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ROGERS–RAMANUJAN TYPE IDENTITIES AND THREE-LINE ARRAYS

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Abstract: In this paper, we present the combinatorial interpretations of many Rogers–Ramanujan type identities using three-line arrays. Bijections between restricted three-line arrays and restricted overpartitions are given, and as a consequence, we get fourteen combinatorial identities.

Keywords and Phrases: Rogers–Ramanujan type identities, $(n+t)$–color overpartitions, three-line arrays.

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1. Introduction

In [9], Santos et al. introduced the concept of a two-line array. For a positive integer ν, let

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_r \\ \beta_1 & \beta_2 & \dots & \beta_r \end{pmatrix}, \qquad \alpha_i,\, \beta_i \geq 0 \quad \text{for } 1 \leq i \leq r \tag{1.1}$$

be a two-line array such that

$$\nu = \sum_{i=1}^{r} \alpha_i + \sum_{i=1}^{r} \beta_i. \tag{1.2}$$

Using the above two-line array representation and by imposing certain restrictions on α_i and β_i, Santos and his collaborators interpreted many identities from Slater's list [10] including Rogers–Ramanujan identities and Lebesgue's partition identities in [9]. They also presented several bijective proofs for partition identities. The work is further related to three-quadrant Ferrers graphs in [4]. Continuing with the above work, Brietzke et al. [5] found the combinatorial interpretations of a number of mock theta functions using a two-line array, some of which were already interpreted using $(n+t)$–color partitions. In [2], Alegri introduced a third row of γ_i in a line array representation, such that $\nu = \sum_i \alpha_i + \sum_i \beta_i + \sum_i \gamma_i$, to construct a correspondence between three-line arrays and overpartitions. In [3], a bijection from certain classes of plane partitions to overpartitions and unrestricted partitions was presented. The purpose of this paper is to study combinatorial interpretations of Rogers–Ramanujan type identities (RRTIs), given in [6, 10], using the three-line arrays and establish bijections with $(n+t)$–color overpartitions. Before proceeding further, we recall:

Definition 1.1. [1] *An $(n+t)$–color partition (also called a partition with "$n+t$ copies of n") $t \geq 0$, is a partition in which a part of size $n, n \geq 0$, can come in $n+t$ different colors denoted by $n_1, n_2, \ldots, n_{n+t}$. Note that, only one copy of zero is allowed and it cannot repeat. The weighted difference is defined as $(((m_i)_{x_i} - (m_j)_{x_j})) = m_i - m_j - x_i - x_j$ where $(m_i)_{x_i}$ and $(m_j)_{x_j} (m_i \geq m_j)$ are two parts in an $(n+t)$–color partition $(m_1)_{x_1} + (m_2)_{x_2} + \ldots + (m_r)_{x_r}$ such that $m_1 \geq m_2 \geq \ldots \geq m_r$. For $\nu = 2$, the $(n+2)$–color partitions are* 2_4, 2_3, 2_2, 2_1, $1_3 1_3$, $1_3 1_2$, $1_3 1_1$, $1_2 1_2$, $1_2 1_1$, $1_1 1_1$. *For convenience, we denote $\delta_i = (((m_i)_{x_i} - (m_{i+1})_{x_{i+1}}))$ where $m_i \geq m_{i+1}$.*

Definition 1.2. [7, 8] *An $(n+t)$–color overpartition is an $(n+t)$–color partition in which the final occurrence of a part $(m_j)_{x_j}$ may be overlined. Here zero is not permitted to repeat and only one copy of 0 is allowed so either 0_x or $\overline{0}_x$ is used.*

In Section 2, we present combinatorial interpretations of fourteen RRTIs with three-line arrays and $(n+t)$–color overpartitions. We proceed constructively to obtain three-line array interpretations and translate these results into $(n+t)$–color overpartitions using certain bijections. Based on these bijections we classify the fourteen identities into five groups: Group 1 contains 2, Group 2 contains 4, Group 3 contains 3, Group 4 contains 2 and Group 5 contains 3 RRTIs. In Section 3, we provide alternate proofs of sum side in RRTIs involving three-line arrays using classical approach. We provide one full proof of a theorem and for remaining theorems we give sketch proofs due to the similarity of the proofs associated with the theorems.

2. Combinatorial interpretations

Group 1

The following RRTIs (2.1) appears in [10] and (2.2) in [6], as Identity No. 29:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q;q)_{2n}} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^2,-q^4,q^6;q^6]_\infty, \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(-q;q^2)_n (q^4;q^4)_n} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^2,-q^3,q^5;q^5]_\infty. \tag{2.2}$$

Here we have employed the standard q-series notation

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),$$

$$[a_1,a_2,,a_k;q_n] = (a_1;q)_n(a_2;q)_n \cdots (a_k;q)_n,$$

$$\text{and} \quad [a_1,a_2,\ldots,a_k;q]_\infty = (a_1;q)_\infty(a_2;q)_\infty \cdots (a_k;q)_\infty.$$

Throughout the paper, sums in an RRTIs represent generating functions for either $A_i(\nu)$, which count partitions in terms of three-line arrays, or $\overline{A}_i(\nu)$, which count $(n+t)$–color overpartitions, where $1 \le i \le 14$. The generating function for $B_i(\nu)$, which count ordinary partitions, is expressed without a sum and instead only uses products from the q-series notation described above. These lead to 3-way combinatorial interpretations satisfying

$$\sum_{\nu=0}^{\infty} A_i(\nu)q^\nu = \sum_{\nu=0}^{\infty} \overline{A}_i(\nu)q^\nu = \sum_{\nu=0}^{\infty} B_i(\nu)q^\nu, \quad 1 \le i \le 14. \tag{2.3}$$

For Group 1, we construct a bijection between three-line arrays and n–color overpartitions. Let $(m_i)_{x_i}$ and $(m_{i+1})_{x_{i+1}}$ be two consecutive parts of an n–color overpartition. Then the corresponding column $\begin{pmatrix}\alpha_i\\ \beta_i\\ \gamma_i\end{pmatrix}$ in the three-line array is given by

$$\phi : (m_i)_{x_i} \to \begin{cases} \begin{pmatrix} m_{i+1}+x_{i+1}+1 \\ x_i - 1 \\ \delta_i \end{pmatrix} & \text{if } m_i \text{ is not overlined,} \\ \\ \begin{pmatrix} m_{i+1}+x_{i+1}+2 \\ x_i - 2 \\ \delta_i \end{pmatrix} & \text{if } m_i \text{ is overlined.} \end{cases} \tag{2.4}$$

In the reverse implication, let $\begin{pmatrix}\alpha_i\\ \beta_i\\ \gamma_i\end{pmatrix}$ be any column in the three-line array. Then the corresponding part in an n–color overpartitions, $(m_i)_{x_i}$, is given by

$$\phi^{-1}:\begin{pmatrix}\alpha_i\\ \beta_i\\ \gamma_i\end{pmatrix}\to\begin{cases}(\overline{\alpha_i+\beta_i+\gamma_i})_{\beta_i+2}, & \text{if } \alpha_i\equiv\ 0 \pmod 2,\\ \\ (\alpha_i+\beta_i+\gamma_i)_{\beta_i+1}, & \text{if } \alpha_i\not\equiv\ 0 \pmod 2.\end{cases}\tag{2.5}$$

We now provide the combinatorial interpretations in terms of three-line arrays and n–color overpartitions for (2.1) and (2.2) in Theorem 2.1 and Theorem 2.2 respectively.

Theorem 2.1. *Let $A_1(\nu)$ represent the number of three-line arrays into r columns satisfying*

(2.1.a) $\alpha_r\in\{1,2\}$,

(2.1.b) $\gamma_i\equiv 0 \pmod 2$, $\forall\ i$

$$(2.1.c)\ \ \alpha_i=\begin{cases}2+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1} & \underline{\textit{if}}\ \alpha_i \textit{ and } \alpha_{i+1} \textit{ are odd },\\ 3+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1} & \underline{\textit{if}}\ \alpha_i \textit{ and } \alpha_{i+1} \textit{ have opposite parity},\\ 4+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1} & \underline{\textit{if}}\ \alpha_i \textit{ and } \alpha_{i+1} \textit{ are even},\end{cases}$$

where $1\le i\le r-1$.

