

Further, it is well-known that if function

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{5}$$

belongs to $\mathcal{S}$, then also

$$f_2(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \cdots$$

belongs to the class $\mathcal{S}$. For the function f_2 we have the appropriate Grunsky's coefficients of the form $\omega_{2p-1,2q-1}$, and the inequality (4) has the form

$$\sum_{q=1}^{\infty} (2q-1) \left| \sum_{p=1}^{\infty} \omega_{2p-1,2q-1} x_{2p-1} \right|^2 \leq \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p-1}. \tag{6}$$

As it has been shown in [4, p. 57], if f is given by (1) then the coefficients a_2, a_3, a_4 and a_5 are expressed by Grunsky's coefficients $\omega_{2p-1,2q-1}$ of the function f_2 given by (5) in the following way:

$$\begin{aligned}
a_2 &= 2\omega_{11}, \\
a_3 &= 2\omega_{13} + 3\omega_{11}^2, \\
a_4 &= 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}\omega_{11}^3, \\
a_5 &= 2\omega_{35} + 8\omega_{11}\omega_{33} + 5\omega_{13}^2 + 18\omega_{11}^2\omega_{13} + \frac{7}{3}\omega_{11}^4, \\
0 &= 3\omega_{15} - 3\omega_{11}\omega_{13} + \omega_{11}^3 - 3\omega_{33}, \\
0 &= \omega_{17} - \omega_{35} - \omega_{11}\omega_{33} - \omega_{13}^2 + \frac{1}{3}\omega_{11}^4.
\end{aligned} \tag{7}$$

We note that in the book of Lebedev [4] there exists a typing mistake for the coefficient a_5. Namely, instead of the term $5\omega_{13}^2$, there is $5\omega_{15}^2$.

2. Main Results

Theorem 1. *Let γ_2, γ_3 and γ_4 be the logarithmic coefficients of function $f \in \mathcal{S}$. Then*

$$|\gamma_3| - |\gamma_2| \leq \frac{1}{\sqrt{5}} \quad \text{and} \quad |\gamma_4| - |\gamma_3| \leq \frac{1}{\sqrt{7}}.$$

Proof. For our consideration we need the following facts. From (6), choosing $x_{2p-1} = 0$ when $p = 3, 4, \ldots$, we have

$$\begin{aligned} &|\omega_{11}x_1 + \omega_{31}x_3|^2 + 3|\omega_{13}x_1 + \omega_{33}x_3|^2 + 5|\omega_{15}x_1 + \omega_{35}x_3|^2 \\ &+7|\omega_{17}x_1 + \omega_{37}x_3|^2 \leq |x_1|^2 + \frac{|x_3|^2}{3}. \end{aligned}$$

If additionally, $x_1 = 1$ and $x_3 = 0$, then

$$|\omega_{11}|^2 + 3|\omega_{13}|^2 + 5|\omega_{15}|^2 + 7|\omega_{17}|^2 \leq 1,$$

and from here also

$$\begin{aligned} |\omega_{11}|^2 + 3|\omega_{13}|^2 + 5|\omega_{15}|^2 &\leq 1, \\ |\omega_{11}|^2 + 3|\omega_{13}|^2 &\leq 1, \end{aligned}$$

and

$$|\omega_{11}|^2 \leq 1.$$

From the previous inequalities we have

$$\begin{aligned} &|\omega_{11}| \leq 1, \\ &|\omega_{13}| \leq \frac{1}{\sqrt{3}}\sqrt{1 - |\omega_{11}|^2}, \\ &|\omega_{15}| \leq \frac{1}{\sqrt{5}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2}, \\ &|\omega_{17}| \leq \frac{1}{\sqrt{7}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2 - 5|\omega_{15}|^2}. \end{aligned} \tag{8}$$

Also, from the fifth relation in (7) we obtain

$$\omega_{33} = \omega_{15} - \omega_{11}\omega_{13} + \frac{1}{3}\omega_{11}^3. \tag{9}$$

Now, for the first estimate of the theorem, using (3) and (7), we have

$$\begin{aligned} |\gamma_3| - |\gamma_2| &= \frac{1}{2}\left|a_4 - a_2a_3 + \frac{1}{3}a_2^3\right| - \frac{1}{2}\left|a_3 - \frac{1}{2}a_2^2\right| \\ &= |\omega_{33} + 2\omega_{11}\omega_{13}| - \left|\omega_{13} + \frac{1}{2}\omega_{11}^2\right|, \end{aligned} \tag{10}$$

and after applying $|\omega_{11}| \le 1$, brings

$$\begin{aligned} |\gamma_3| - |\gamma_2| &\le |\omega_{33} + 2\omega_{11}\omega_{13}| - |\omega_{11}| \cdot \left|\omega_{13} + \frac{1}{2}\omega_{11}^2\right| \\ &\le \left|(\omega_{33} + 2\omega_{11}\omega_{13}) - (\omega_{11})\left(\omega_{13} + \frac{1}{2}\omega_{11}^2\right)\right| \\ &= \left|\omega_{33} + \omega_{11}\omega_{13} - \frac{1}{2}\omega_{11}^3\right|. \end{aligned}$$

