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Integration of modified multivariable H-function with respect to their parameters

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Abstract: The object of the present paper is to obtain some interesting results by integrating the modified multivariable H-function with respect to its parameters. Such integrals are useful in the study of certain boundary value problems.

1. Introduction

The modified multi-variable H-function employed as kernel of multi-dimensional transform defined by Prasad and Singh [5] on the lines of Srivastava and Panda [7], Prasad and Maurya [4] is as follows:

$$H^{m,n:|R':m_1,n_1;,\ldots;m_r,n_r}_{p,q:|R:p_1,q_1;\ldots;\ p_r,\ q_r}$$

$$\left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\middle| \begin{array}{l} (a_j;\alpha'_j,\ldots,\alpha_j^{(r)})_{1,p}:(e_j;u'_j g'_j,\ldots,u_j^{(r)}g_j^{(r)})_{1,|R'}:(c'_j,\gamma'_j)_{1,p_1};\ldots;(c_j^{(r)},\gamma_j^{(r)})_{1,p_r} \\ (b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,q}:(l_j;U'_j f'_j,\ldots,U_j^{(r)}f_j^{(r)})_{1,|R}:(d'_j,\delta'_j)_{1,q_1};\ldots;(d_j^{(r)},\delta_j^{(r)})_{1,q_r} \end{array}\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\Phi\ (\xi_1)\ldots\Phi_r(\xi_r)\,\psi(\xi_1\ldots\xi_r)\,z_1^{\xi_1}\ldots z_r^{\xi_r}d\xi_1\ldots d\xi_r. \qquad (1)$$

where $\omega=\sqrt{-1}$

$$\Phi_i(\xi_i)=\frac{\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-\delta_j^{(i)}\xi_i\right)\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}-\gamma_j^{(i)}\xi_i\right)}{\prod_{j=m_i+1}^{q_i}\Gamma\left(1-d_j^{(i)}+\delta_j^{(i)}\xi_i\right)\prod_{j=n_i+1}^{p_i}\Gamma\left(c_j^{(i)}-\gamma_j^{(i)}\xi_i\right)}(i=1,2,\ldots,r) \qquad (2)$$

$$\Psi(\xi_1,...,\xi_r) = \frac{\prod_{j=1}^{m_i}\Gamma\left(b_j - \sum_{i=1}^{r}\beta_j^{(i)}\xi_i\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i\right)}{\prod_{j=m+1}^{p}\Gamma\left(a_j - \sum_{i=1}^{r}\alpha_j^{(i)}\xi_i\right)\prod_{j=n+1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i\right)}$$

$$\times\frac{\prod_{j=1}^{R'}\Gamma\left(e_j+\sum_{i=1}^{r}u_j^{(i)}g_j^{(i)}\xi_i\right)}{\prod_{j=1}^{R}\Gamma\left(I_j+\sum_{i=1}^{r}U_j^{(i)}f_j^{(i)}\xi_i\right)} \tag{3}$$

The multiple integral (1) converges absolutely if
$|\arg z_1| < \frac{1}{2}U_i\pi, \quad (i = 1,2,...,r)$
where

$$U_i = \sum_{j=1}^{m}\beta_j^{(i)} - \sum_{j=m+1}^{q}\beta_j^{(i)} + \sum_{j=1}^{n}\alpha_j^{(i)} - \sum_{j=n+1}^{p}\alpha_j^{(i)}\sum_{j=1}^{m}\delta_j^{(i)} -$$

$$\sum_{j=m_1+1}^{q_i}\delta_j^{(i)}\sum_{j=1}^{n_1}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i}\gamma_j^{(i)} + \sum_{j=1}^{R'}g_j^{(i)} - \sum_{j=1}^{R}f_j^{(i)} > 0 \tag{3a}$$

(i=1,2,...,r)
The modified multivariable H-function is the most general special function and most of the function occurring in pure and applied mathematics are particular cases of it can be derived easily from our results.