Let $\overline{A}_1(\nu)$ count the number of n–color overpartitions of ν into r parts such that

(2.1.d) $m_i-x_i\equiv 0 \pmod 2$ $\forall\ i$,

(2.1.e) *if $x_i=1$, then the occurrence of the part is not overlined.*

Let $B_1(\nu)$ is the number of partitions of ν such that the odd parts are distinct, the even parts are $\equiv\pm2 \pmod 6$ and two copies of the parts $\equiv\pm2 \pmod{12}$ are allowed. Then $A_1(\nu)=\overline{A}_1(\nu)=B_1(\nu)$, $\forall\ \nu\ge 0$.

Proof. We begin by expanding left hand side of (2.1)

$$\sum_{r=0}^{\infty}\frac{(-q;q^2)_r q^{r^2}}{(q;q)_{2r}} = \sum_{r=0}^{\infty}\frac{(1+q)(1+q^3)\cdots(1+q^{2r-1})q^{1+3+\ldots+(2r-1)}}{(1-q)(1-q^3)\cdots(1-q^{2r-1})(1-q^2)(1-q^4)\cdots(1-q^{2r})}.$$

Clearly the factor $q^{r^2}=q^{1+3+5\ldots+(2r-1)}$ in the above expression generates the partition into r odd parts. It corresponds to the following three-line array

$$\begin{pmatrix}2r-1 & \ldots & 3 & 1\\ 0 & \ldots & 0 & 0\\ 0 & \ldots & 0 & 0\end{pmatrix}.$$

The factor $(-q;q^2)_r$ generates the partition into distinct odd parts $\le 2r-1$, say, $1\cdot a_1, 3\cdot a_2,\ldots,(2r-1)\cdot a_r$, where $a_i\in\{0,1\}$, $\forall\ i$. Thus the transformed three-line array becomes

$$\begin{pmatrix} 2r-1+a_1+\sum_{i=2}^{r} 2a_i & \dots & 3+a_{r-1}+2a_r & 1+a_r \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

The factor $(q;q^2)_r^{-1}$ generates the partition into odd parts $\leq 2r-1$, say, $1\cdot b_1, 3\cdot b_2, \dots, (2r-1)\cdot b_r$, where $b_i \geq 0,\ \forall\, i$. The three-line array transforms to

$$\begin{pmatrix} 2r-1+a_1+\sum_{i=2}^{r} 2(a_i+b_i) & \dots & 3+a_{r-1}+2a_r+2b_r & 1+a_r \\ b_1 & \dots & b_{r-1} & b_r \\ 0 & \dots & 0 & 0 \end{pmatrix}$$

The factor $(q^2;q^2)_r^{-1}$ generates the partition into even parts $\leq 2r$, say, $2\cdot c_1, 4\cdot c_2, \dots, 2r\cdot c_r$, where $c_i \geq 0$, $\forall\, i$. Hence, the three-line array transforms to

$$\begin{pmatrix} \begin{array}{c} 2r-1+a_1+ \\ \sum_{i=2}^{r} 2(a_i+b_i+c_i) \end{array} & \dots & \begin{array}{c} 3+a_{r-1}+2a_r+ \\ 2b_r+2c_r \end{array} & 1+a_r \\ b_1 & \dots & b_{r-1} & b_r \\ 2c_1 & \dots & 2c_{r-1} & 2c_r \end{pmatrix}$$

and $\nu = 1\cdot(a_1+b_1+1)+3\cdot(a_2+b_2+1)+\dots+(2r-1)\cdot(a_r+b_r+1)+2c_1+4c_2+\dots+2rc_r$. Therefore, $A_1(\nu)$ enumerates the three-line arrays with $\beta_i = b_i$, $\gamma_i = 2c_i\ \forall\ i$ and satisfying (2.1.a)–(2.1.c).

Table 1

$A_1(6)$	$\overline{A}_1(6)$	$B_1(6)$	$A_1(6)$	$\overline{A}_1(6)$	$B_1(6)$
$\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$	6_6	51	$\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$	$\overline{6}_6$	42_1
$\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$	6_4	42_2	$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$	$\overline{6}_4$	$32_1 1$
$\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$	6_2	$32_2\ 1$	$\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$	$\overline{6}_2$	$2_1 2_1 2_1$
$\begin{pmatrix} 3 & 1 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}$	$5_3 1_1$	$2_1 2_1 2_2$	$\begin{pmatrix} 4 & 1 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\overline{5}_3 1_1$	$2_1 2_2 2_2$
$\begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}$	$5_1 1_1$	$2_2 2_2 2_2$			

Example 2.1. For $\nu = 6$, the n–color overpartitions and three-line arrays satisfying Theorem 2.1 are listed in Table 1:

Theorem 2.2. *Let $A_2(\nu)$ represent the number of three-line arrays into r columns satisfying* (2.1.a), (2.1.c), *and* $\gamma_i \equiv 0 \pmod 4$ $\forall$ i. *Let $\overline{A}_2(\nu)$ count the number of n–color overpartitions of ν into r parts satisfying* (2.1.e), $m_r - x_r \equiv 0 \pmod 4$, *and* $m_i - x_i \equiv 0 \pmod 2$, *for* $1 \leq i \leq r-1$. $\delta_i \geq 0$, *and* $\delta_i \equiv 0 \pmod 4$ *for* $i < r$. *Let* $B_2(\nu) = \sum_{k=0}^{\nu} C_2(\nu - k)D_2(k)$, *where $C_2(\nu)$ is the number of partitions of ν into distinct parts that are either* $\equiv 5 \pmod{10}$ *or* $\equiv \pm 1 \pmod{10}$ *with two copies of parts that are* $\equiv 5 \pmod{10}$ *allowed and one copy of parts that are* $\equiv \pm 1 \pmod{10}$ *allowed and $D_2(\nu)$ is the number of partitions of ν into parts that are* $\equiv \pm 2 \pmod{10}$ *with two copies of each part allowed. Then* $A_2(\nu) = \overline{A}_2(\nu) = B_2(\nu)$ $\forall$ $\nu \geq 0$.

Group 2

The RRTIs (2.6)–(2.9) below appear in [6] with Identity No. 104, 102, 27, 25 respectively. In this group, the bijection used for (2.7)–(2.9) to establish the connection between three-line arrays and overpartitions is the same as defined in Group 1. The RRTIs for this group are:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)}}{(q;q^2)_{n+1}(q^2;q^2)_n} = \frac{1}{(q;q)_\infty}[q^4, q^8, q^{12}; q^{12}]_\infty, \tag{2.6}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+2)}}{(q;q^2)_{n+1}(q^2;q^2)_n} = \frac{1}{(q;q)_\infty}[q^2, q^{10}, q^{12}; q^{12}]_\infty, \tag{2.7}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n(n+2)}}{(-q;q^2)_n (q^4;q^4)_n} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[-q, -q^4, q^5; q^5]_\infty, \tag{2.8}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n(n+2)}}{(-q;q^2)_{n+1} (q^4;q^4)_n} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^5, -q^5, q^5; q^5]_\infty. \tag{2.9}$$

The combinatorial interpretations in terms of three-line arrays and n–color overpartitions of (2.6)–(2.9) are given in Theorems 2.3–2.6 respectively.