From here, having in mind the relation (9) and inequalities (8), we receive

$$\begin{aligned} |\gamma_3| - |\gamma_2| &\le \left|\omega_{15} - \frac{1}{6}\omega_{11}^3\right| \le |\omega_{15}| + \frac{1}{6}|\omega_{11}|^3 \\ &\le \frac{1}{\sqrt{5}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2} + \frac{1}{6}|\omega_{11}|^3 \\ &=: \Phi(|\omega_{11}|, |\omega_{13}|), \end{aligned} \tag{11}$$

where

$$\Phi(u, v) = \frac{1}{\sqrt{5}}\sqrt{1 - u^2 - 3v^2} + \frac{1}{6}u^3$$

with domain

$$\Omega = \left\{(u, v) : 0 \le u \le 1,\ 0 \le v \le \frac{1}{\sqrt{3}}\sqrt{1 - u^2}\right\}.$$

It remains to find the maximal value of Φ on the domain Ω.

Is easy to verify that the function $\Phi(u, v)$ has no interior singular points in the domain Ω ($\Phi'_v(u, v) = 0$, if, and only of, $v = 0$). On the edges:

$$\begin{aligned} \Phi(u, 0) &= \frac{1}{\sqrt{5}}\sqrt{1 - u^2} + \frac{1}{6}u^3 \le \Phi(0, 0) = \frac{1}{\sqrt{5}} = 0.44721\ldots, \\ \Phi(0, v) &= \frac{1}{\sqrt{5}}\sqrt{1 - 3v^2} \le \frac{1}{\sqrt{5}} = 0.44721\ldots, \\ \Phi\left(u, \frac{1}{\sqrt{3}}\sqrt{1 - u^2}\right) &= \frac{1}{6}u^3 \le \frac{1}{6} = 0.1(6). \end{aligned}$$

Using all previous facts and (11), we finally conclude that

$$|\gamma_3| - |\gamma_2| \le \frac{1}{\sqrt{5}}.$$

As for second estimate of this theorem, from (2) we receive

$$\gamma_4 = \frac{1}{2}\left(a_5 - a_2a_4 - \frac{1}{2}a_3^2 + a_2^2a_3 - \frac{1}{4}a_2^4\right),$$

and by using the relations (7),

$$\gamma_4 = \frac{1}{2}\left(2\omega_{35} + 3\omega_{13}^2 + 4\omega_{11}\omega_{33} + 4\omega_{11}^2\omega_{13} - \frac{5}{6}\omega_{11}^4\right). \tag{12}$$

Next, from the last two relations in (7) we can express ω_{33} and ω_{35} and apply them in (12). After some calculations, we receive

$$\gamma_4 = \omega_{17} + \omega_{11}\omega_{15} + \omega_{11}^2\omega_{13} + \frac{1}{2}\omega_{13}^2 + \frac{1}{4}\omega_{11}^4.$$

Now, using the above expression for γ_4, the expression for γ_3 obtained within (10), $\gamma_3 = \omega_{33} + 2\omega_{11}\omega_{13}$, and the expressions for ω_{33} and ω_{35} from the last two lines from (7), having in mind that $|\omega_{11}| \le 1$, we receive:

$$\begin{aligned}
|\gamma_4| - |\gamma_3| &\le |\gamma_4| - |\omega_{11}||\gamma_3| \\
&\le |\gamma_4 - \omega_{11}\gamma_3| \\
&= \left|\omega_{17} + \frac{1}{2}\omega_{13}^2 - \frac{1}{12}\omega_{11}^4\right| \\
&\le |\omega_{17}| + \frac{1}{2}|\omega_{13}|^2 + \frac{1}{12}|\omega_{11}|^4 \\
&\le \frac{1}{\sqrt{7}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2 - 5|\omega_{15}|^2} + \frac{1}{2}|\omega_{13}|^2 + \frac{1}{12}|\omega_{11}|^4 \\
&\le \frac{1}{\sqrt{7}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2} + \frac{1}{2}|\omega_{13}|^2 + \frac{1}{12}|\omega_{11}|^4 \\
&=: \Psi(|\omega_{11}|^2, |\omega_{13}|^2),
\end{aligned} \tag{13}$$

where

$$\Psi(s,t) = \frac{1}{\sqrt{7}}\sqrt{1 - s - 3t} + \frac{1}{2}t + \frac{1}{12}s^2,$$

$0 \le s \le 1$, $0 \le t \le \frac{1}{3}(1-s)$. It is easily verified that the function $\Psi(s,t)$ attains its maximum $\frac{1}{\sqrt{7}}$ for $s = t = 0$. Finally, from (13),

$$|\gamma_4| - |\gamma_3| \le \frac{1}{\sqrt{7}}.$$

3. Conclusion

Previous theorem leads to the conjecture for the difference of the moduli of two consecutive logarithmic coefficients of univalent functions.

Conjecture 1. *If γ_n is logarithmic coefficient of function $f \in \mathcal{S}$, then*

$$|\gamma_n| - |\gamma_{n-1}| \le \frac{1}{\sqrt{2n-1}}, \quad n = 3, 4, \ldots .$$

One possible direction for studying this conjecture is by using techniques and methods from [1, 2, 5, 12].

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