2. Main Integral

$$\mathrm{I} = \frac{1}{(2\pi\omega)}\int_{-\omega\infty}^{+\omega\infty}\Gamma(\mathrm{a+x})\,\Gamma(\mathrm{b-x})\,\Gamma(\mathrm{c-x})\quad \mathrm{e}^{\pm\omega\pi\mathrm{x}}\quad \mathrm{H}^{0,\mathrm{n}:|\mathrm{R}':\mathrm{m}_1,\mathrm{n}_1;,...;\mathrm{m}_\mathrm{r},\mathrm{n}_\mathrm{r}}_{\mathrm{p,q+1}:|\mathrm{R}:\mathrm{p}_1,\mathrm{q}_1;....;\ \mathrm{p}_\mathrm{r},\ \mathrm{q}_\mathrm{r}}$$

$$\left[\begin{array}{c|c} z_1 & (a_j;\alpha'_j,...,\alpha_j^{(r)})_{1,p} : (e_j;u'_jg'_j,...,u_j^{(r)}g_j^{(r)})_{1,R'} : \\ \vdots & \\ z_r & (b_j;\beta'_j,...,\beta_j^{(r)})_{1,q} : (I_j;U'_jf'_j,...,U_j^{(r)}f_j^{(r)})_{1,IR} : \end{array}\right.$$

$$: (c'_j, \gamma'_j)_{1,p_1}; ...; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r}$$

$$: (1-d+x; h_1...h_r) : (d'_j, \delta'_j)_{1,q_1}; ...; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \Bigg] dx$$

$$= \Gamma(\mathrm{a+b})\,\Gamma(\mathrm{a+c})\exp(\pm \mathrm{a}\pi)\;\; \mathrm{H}^{0,\mathrm{n}+1:|\mathrm{R}':\mathrm{m}_1,\mathrm{n}_1;,...;\mathrm{m}_r,\mathrm{n}_r}_{\mathrm{p}+1,\mathrm{q}+2:|\mathrm{R}:\mathrm{p}_1,\mathrm{q}_1;....;\;\mathrm{p}_r,\;\mathrm{q}_r}$$

$$\left[\begin{array}{c|l} z_1 & (1+a+b+c-d : h_1...h_r), (a_j; \alpha'_j, ..., \alpha_j^{(r)})_{1,p} : (e_j; u'_j g'_j, ..., u_j^{(r)} g_j^{(r)})_{1,R'} : \\ \vdots & \\ z_r & (1+b-d : h_1...h_r), (1+c-d : h_1...h_r), (b_j; \beta'_j, ..., \beta_j^{(r)})_{1,q} : \end{array}\right.$$

$$: (c'_j, \gamma'_j)_{1,p_1}; ...; (c_j^{(r)}, \gamma_j^{(r)})_{1,p_r}$$

$$\left. : (I_j; U'_j f'_j, ..., U_j^{(r)} f_j^{(r)})_{1,IR}; (1-d+x : h_1...h_r) : (d'_j, \delta'_j)_{1,q_1}; ...; (d_j^{(r)}, \delta_j^{(r)})_{1,q_r} \right]$$

Provided that

1. $\mathrm{h_i}>0$ $(i = 1, 2, \ldots, r)$
2. $\mathrm{Re}\left[d - a - b - c + \mathrm{h_i}\,\frac{\mathrm{d_j^{(i)}}}{\delta_\mathrm{j}^{(\mathrm{i})}}\right] > 0,$

For i = 1,2,. . . ,r ; j =1, 2, . . . , $\mathrm{m_i}$

Proof :