Theorem 2.3. *Let $A_3(\nu)$ represent the number of three-line arrays into r columns satisfying* (2.1.b) *along with*

(2.4.a) $\alpha_r = 0 = \gamma_r$,

(2.4.b) $\alpha_{r-1} \geq 2$,

$$(2.4.c)\ \ \alpha_i = \begin{cases} 2+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1} & \underline{if}\ \alpha_i \text{ and } \alpha_{i+1} \text{ are even },\\ 3+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1} & \underline{if}\ \alpha_i \text{ and } \alpha_{i+1} \text{ have opposite parity},\\ 4+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1} & \underline{if}\ \alpha_i \text{ and } \alpha_{i+1} \text{ are odd}, \end{cases}$$

where $1 \le i \le r-2$,

$(2.4.d)$ $\gamma_i \equiv 0 \pmod 2$, $\forall\ i$.

Let $\overline{A}_3(\nu)$ *count the number of* $(n+1)$*–color overpartitions of* ν *into* r *parts satisfying*

$(2.4.e)$ $x_r = m_r + 1$,

$(2.4.f)$ $(m_r)_{x_r}$ *is not overlined,*

$(2.4.g)$ $m_i - x_i \equiv 1 \pmod 2$, $\forall\ i$

$(2.4.h)$ *if* $x_i = 1$, *then the occurrence of the part is not overlined.*

Let $B_3(\nu)$ *is the number of partitions of* ν *in which the parts are* $\not\equiv 0, \pm 4 \pmod{12}$. *Then* $A_3(\nu) = \overline{A}_3(\nu) = B_3(\nu)\ \forall\ \nu \ge 0$.

Remark 2.1. *Here we observe that the mapping used for first* $r-1$ *parts is same as defined in* (2.4) *and*

$$\phi^{-1} : \begin{pmatrix}\alpha_i\\ \beta_i\\ \gamma_i\end{pmatrix} \to \begin{cases} (\overline{\alpha_i+\beta_i+\gamma_i})_{\beta_i+2}, & \text{if } \alpha_i \equiv\ 0 \pmod 2,\\ \\ (\alpha_i+\beta_i+\gamma_i)_{\beta_i+1}, & \text{if } \alpha_i \not\equiv\ 0 \pmod 2. \end{cases}$$

For the r^{th} *part the mapping is* $\phi : (m_r)_{x_r} \to \begin{pmatrix}0\\ x_r-1\\ 0\end{pmatrix}$ *and the inverse mapping is* $\phi^{-1} : \begin{pmatrix}0\\ \beta_r\\ 0\end{pmatrix} \to (\beta_r)_{\beta_r+1}$.

Example 2.2. For $\nu = 6$, $A_3(6) = 9 = \overline{A}_3(6)$. The three-line arrays corresponding to $A_3(6)$ are

$$\begin{pmatrix}0\\6\\0\end{pmatrix}, \begin{pmatrix}2&0\\4&0\\0&0\end{pmatrix}, \begin{pmatrix}2&0\\2&0\\2&0\end{pmatrix}, \begin{pmatrix}2&0\\0&0\\4&0\end{pmatrix}, \begin{pmatrix}4&0\\1&1\\0&0\end{pmatrix},$$
$$\begin{pmatrix}3&0\\3&0\\0&0\end{pmatrix}, \begin{pmatrix}3&0\\1&0\\2&0\end{pmatrix}, \begin{pmatrix}3&0\\3&1\\0&0\end{pmatrix}, \begin{pmatrix}4&2&0\\0&0&0\\0&0&0\end{pmatrix}.$$

The $(n+1)$–color overpartitions corresponding to $\overline{A}_3(6)$ are 6_7, $6_5 0_1$, $6_3 0_1$, $6_1 0_1$, $5_2 1_2$, $\overline{6}_5 0_1$, $\overline{6}_3 0_1$, $\overline{5}_2 1_2$, $4_1 2_1 0_1$. The partitions corresponding to $B_3(6)$ are 6, 51, 33, 321, 3111, 222, 2211, 21111, 111111.

Theorem 2.4. *Let $A_4(\nu)$ represent the number of three-line arrays into r columns satisfying (2.4.a) and (2.4.d) along with:*

(2.7.a) $\alpha_{r-1} \geq 3$,

(2.7.b) $\alpha_i = \begin{cases} 2+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1} & \underline{\text{if}}\ \alpha_i \text{ and } \alpha_{i+1} \text{ are odd },\\ 3+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1} & \underline{\text{if}}\ \alpha_i \text{ and } \alpha_{i+1} \text{ have opposite parity},\\ 4+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1} & \underline{\text{if}}\ \alpha_i \text{ and } \alpha_{i+1} \text{ are even}, \end{cases}$
where $1 \leq i \leq r-2$.

Let $\overline{A}_4(\nu)$ count the number of $(n+2)$–color overpartitions of ν into r parts satisfying (2.4.f) and (2.4.h) along with:

(2.7.c) $m_i - x_i \equiv 0 \pmod 2\ \forall\ i$,

(2.7.d) $x_r = m_r + 2$.

Let $B_4(\nu)$ is the number of partitions of ν in which the parts are $\not\equiv \pm 2 \pmod{12}$. Then $A_4(\nu) = \overline{A}_4(\nu) = B_4(\nu)\ \forall\ \nu \geq 0$.

Remark 2.2. *Here we observe that the mapping used for first $r-1$ parts is same as defined in (2.4)–(2.5) and for the r^{th} part is $\phi : (m_r)_{x_r} \to \begin{pmatrix} 0 \\ x_r - 2 \\ 0 \end{pmatrix}$ and the inverse mapping is $\phi^{-1} : \begin{pmatrix} 0 \\ \beta_r \\ 0 \end{pmatrix} \to (\beta_r)_{\beta_r+2}$.*

Theorem 2.5. *Let $A_5(\nu)$ represent the number of three-line arrays into r columns satisfying (2.7.b) for $1 \leq i \leq r-1$, along with:*

(2.9.a) $\alpha_r \in \{3, 4\}$,

(2.9.b) $\gamma_i \equiv 0 \pmod 4,\ \forall\, i$.

Let $\overline{A}_5(\nu)$ count the number of n–color overpartitions of ν into r parts satisfying (2.4.h) and (2.7.c) along with:

(2.9.c) $m_r \geq 3$,

(2.9.d) $m_r - x_r \equiv 2 \pmod 4$,

(2.9.e) $\delta_i \geq 0$, *and* $\delta_i \equiv 0 \pmod 4$, $\forall\ i < r$.

Let $B_5(\nu) = \sum_{k=0}^{\nu} C_5(\nu-k) D_5(k)$, where $C_5(\nu)$ is the number of partitions of ν into distinct parts that are $\equiv 5 \pmod{10}$ or $\equiv \pm 3 \pmod{10}$ with two copies of parts that are $\equiv 5 \pmod{10}$ allowed and one copy of parts that are $\equiv \pm 3 \pmod{10}$

allowed and $D_5(\nu)$ is the number of partitions of ν into parts that are $\equiv \pm 4 \pmod{10}$ *with two copies of each part allowed. Then $A_5(\nu) = \overline{A}_5(\nu) = B_5(\nu)\ \forall\ \nu \geq 0$.*

Remark 2.3. *In above the mapping for the r^{th} part is*

$$\phi : (m_r)_{x_r} \to \begin{cases} \begin{pmatrix} 3 \\ x_r - 1 \\ m_r - x_r - 2 \end{pmatrix} & \text{if } m_r \text{ is not overlined,} \\ \\ \begin{pmatrix} 4 \\ x_r - 2 \\ m_r - x_r - 2 \end{pmatrix} & \text{if } m_r \text{ is overlined.} \end{cases}$$

And the inverse mapping is same as given in (2.5).

Theorem 2.6. *Let $A_6(\nu)$ represent the number of three-line arrays into r columns satisfying* (2.4.*a*), (2.7.*a*), (2.7.*b*), *and* (2.9.*b*). *Let $\overline{A}_6(\nu)$ count the number of $(n+2)$–color overpartitions of ν into r parts satisfying* (2.4.*f*), (2.4.*h*), (2.7.*c*), (2.7.*d*) (2.9.*e*). *Let $B_6(\nu) = \sum_{k=0}^{\nu} C_6(\nu - k)D_6(k)$, where $C_6(\nu)$ is the number of partitions of ν into distinct parts in which the parts are* $\equiv \pm 1, \pm 3 \pmod{10}$ *and $D_6(\nu)$ is the number of partitions of ν in which the parts are* $\equiv \pm 2, \pm 4 \pmod{10}$. *Then $A_6(\nu) = \overline{A}_6(\nu) = B_6(\nu)\ \forall\ \nu \geq 0$.*

Group 3

The RRTIs (2.10)–(2.12) appear in [6] as Identity No. 195, 45, 46 respectively:

$$\sum_{n=1}^{\infty} \frac{(-q^2;q^2)_{n-1} q^{n^2}}{(q;q)_{2n}} = \frac{[q^2, q^{14}, q^{16}; q^{16}]_\infty [q^{12}, q^{20}; q^{32}]_\infty}{(q;q)_\infty}, \tag{2.10}$$

$$\sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n(n+1)}}{(q;q)_{2n}} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} [-q^3, -q^3, q^6; q^6]_\infty, \tag{2.11}$$

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n(n+1)}}{(q;q)_{2n+1}} = \frac{(-q^2,q^2)_\infty}{(q^2;q^2)_\infty} [-q, -q^5, q^6; q^6]_\infty. \tag{2.12}$$

The bijection between three-line arrays and $(n+t)$–color overpartitions for (2.10)–(2.12) is given by

$$\phi : (m_i)_{x_i} \to \begin{cases} \begin{pmatrix} m_{i+1} + x_{i+1} + 1 \\ x_i - 1 \\ \delta_i \end{pmatrix} & \text{if } m_i \text{ is not overlined,} \\ \\ \begin{pmatrix} m_{i+1} + x_{i+1} + 3 \\ x_i - 1 \\ \delta_i - 2 \end{pmatrix} & \text{if } m_i \text{ is overlined,} \end{cases} \tag{2.13}$$

and

$$\phi^{-1}:\begin{pmatrix}\alpha_i\\\beta_i\\\gamma_i\end{pmatrix}\to\begin{cases}(\alpha_i+\beta_i+\gamma_i)_{\beta_i+1} & \text{if } \alpha_i=2+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1},\\ \overline{(\alpha_i+\beta_i+\gamma_i)}_{\beta_i+1} & \text{if } \alpha_i=4+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1}.\end{cases} \tag{2.14}$$

The combinatorial interpretations in terms of three-line arrays and n–color overpartitions of (2.10)–(2.12) are given in Theorems 2.7–2.9 respectively.

Theorem 2.7. *Let $A_7(\nu)$ represent the number of three-line arrays into r columns satisfying*

(2.12.a) $\alpha_r=1$,

(2.12.b) $\alpha_i\in\{2+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1},\ 4+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1}\}$, *for* $1\le i\le r-1$

(2.12.c) $\gamma_i\equiv 0 \pmod 2$, $\forall\ i$.

Let $\overline{A}_7(\nu)$ count the number of n–color overpartitions of ν into r parts satisfying

(2.12.d) $m_i-x_i\equiv 0 \pmod 2)\ \forall\ i$,

(2.12.e) $(m_r)_{x_r}$ *is not overlined,*

(2.12.f) $\delta_i\ge 0$, *and* $\delta_i\equiv 0 \pmod 2)\ \forall\ i<r$. *For* $\delta_i=0$, m_i *is not overlined.*

Let $B_7(\nu)$ is the number of partitions of ν in which the parts are $\not\equiv \pm2,\pm12,\pm14,16 \pmod{32}$. *Then* $A_7(\nu)=\overline{A}_7(\nu)=B_7(\nu)\ \forall\ \nu\ge 0$.

For the combinatorial interpretation of left hand side of (2.11), we write

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1;q^2)_nq^{n(n+1)}}{(q;q)_{2n}}&=1+2\sum_{n=1}^{\infty}\frac{(-q^2;q^2)_{n-1}q^{n(n+1)}}{(q;q)_{2n}}\\&=1+2\sum_{\nu=1}^{\infty}\hat{A}_8(\nu)q^{\nu},\end{aligned}\tag{2.15}$$

where $\sum_{\nu=1}^{\infty}\hat{A}_8(\nu)q^{\nu}=\sum_{n=1}^{\infty}\frac{(-q^2;q^2)_{n-1}q^{n(n+1)}}{(q;q)_{2n}}$. Now we give the combinatorial interpretation of (2.11) in the following theorem.

Theorem 2.8. *Let $\hat{A}_8(\nu)$ represent the number of three-line arrays into r columns satisfying* (2.12.b), (2.12.c) *and* $\alpha_r=2$. *Let $\overline{\hat{A}}_8(\nu)$ count the number of n–color overpartitions of ν into r parts satisfying* (2.12.d), (2.12.e), m_r, $x_r>1$, $\delta_i\ge -2$, *and* $\delta_i\equiv 0 \pmod 2$, $\forall\ i<r$. *For* $\delta_i=-2$, m_i *is not overlined.*
Let $B_8(\nu)=\sum_{k=0}^{\nu}C_8(\nu-k)D_8(k)$, *where $C_8(\nu)$ is the number of partitions ν in*

which the parts are $\equiv \pm 2, \pm 3, \pm 4 \pmod{12}$ *and* $D_8(\nu)$ *is the number of partitions of* ν *into distinct parts that are* $\equiv \pm 2, \pm 3, \pm 4 \pmod{12}$. *Then,*
$2\hat{A}_8(\nu) = A_8(\nu) = 2\overline{\hat{A}}_8(\nu) = \overline{A}_8(\nu) = B_8(\nu)\ \forall\ \nu \geq 1.$

Remark 2.4. *Here for the* r^{th} *part, we use the following mapping:*

$$\phi : (m_r)_{x_r} \to \begin{pmatrix} 2 \\ x_r - 1 \\ m_r - x_r - 1 \end{pmatrix}.$$

And the inverse mapping is same as in (2.4).

Example 2.3. For $\nu = 6$, the three-line arrays corresponding to $\hat{A}_8(6)$ are

$$\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

And the n–color overpartitions corresponding to $\overline{\hat{A}}_8(6)$ are 6_6, 6_4, 6_2, $5_3 1_1$, $5_1 1_1$, $\overline{5}_1 1_1$. Hence $\hat{A}_8(6) = \overline{\hat{A}}_8(6) = 6$ and $A_8(6) = \overline{A}_8(6) = 12$.

Theorem 2.9. *Let* $A_9(\nu)$ *represent the number of three-line arrays into* r *columns satisfying* (2.12.b) *and* (2.12.c) *along with:*

(2.16.a) $\alpha_r = 0 = \gamma_r$,

(2.16.b) $\alpha_{r-1} \geq 2$.

Let $\overline{A}_9(\nu)$ *counts the number of* $(n+1)$*–color overpartitions of* ν *into* r *parts satisfying* (2.12.f) *along with:*

(2.16.c) $x_r = m_r + 1$,

(2.16.d) $(m_r)_{x_r}$ *is not overlined,*

(2.16.e) $m_i - x_i \equiv 1 \pmod 2$, $\forall\ i$.

Let $B_9(\nu) = \sum_{k=0}^{\nu} C_9(\nu - k) D_9(k)$, *where* $C_9(\nu)$ *is the number of partitions of* ν *in which the parts are* $\equiv 2,\ 4 \pmod 6$ *and* $D_9(\nu)$ *is the number of partitions of* ν *into distinct parts that are* $\equiv 0, \pm 1, \pm 2 \pmod 6$. *Then* $A_9(\nu) = \overline{A}_9(\nu) = B_9(\nu)\ \forall$ $\nu \geq 0$.