In the integrand of (2.1) we replace the modified multivariable by (1.1), change the order of integration which is justified under the condition stated above, we get

$$= \frac{1}{(2\pi\omega)^{\mathrm{r}}} \int_{\mathrm{L}_1} \cdots \int_{\mathrm{L}_\mathrm{r}} \Phi\;(\xi_1)\ldots\Phi_\mathrm{r}(\xi_\mathrm{r})\,\psi(\xi_1\ldots\xi_\mathrm{r})\,\mathrm{z}_1^{\xi_1}\ldots\mathrm{z}_\mathrm{r}^{\xi_\mathrm{r}}\mathrm{d}\xi_1\ldots\mathrm{d}\xi_\mathrm{r}$$

$$\frac{1}{(2\pi\omega)} \int_{-\omega\infty}^{+\omega\infty} \frac{\Gamma(\mathrm{a+x})\,\Gamma(\mathrm{b-x})\,\Gamma(\mathrm{c-x})}{(d\;x + \mathrm{h_i}\xi_\mathrm{i} + \ldots. + \mathrm{h_r}\xi_\mathrm{r})}\;\; \mathrm{e}^{\pm\pi\omega\mathrm{x}}\;\mathrm{dx}$$

Now we evaluate the inner integral using a known integral formula [8,p.289], noting that it is a hypergeometric function with unit argument. On expressing the resulting expression with the help of (1.1), we obtain (2.1).

3. Particular Cases

(i) If we take all Greek letters equal to unity in (1.1), the modified multivariable H-function would reduce to all the relatively more familiar G-function of two variables. Thus our result will yield similar integrals involving the double G-function.
(ii) If we put m $= |R' = |R = p = q = 0$ in (1.1), the modified multivariable h-function degenerates into r (Fox's) H-functions.
(iii) If we set r=2, m_2=1, $q_2 = q_2+1$ in (2.1) we get an integral relation for the H-function of two variables in the form:

$$\frac{1}{(2\pi\omega)}\int_{-\omega\infty}^{+\omega\infty}\Gamma(a+x)\,\Gamma(b-x)\,\Gamma(c-x)\quad e^{\pm\omega\pi x}\quad H^{0,n:|R':m_1,n_1,m_2,n_2}_{p,q:|R:p_1,q_1,p_2,\ q_2}$$

$$\left[\begin{array}{c|l} z_1 & (a_j;\alpha'_j,\alpha''_j)_{1,p}:(e_j;u'_jg'_j,u''_jg''_j)_{1,R'}: \\ \vdots & \\ z_r & (b_j;\beta'_j,\beta''_j)_{1,q}:(I_j;U'_jf'_j,U''_jf''_j)_{1,R}: \end{array}\right.$$

$$\left.\begin{array}{l} :(c'_j,\gamma'_j)_{1,p_1};(c''_j,\gamma''_j)_{1,p_2} \\ \\ :(1-d+x:h_1,h_2):(d'_j,\delta'_j)_{1,q_1};(d^2_j,\delta^2_j)_{1,q_2} \end{array}\right]dx$$

$$=\Gamma(a+b)\,\Gamma(a+c)\exp(\pm a\pi)\ H^{0,n+1:|R':m_1,n_1,1,n_2}_{p+1,q+2:|R:p_1,q_1,p_2,\ q_2+1}$$

$$\left[\begin{array}{c|l} z_1 & (1+a+b+c-d:h_1,h_2),(a_j;\alpha'_j,\alpha''_j)_{1,p}:(e_j;u'_jg'_j,u''_jg''_j)_{1,R'}: \\ \vdots & \\ z_r & (1+b-d:h_1,h_2),(1+c-d:h_1,h_2),(b_j;\beta'_j,\beta''_j)_{1,q}: \end{array}\right.$$

$$\left.\begin{array}{l} :(c'_j,\gamma'_j)_{1,p_1};(c''_j,\gamma''_j)_{1,p_2} \\ \\ :(I_j;U'_jf'_j,U''_jf''_j)_{1,R}:(1-d+x;h_1,h_2):(d'_j,\delta'_j)_{1,q_1};(d''_j,\delta''_j)_{1,q_2} \end{array}\right] \tag{3.1}$$

With the conditions

1. $h_1 > 0,\ h_2 > 0$
2. $\mathrm{Re}\left[d-a-b-c+h_1\ \sum_{j=1}^{m_1}\ \frac{d'_j}{\delta'_j}+h_2\ \sum_{j=1}^{m_2}\ \frac{d''_j}{\delta''_j}\right]\ > 0$

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