Remark 2.5. *Here for the* r^{th} *part, the mapping and inverse mapping are defined as*

$$\phi : (m_r)_{x_r} \to \begin{pmatrix} 0 \\ x_r - 1 \\ 0 \end{pmatrix},\ \phi^{-1} : \begin{pmatrix} 0 \\ \beta_r \\ 0 \end{pmatrix} \to (\beta_r)_{\beta_r + 1}.$$

Group 4

The RRTIs (2.16)–(2.17) below appear in [6] as Identity No. 11, 12 respectively.

$$\sum_{n=0}^{\infty}\frac{(-1;q^4)_n q^{n^2}}{(q;q^2)_n(q^4;q^4)_n}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[-q,-q^4,q^3;-q^4]_\infty, \tag{2.16}$$

$$\sum_{n=0}^{\infty}\frac{(-1;q^4)_n q^{n(n+2)}}{(q;q^2)_n(q^4;q^4)_n}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}[-q^3,-q^4,q;-q^4]_\infty. \tag{2.17}$$

To establish the bijection between three-line arrays and n–color overpartitions for (2.16) and (2.17), we use

$$\phi:(m_i)_{x_i}\to\begin{cases}\begin{pmatrix}m_{i+1}+x_{i+1}+1\\ x_i-1\\ \delta_i\end{pmatrix} & \text{if } m_i \text{ is not overlined,}\\[2ex] \begin{pmatrix}m_{i+1}+x_{i+1}+5\\ x_i-1\\ \delta_i-4\end{pmatrix} & \text{if } m_i \text{ is overlined,}\end{cases} \tag{2.18}$$

and

$$\phi^{-1}:\begin{pmatrix}\alpha_i\\ \beta_i\\ \gamma_i\end{pmatrix}\to\begin{cases}(\alpha_i+\beta_i+\gamma_i)_{\beta_i+1} & \text{if } \alpha_i=2+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1},\\ (\overline{\alpha_i+\beta_i+\gamma_i})_{\beta_i+1} & \text{if } \alpha_i=6+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1}.\end{cases} \tag{2.19}$$

The combinatorial interpretations in terms of three-line arrays and n–color overpartitions of (2.16) and (2.17) are given in Theorem 2.10 and Theorem 2.11 respectively.

Theorem 2.10. *Let $\hat{A}_{10}(\nu)$ represent the number of three-line arrays into r columns satisfying*

(2.18.a) $\alpha_r=1$,

(2.18.b) $\alpha_i\in\{2+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1},6+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1}\}$, *for* $1\le i\le r-1$

(2.18.c) $\gamma_i\equiv 0 \pmod 4$, $\forall$ i.

Let $\overline{\hat{A}}_{10}(\nu)$ count the number of n–color overpartitions of ν into r satisfying

(2.18.d) $m_r-x_r\equiv 0 \pmod 4$,

(2.18.e) $(m_r)_{x_r}$ *is not overlined,* (2.18.f) $\delta_i \geq 0$ *and* $\delta_i \equiv 0 \pmod 4$, $\forall\ i < r$. *For* $\delta_i = 0$, m_i *is not overlined.*

Let $B_{10}(\nu) = \sum_{k=0}^{\nu} C_{10}(\nu-k)D_{10}(k)$, *where* $C_{10}(\nu)$ *is the number of partitions of* ν *in which the parts are* $\equiv \pm 1, 4 \pmod 8$ *and* $D_{10}(\nu)$ *is the number of partitions of* ν *in which the parts are* $\equiv \pm 1, 4 \pmod 8$. *Then,*
$2\hat{A}_{10}(\nu) = A_{10}(\nu) = 2\overline{\hat{A}}_{10}(\nu) = \overline{A}_{10}(\nu) = B_{10}(\nu)\ \forall\ \nu \geq 1.$

Remark 2.6. *In the above theorem, we used similar argument as given in* (2.15) *and letting* $\sum_{\nu=1}^{\infty} \hat{A}_{10}(\nu)q^{\nu} = \sum_{n=1}^{\infty} \frac{(-q^4;q^4)_{n-1}q^{n^2}}{(q;q^2)_n(q^4,q^4)_n}$.

Theorem 2.11. *Let* $\hat{A}_{11}(\nu)$ *represent the number of three-line arrays into* r *columns satisfying* $\alpha_r \geq 3$ *and* (2.18.b)–(2.18.c). *Let* $\overline{\hat{A}}_{11}(\nu)$ *count the number of* n*–color overpartitions of* ν *into* r *parts satisfying along with* (2.18.f)*:*

(2.20.a) $m_r > 2$,

(2.20.b) $m_r - x_r \equiv 2 \pmod 4$, $(m_r)_{x_r}$ *is not overlined.*

Let $B_{11}(\nu) = \sum_{k=0}^{\nu} C_{11}(\nu-k)D_{11}(k)$, *where* $C_{11}(\nu)$ *is the number of partitions of* ν *in which the parts are* $\equiv \pm 3, 4 \pmod 8$ *and* $D_{11}(\nu)$ *is the number of partitions of* ν *into distinct parts that are* $\equiv \pm 3, 4 \pmod 8$. *Then,*
$2\hat{A}_{11}(\nu) = A_{11}(\nu) = 2\overline{\hat{A}}_{11}(\nu) = \overline{A}_{11}(\nu) = B_{11}(\nu)\ \forall\ \nu \geq 1.$

Remark 2.7. *In the above theorem, we used similar argument as given in* (2.15) *and letting* $\sum_{\nu=1}^{\infty} \hat{A}_{11}(\nu)q^{\nu} = \sum_{n=1}^{\infty} \frac{(-q^4;q^4)_{n-1}q^{n(n+2)}}{(q;q^2)_n(q^4,q^4)_n}$. *The mapping for the* r^{th} *part is* $\phi : (m_r)_{x_r} \to \begin{pmatrix} 3 \\ x_r - 1 \\ m_r - x_r - 2 \end{pmatrix}$. *And the inverse mapping is same as in* (2.19).

Group 5

The RRTIs (2.20)–(2.22) below appear in [6] as Identity No. 37, 106, 40 respectively.

$$\sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n^2}}{(q;q^2)_n(q;q)_n} = \frac{(-q;q)_\infty}{(q;q)_\infty}[q^3, q^3, q^6; q^6]_\infty, \tag{2.20}$$

$$\sum_{n=1}^{\infty} \frac{(-q;q)_{n-1} q^{n^2}}{(q;q^2)_n(q;q)_n} = \frac{[-q^5, -q^7, q^{12}; q^{12}]_\infty}{(q;q)_\infty}, \tag{2.21}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+1)}}{(q;q^2)_{n+1}(q;q)_n} = \frac{(-q;q)_\infty}{(q;q)_\infty}[q, q^5, q^6; q^6]_\infty. \tag{2.22}$$

To establish the bijection between three-line arrays and $(n+t)$–color overpartitions for (2.20)–(2.22), we use

$$\phi : (m_i)_{x_i} \to \begin{cases} \begin{pmatrix} m_{i+1}+x_{i+1}+1 \\ x_i - 1 \\ \delta_i \end{pmatrix} & \text{if } m_i \text{ is not overlined,} \\ \\ \begin{pmatrix} m_{i+1}+x_{i+1}+2 \\ x_i - 1 \\ \delta_i - 1 \end{pmatrix} & \text{if } m_i \text{ is overlined,} \end{cases}$$

and

$$\phi^{-1} : \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{pmatrix} \to \begin{cases} (\alpha_i+\beta_i+\gamma_i)_{\beta_i+1} & \text{if } \alpha_i = 2+\alpha_{i+1}+\beta_{i+1}+\gamma_{i+1}, \\ \overline{(\alpha_i+\beta_i+\gamma_i)}_{\beta_i+2} & \text{if } \alpha_i = 3+\alpha_{i+1}+\beta_{i+1}+\gamma_{i+1}. \end{cases}$$

The combinatorial interpretations in terms of three-line arrays and n–color overpartitions of (2.20)–(2.22) are given in Theorems 2.12–2.14 respectively.

Theorem 2.12. *Let $\hat{A}_{12}(\nu)$ represent the number of three-line arrays into r columns satisfying*

(2.22.a) $\alpha_r = 1$,

(2.22.b) $\alpha_i \in \{2+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1},\ 3+\alpha_{i+1}+2\beta_{i+1}+\gamma_{i+1}\}$ *for* $1 \le i \le r-1$.

Let $\overline{\hat{A}}_{12}(\nu)$ count the number of n–color overpartitions of ν into r parts satisfying

(2.22.c) $(m_r)_{x_r}$ *is not overlined,*

(2.22.d) $\delta_i \ge 0$, $\forall\ i < r$. *For* $\delta_i = 0$, m_i *is not overlined.*

Let $B_{12}(\nu) = \sum_{k=0}^{\nu} C_{12}(\nu-k)D_{12}(k)$, where $C_{12}(\nu)$ is the number of partitions of ν in which the parts are $\equiv \pm1, \pm2 \pmod 6$ and $D_{12}(\nu)$ is the number of partitions of ν into the distinct parts that are $\equiv \pm1, \pm2 \pmod 6$. Then,
$2\hat{A}_{12}(\nu) = A_{12}(\nu) = 2\overline{\hat{A}}_{12}(\nu) = \overline{A}_{12}(\nu) = B_{12}(\nu)\ \forall\ \nu \ge 1.$

Remark 2.8. *In the above theorem, we used similar argument as given in* (2.15) *and letting* $\sum_{\nu=1}^{\infty} \hat{A}_{12}(\nu)q^{\nu} = \sum_{n=1}^{\infty} \frac{(-q;q)_{n-1}q^{n^2}}{(q;q^2)_n(q,q)_n}$.

Theorem 2.13. *Let $A_{13}(\nu)$ represent the number of three-line arrays into r columns and $\overline{A}_{13}(\nu)$ counts the number of n–color overpartitions of ν into r parts* *satisfying all the conditions of* $\hat{A}_{12}(\nu)$ *and* $\overline{\hat{A}}_{12}(\nu)$ *defined in Theorem* 2.22 *respectively. Let* $B_{13}(\nu) = \sum_{k=0}^{\nu} C_{13}(\nu - k)D_{13}(k)$, *where* $C_{13}(\nu)$ *is the number of partitions of* ν *and* $D_{13}(\nu)$ *is the partitions of* ν *in which the parts are* $\equiv \pm 5 \pmod{12}$. *Then* $A_{13}(\nu) = \overline{A}_{13}(\nu) = B_{13}(\nu)\ \forall\ \nu \geq 1$.

Theorem 2.14. *Let* $A_{14}(\nu)$ *represent the number of three-line arrays into* r *columns satisfying* (2.22.b) *along with:*

(2.25.a) $\alpha_r = 0 = \gamma_r$,

(2.25.b) $\alpha_{r-1} \geq 2$.

Let $\overline{A}_{14}(\nu)$ *counts the number of* $(n+1)$*–color overpartitions of* ν *into* r *parts satisfying* (2.22.d) *along with:*

(2.25.c) $x_r = m_r + 1$,

(2.25.d) $(m_r)_{x_r}$ *is not overlined,*

(2.25.e) $m_i - x_i \equiv 1 \pmod{2}$, $\forall\ i$.

Let $B_{14}(\nu) = \sum_{k=0}^{\nu} C_{14}(\nu - k)D_{14}(k)$, *where* $C_{14}(\nu)$ *is the number of partitions of* ν *in which the parts are* $\equiv \pm 2, 3 \pmod{6}$ *and* $D_{14}(\nu)$ *is the number of partitions of* ν *into distinct parts. Then* $A_{14}(\nu) = \overline{A}_{14}(\nu) = B_{14}(\nu)\ \forall\ \nu \geq 0$.

The Proof of Theorems 2.2–2.14 can be supplied by reader on lines of Theorem 2.1, hence omitted.

3. Alternative Proofs

In this section we provide an alternate proof for arrays enumerated by $A_i(\nu)$, where $1 \leq i \leq 14$. Throughout this section, if $A_i(\nu)$ denote the arrays with some conditions in any number of columns, then $A_i(r, \nu)$ will denote the arrays with same conditions into r columns. In the proofs we follow the method of proof of [1]. Due to the similarity of the proofs of the Theorem 2.1–Theorem 2.14 presented so far only detailed proof of Theorem 2.1 is given below and for rest we will only make an outline of them.

Proof of Theorem 2.1. Split the arrays enumerated by $A_1(r, \nu)$ into the following four classes:

(i) those arrays in which r^{th} column is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,

(ii) those arrays in which r^{th} column is $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$,

(iii) those arrays in which $\gamma_r \neq 0$,

(iv) those arrays in which $\gamma_r = 0$, and $\beta_r \neq 0$.

Transform the arrays of class (i) by eliminating the r^{th} column $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ of the array and subtracting 2 from all α_i, keeping β_i and γ_i as same for $1 \leq i \leq r-1$. We see that transformed arrays are enumerated by $A_1(r-1, \nu - 2r + 1)$. In class (ii) deleting r^{th} column $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ of the array and subtracting 4 from all α_i, keeping β_i and γ_i as same for $1 \leq i \leq r-1$. The transformed arrays are enumerated by $A_1(r-1, \nu - 4r + 2)$. In class (iii) subtracting 2 from α_i, $1 \leq i \leq r-1$, keeping β_i $\forall$ i as same and subtract 2 from γ_r. Remaining γ_i, $1 \leq i \leq r-1$ remains same. The transformed arrays are enumerated by $A_1(r, \nu - 2r)$.
Finally, we transform the arrays of class (iv) by subtracting 2 from α_i, $\forall$ i, and subtracting 1 from β_r and remaining β_i for $1 \leq i \leq r-1$ and γ_i $\forall$ i are same, we see that transformed arrays are enumerated by $A_1(r, \nu - 2r + 1)$ having the r^{th} column as $\begin{pmatrix} \alpha_i \\ \beta_i \\ 0 \end{pmatrix}$, $\beta_i \neq 0$. Thus number of arrays in class (iv) are obtained by subtracting the number of array which are enumerated by $A_1(r, \nu - 2r + 1)$ with the r^{th} column as $\begin{pmatrix} \alpha_r \\ \beta_r \\ \gamma_r \end{pmatrix}$ where $\gamma_r \neq 0$ from $A_1(r, \nu - 2r + 1)$. Thus the transformed arrays are enumerated by $A_1(r, \nu - 2r + 1) - A_1(r, \nu - 4r + 1)$. Hence we get the following recurrence formula for $A_1(r, \nu)$:

$$\begin{aligned} A_1(r, \nu) = A_1(r-1, \nu - 2r + 1) + A_1(r-1, \nu - 4r + 2) + A_1(r, \nu - 2r) \\ + A_1(r, \nu - 2r + 1) - A_1(r, \nu - 4r + 1), \end{aligned} \tag{3.1}$$

where $A_1(0, 0) = 1$ and $A_1(r, \nu) = 0$ for $\nu < 0$.
For $|q| < 1$ and $|z| < |q|^{-1}$, let $g_1(z, q)$ be defined by

$$g_1(z, q) = \sum_{\nu=0}^{\infty} \sum_{r=0}^{\infty} A_1(r, \nu) z^r q^\nu. \tag{3.2}$$

Substitute $A_1(r, \nu)$ from (3.1) in (3.2), we get q–functional equation

$$\begin{aligned} g_1(z, q) = zqg_1(zq^2, q) + zq^2 g_1(zq^4, q) + g_1(zq^2, q) \\ + q^{-1} g_1(zq^2, q) - q^{-1} g_1(zq^4, q). \end{aligned} \tag{3.3}$$

Setting

$$g_1(z,q) = \sum_{n=0}^{\infty} w_1(n,q) z^n. \tag{3.4}$$

Using (3.3) in (3.4) and then examining the coefficients of z^n, we get

$$w_1(n,q) = \frac{q^{2n-1}(1+q^{2n-1})}{(1-q^{2n})(1-q^{2n-1})} w_1(n-1,q). \tag{3.5}$$

Iterating (3.5) n times and observing $w_1(0,q) = 1$, we find that

$$w_1(n,q) = \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n (q;q^2)_n}. \tag{3.6}$$

Therefore,

$$g_1(z,q) \;=\; \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n (q;q^2)_n} z^n$$

and

$$\begin{aligned}
\sum_{\nu=0}^{\infty} A_1(\nu) q^{\nu} &= \sum_{\nu=0}^{\infty} \left(\sum_{r=0}^{\infty} A_1(r,\nu) \right) q^{\nu} \\
&= g_1(1,q) \\
&= \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n (q;q^2)_n}.
\end{aligned}$$

Hence the proof. Now we give a concise proof of remaining theorems.

Sketch Proof of Theorem 2.2.
The classes of $A_2(r,\nu)$ are same as defined for $A_1(r,\nu)$. Using the similar transformation we get the following recurrence relation for $A_2(r,\nu)$.

$$\begin{aligned}
A_2(r,\nu) &= A_2(r-1,\nu-2r+1) + A_2(r-1,\nu-4r+2) + A_2(r,\nu-4r) \\
&\quad + A_2(r,\nu-2r+1) - A_2(r,\nu-6r-1).
\end{aligned}$$

The remaining proof can be supplied by the reader as done earlier.

Sketch Proof of Theorem 2.3.
Split the arrays into two classes, first containing the arrays with $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ as their

r^{th} column and second containing the arrays with $\begin{pmatrix} 0 \\ \beta_r \\ 0 \end{pmatrix}$ as r^{th} column. After transformation, the arrays in the first class are enumerated by $A_1(r-1, \nu-r+1)$ and in the second class are enumerated by $A_3(r, \nu-2r+1)$. Thus the recurrence relation becomes

$$A_3(r, \nu) = A_1(r-1, \nu-r+1) + A_3(r, \nu-2r+1).$$

The remaining proofs for Theorems 2.2-2.3 can be supplied by the reader as done earlier.

For upcoming lemmas and theorems we only provide the recurrence relations and the corresponding classes are given in Table 2. As the detailed proof is on similar lines as done earlier, hence omitted.

Sketch proof of Theorems 2.4–2.6.

The recurrence relations for the enumerates $A_i(r, \nu)$, $4 \le i \le 6$, are:

$$\begin{aligned} A_4(r, \nu) &= A_1(r-1, \nu-2r+2) + A_4(r, \nu-2r+1), \\ A_5(r, \nu) &= A_5(r-1, \nu-2r-1) + A_5(r-1, \nu-4r) + A_5(r, \nu-4r) \\ &\quad +A_5(r, \nu-2r+1) - A_5(r, \nu-6r+1), \\ A_6(r, \nu) &= A_5(r-1, \nu) + A_6(r, \nu-2r+1). \end{aligned}$$

Sketch Proof of Theorem 2.7.

To obtain the recurrence relation for the enumerator $A_7(r, \nu)$, we consider the following q–series:

$$\sum_{\nu=0}^{\infty} M_1(\nu) q^{\nu} = \sum_{n=0}^{\infty} \frac{(-q^2; q^2)_n q^{n^2}}{(q; q)_{2n}}. \tag{3.7}$$

The interpretation of (3.7) in terms of three-line arrays is given in following lemma.

Lemma 3.1. *For $\nu \ge 0$, let $M_1(\nu)$ represent the number of three-line arrays satisfying $\alpha_i \in \{1, 3\}$, (2.12.b) and (2.12.c).*

Sketch Proof of Lemma 3.1.

Let $M_1(r, \nu)$ represent the number of three-line arrays enumerated by $M_1(\nu)$ of ν into r columns. We split the arrays enumerated by $M_1(r, \nu)$ into four classes and get the recurrence relation:

$$\begin{aligned} M_1(r, \nu) &= M_1(r-1, \nu-2r+1) + M_1(r-1, \nu-4r+1) + M_1(r, \nu-2r) \\ &\quad +M_1(r, \nu-2r+1) - M_1(r, \nu-4r+1), \end{aligned}$$

where $M_1(0, 0) = 1$ and $M_1(r, \nu) = 0$ for $\nu < 0$.

We now find the recurrence relation for $A_7(r,\nu)$ using the classes of $M_1(r,\nu)$:

$$\begin{aligned}A_7(r,\nu) &= M_1(r-1,\nu-2r+1)+M_1(r-1,\nu-4r+1)+A_7(r,\nu-4r)\\ &\quad +A_7(r,\nu-2r+1)-A_7(r,\nu-6r+1).\end{aligned}$$

Sketch Proof of Theorems 2.8 and 2.9
To obtain the recurrence relation for the enumerator $\hat{A}_8(r,\nu)$ and $A_9(r,\nu)$, where $\hat{A}_8(r,\nu)$ is $\hat{A}_8(\nu)$ into r columns, we consider the following q–series:

$$\sum_{\nu=0}^{\infty} M_2(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n(n+1)}}{(q;q)_{2n}}. \tag{3.8}$$

The combinatorial interpretation of (3.8) is given in Lemma 3.2.

Lemma 3.2. *For $\nu \geq 0$, let $M_2(\nu)$ represent the number of three-line arrays satisfying $\alpha_i \in \{2,4\}$, (2.12.b) and (2.12.c).*
Sketch Proof of Lemma 3.2.
The recurrence relation for $M_2(r,\nu)$, where $M_2(r,\nu)$ represent $M_2(\nu)$ into r columns is:

$$\begin{aligned}M_2(r,\nu) &= M_2(r-1,\nu-2r)+M_2(r-1,\nu-4r)+M_2(r,\nu-2r)\\ &\quad +M_2(r,\nu-2r+1)-M_2(r,\nu-4r+1).\end{aligned}$$

Now we find the recurrence relations for $\hat{A}_8(r,\nu)$ and $A_9(r,\nu)$:

$$\begin{aligned}\hat{A}_8(r,\nu) &= M_2(r-1,\nu-2r)+M_2(r-1,\nu-4r)+\hat{A}_8(r,\nu-4r)\\ &\quad +\hat{A}_8(r,\nu-2r+1)-\hat{A}_8(r,\nu-6r+1),\\ A_9(r,\nu) &= M_2(r-1,\nu-2r+1)+A_9(r,\nu-2r+1).\end{aligned}$$

Sketch Proof of Theorem 2.10.
To obtain the recurrence relation for the enumerator $\hat{A}_{10}(r,\nu)$, where $\hat{A}_{10}(r,\nu)$ is $\hat{A}_{10}(\nu)$ into r columns, we consider the following q–series:

$$\sum_{\nu=0}^{\infty} M_3(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{(-q^4;q^4)_n q^{n^2}}{(q;q^2)_n(q^4;q^4)_n}. \tag{3.9}$$

The combinatorial interpretation of (3.9) is given in Lemma 3.3.

Lemma 3.3. *For $\nu \geq 0$, let $M_3(\nu)$ represent the number of three-line arrays satisfying $\alpha_i \in \{1,5\}$, (2.18.b) and (2.18.c).*
Sketch Proof of Lemma 3.3.
The recurrence relation for $M_3(r,\nu)$, where $M_3(r,\nu)$ represent $M_3(\nu)$ into r columns is:

$$\begin{aligned}M_3(r,\nu) &= M_3(r-1,\nu-2r+1)+M_3(r-1,\nu-6r+1)+M_3(r,\nu-4r)\\&\quad+M_3(r,\nu-2r+1)-M_3(r,\nu-6r+1).\end{aligned}$$

Now we find the recurrence relations for $\hat{A}_{10}(r,\nu)$:

$$\begin{aligned}\hat{A}_{10}(r,\nu) &= M_3(r-1,\nu-2r+1)+M_3(r-1,\nu-6r+1)+\hat{A}_{10}(r,\nu-8r)\\&\quad+\hat{A}_{10}(r,\nu-2r+1)-\hat{A}_{10}(r,\nu-10r+1).\end{aligned}$$

Sketch Proof of Theorem 2.11.
To obtain the recurrence relation for the enumerator $\hat{A}_{11}(r,\nu)$, where $\hat{A}_{11}(r,\nu)$ is $\hat{A}_{11}(\nu)$ into r columns, we consider the following q–series:

$$\sum_{\nu=0}^{\infty} M_4(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{(-q^4;q^4)_n q^{n(n+2)}}{(q;q^2)_n(q^4;q^4)_n}. \tag{3.10}$$

The combinatorial interpretation of (3.10) is given in Lemma 3.4.

Lemma 3.4. *For $\nu \geq 0$, let $M_4(\nu)$ represent the number of three-line arrays satisfying $\alpha_i \in \{3,7\}$, (2.18.b) and (2.18.c).*
Sketch Proof of Lemma 3.4.
The recurrence relation for $M_4(r,\nu)$, where $M_4(r,\nu)$ represent $M_4(\nu)$ into r columns is:

$$\begin{aligned}M_4(r,\nu) &= M_4(r-1,\nu-2r-1)+M_4(r-1,\nu-6r-1)+M_4(r,\nu-4r)\\&\quad+M_4(r,\nu-2r+1)-M_4(r,\nu-6r+1).\end{aligned}$$

And the recurrence relations for $\hat{A}_{11}(r,\nu)$ is:

$$\begin{aligned}\hat{A}_{11}(r,\nu) &= M_4(r-1,\nu-2r-1)+M_4(r-1,\nu-6r-1)+\hat{A}_{11}(r,\nu-8r)\\&\quad+\hat{A}_{11}(r,\nu-2r+1)-\hat{A}_{11}(r,\nu-10r+1).\end{aligned}$$

Sketch proof of Theorem 2.12.
To obtain the recurrence relation for the enumerator $\hat{A}_{12}(r,\nu)$, where $\hat{A}_{12}(r,\nu)$ is $\hat{A}_{12}(\nu)$ into r columns, we consider the following q–series:

$$\sum_{\nu=0}^{\infty} M_5(\nu)q^{\nu} = \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n^2}}{(q;q^2)_n(q;q)_n}. \tag{3.11}$$

The combinatorial interpretation of (3.11) is given in Lemma 3.5.

Lemma 3.5. *For $\nu \geq 0$, let $M_5(\nu)$ represent the number of three-line arrays satisfying $\alpha_i \in \{1,5\}$, and (2.22.b).*
Sketch Proof of Lemma 3.5.
The recurrence relation for $M_5(r,\nu)$, where $M_5(r,\nu)$ represent $M_5(\nu)$ into r columns is:

$$\begin{aligned} M_5(r,\nu) &= M_5(r-1,\nu-2r+1)+M_5(r-1,\nu-3r+1)+M_5(r,\nu-r)\\ &\quad +M_5(r,\nu-2r+1)-M_5(r,\nu-3r+1). \end{aligned}$$

And the recurrence relations for $\hat{A}_{12}(r,\nu)$ is:

$$\begin{aligned} \hat{A}_{12}(r,\nu) &= M_5(r-1,\nu-2r+1)+M_5(r-1,\nu-3r+1)+\hat{A}_{12}(r,\nu-2r)\\ &\quad +\hat{A}_{12}(r,\nu-2r+1)-\hat{A}_{12}(r,\nu-4r+1). \end{aligned}$$

As $\hat{A}_{12}(\nu) = A_{13}(\nu)$, so the recurrence relations for the enumerators $\hat{A}_{12}(\nu)$ and $A_{13}(\nu)$ are same.

Sketch proof of Theorem 2.14.
The recurrence relation for the enumerator $A_{14}(r,\nu)$ is

$$A_{14}(r,\nu) = M_5(r-1,\nu-2r+1)+A_{14}(r,\nu-2r+1).$$

Table 2

Enumerator	class 1	class 2	Enumerator	class 1	class 2
$A_4(\nu)$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\\beta_r\\0\end{pmatrix}$	$\hat{A}_{12}(\nu)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$
$A_5(\nu)$	$\begin{pmatrix}3\\0\\0\end{pmatrix}$	$\begin{pmatrix}4\\0\\0\end{pmatrix}$	$A_{13}(\nu)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\end{pmatrix}$
$A_6(\nu)$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\\beta_r\\0\end{pmatrix}$	$A_{14}(\nu)$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\\beta_r\\0\end{pmatrix}$
$\hat{A}_8(\nu)$	$\begin{pmatrix}2\\0\\0\end{pmatrix}$	$\begin{pmatrix}2\\0\\2\end{pmatrix}$	$M_2(\nu)$	$\begin{pmatrix}2\\0\\0\end{pmatrix}$	$\begin{pmatrix}4\\0\\0\end{pmatrix}$
$A_9(\nu)$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\\beta_r\\0\end{pmatrix}$	$M_3(\nu)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}5\\0\\0\end{pmatrix}$
$A_{10}(\nu)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\4\end{pmatrix}$	$M_4(\nu)$	$\begin{pmatrix}3\\0\\0\end{pmatrix}$	$\begin{pmatrix}7\\0\\0\end{pmatrix}$
$\hat{A}_{11}(\nu)$	$\begin{pmatrix}3\\0\\0\end{pmatrix}$	$\begin{pmatrix}3\\0\\4\end{pmatrix}$	$M_5(\nu)$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}2\\0\\0\end{pmatrix}$

In Table 2, we provide the classes and the corresponding enumerator. For $A_i(\nu)$, $i = 4, 6, 9$ and 14 we have only two classes given in the Table 2, and for the remaining enumerators we have four classes: first and second classes are given in the table, third class has those arrays in which $\gamma_r \neq 0$, and fourth class has those arrays in which $\gamma_r = 0$, and $\beta_r \neq 0$.

References

[1] Agarwal A. K., Andrews G. E., Rogers-Ramanujan identities for partitions with "N copies of N", Journal of Combinatorial Theory, Series A, 45(1), (1987), 40-49.

[2] Alegri M., Interpretações para Identidades Envolvendo Sobrepartições e Partições Planas, Ph. D. Thesis, IMECC-Universidade Estadual de Campinas, Campinas-SP, 2010.

[3] Alegri M., Brietzke E. H. M., Santos J. P. O., Silva R., Bijections between lattice paths and plane partitions, Open Journal of Discrete Mathematics, 1(3), (2011), 108-115.

[4] Brietzke E. H. M., Santos J. P. O., Silva R., Bijective proofs using two-line matrix representations for partitions, The Ramanujan Journal, 23(1-3), (2010), 265-295.

[5] Brietzke E. H. M., Santos J. P. O., Silva R., A new approach and generalizations to some results about mock theta functions, Discrete Mathematics, 311(8-9), (2011), 595-615.

[6] Chu W., Zhang W., Bilateral bailey lemma and Rogers–Ramanujan identities, Advances in Applied Mathematics, 42(3), (2009), 358-391.

[7] Lovejoy J., Mallet O., n–color overpartitions, twisted divisor functions and Rogers–Ramanujan identities, South East Asian Journal of Mathematics and Mathematical Science (Andrews' 70th birthday issue), 6, (2008), 23-36.

[8] Mallet O., n–color overpartitions, lattice paths, and multiple basic hypergeometric series, Discrete Mathematics & Theoretical Computer Science, 2008.

[9] Santos J. P. O., Mondek P., Ribeiro A. C., New two-line arrays representing partitions, Annals of Combinatorics, 15(2), (2011), 341.

[10] Slater L. J., Further identities of the Rogers–Ramanujan type, Proceedings of the London Mathematical Society, 2(1), (1952), 147-167.