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TRIANGULAR NEUTROSOPHIC GRACEFUL LABELING OF SOME GRAPHS

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Abstract: In this article, triangular neutrosophic number (TrNN) is applied with the neutrosophic graceful labeling graph and as a result, the triangular neutrosophic graceful labeling (TrNGL) graphs are obtained. Also, operations like union and join are tested with TrNGL path graphs. In addition, some applications and its related algorithm is stated to enrich this new labeling concept.

Keywords and Phrases: Graceful labeling, Triangular neutrosophic number, Neutrosophic labeling graph, Triangular neutrosophic graceful labeling.

2020 Mathematics Subject Classification: 05C78.

1. Introduction

Graph theory acts as an effective phenomena to visualize an idea in a graphical manner using vertices and edges. Though there are many crisp graph models, fuzzy graph approach came into existence to overcome the issues raised during application based on approximation and certainty of output values. Kaufmann [17] relied on Zadeh's [29] fuzzy set theory to formulate the components that form fuzzy graphs. Rosenfeld [22] brought new graph models which is used to carry through some applications using its fuzzified structural components. Later, some researchers entered into fuzzy field and done many innovative works on fuzzy graph theory and its wide applications. Solomon W. Golomb was the first one to name a labeling called "Graceful labeling", which is introduced first in crisp graph and then it is also dealt with fuzzy graphs. Jahir Hussain et al. [15] initiated the proposal of using graceful labeling in fuzzy graphical structure. It is quite useful to the forthcoming researchers to introduce edge and vertex graceful labeling on fuzzy graphs. Sujatha et al. [26] started to incorporate the triangular fuzzy number in fuzzy graceful labeling of some graphs. Atanassov [8] improvised the theories on fuzzy set and coined a brand new set theory in the name "Intuitionistic fuzzy set(InFS)" to deal with ambiguous results attained during final output. Parvathi et al. [19, 20] developed the graphical representations and components using InFS and further called it as intuitionistic fuzzy graph(InFG). Nagoorgani [18] and Akram [1, 2] enlightened some theories and dealt with specific structural findings on InFG. Rabeeh Ahamed et al. [21] implicated the triangular intuitionistic fuzzy number in graceful labeling of intuitionistic fuzzy graphs. Smarandache [23, 24] and Broumi [12] noticed the unsolved ambiguity in InFS & InFG ideas and flourished neutrosophic set & neutrosophic graphs, which replaced the uncertainty with nearly accurate values in real-life implications. Akram [3, 4, 5] illuminated the neutrosophic graphical phenomena by the execution of operations in single valued neutrosophic graphs and his works on neutrosophic competition graphs. Gomathi et al. [14] executed a novel work by considering the labeling hypothesis on neutrosophic graphs. Chakraborty et al. [13] already organized all possible forms of triangular neutrosophic number(TrNN), de-neutrosophication techniques and its applications. Bhimraj Basumatary and Broumi [11] applied the interval valued TrNN in a linear programming problem. Subadhra Srinivas and Prabhakaran [25] enacted to solve neutrosophic transportation problem using TrNN. As mentioned, triangular neutrosophic number was already used in some works related to neutrosophic theory but it was not applied with graceful labeling of neutrosophic kind. This novel implementation of TrNN in neutrosophic graceful labeling graph bridges the gap between the fuzzy approach and neutrosophic extension. It is an effective model since the indeterminacy cases are taken into account, to improvise the certainty in results regarding real-world scenario.

This article presents some new results and discussions on neutrosophic graceful labeling using TrNN. An application and algorithm which deals with triangular neutrosophic graceful (TrNG) labeling graph is also illustrated.

1.1. Motivation

Labeling is a fascinating concept on graph theory, which is studied in a specific manner based on the labeling types. In particular, graceful labeling has a wide discussion on crisp and fuzzy graphical approach. But this concept was not yet applied with neutrosophic graph theory. In this article, a triangular neutrosophic number is used to define the label values for vertices and edges of a neutrosophic graceful labeled graph. Thereby, new applicable areas based on this concept can be excavated.

1.2. Objective and significance

To shorten the research gap between the fuzzy and neutrosophic graph theory, it is necessary to incur the fuzzy graphical establishments in neutrosophic graphical information. Our aim is to introduce the fuzzy labeling concepts and features to neutrosophic labeling graphs. Here, graceful labeling of graphs is priorly taken to deal with neutrosophic environment. This will benefit in addressing the current life events.

1.3. Article structure

This article is framed as follows: The introductory part of fuzzy and its extensional features along with graceful labeling are portrayed in the section 1. Section 2 list the basic definitions of neutrosophic graphs, triangular neutrosophic number(TrNN) and some operations. In section 3, neutrosophic graceful labeled graphs is implemented with TrNN and some results are obtained. Some applications regarding triangular neutrosophic graceful labeling is furnished in section 4.

2. Preliminaries

Definition 2.1. [14] *Consider a neutrosophic graph $Gr = (P, Q)$, where $\sigma = (\alpha_P, \beta_P, \gamma_P)$ and $\mu = (\alpha_Q, \beta_Q, \gamma_Q)$ with the following conditions,*

(i) Let $\alpha_P : P \to [0, 1]$, $\beta_P : P \to [0, 1]$ and $\gamma_P : P \to [0, 1]$ indicates the degree of existence, uncertain and non-existence functions of the element $a_i \in P$, respectively and $0 \leq \alpha_P(a_i) + \beta_P(a_i) + \gamma_P(a_i) \leq 3$, $\forall a_i \in P$.

(ii) Let $\alpha_Q : P \times P \to [0, 1]$, $\beta_Q : P \times P \to [0, 1]$ and $\gamma_Q : P \times P \to [0, 1]$ indicates the degree of existence, uncertain and non-existence functions of the edge (a_i, a_j) respectively, such that
$\alpha_Q(a_i, a_j) \leq [\alpha_P(a_i) \wedge \alpha_P(a_j)]$,
$\beta_Q(a_i, a_j) \leq [\beta_P(a_i) \wedge \beta_P(a_j)]$,
$\gamma_Q(a_i, a_j) \leq [\gamma_P(a_i) \vee \gamma_P(a_j)]$ *and*
$0 \leq \alpha_Q(a_i, a_j) + \beta_Q(a_i, a_j) + \gamma_Q(a_i, a_j) \leq 3$, *for every edge $(a_i, a_j) \in Q$.*

Definition 2.2. [14] *Consider a neutrosophic graph $Gr = (P, Q)$, in which $\sigma = (\alpha_P, \beta_P, \gamma_P)$ and $\mu = (\alpha_Q, \beta_Q, \gamma_Q)$ and it is known as neutrosophic labeling graph, if $\alpha_P : P \to [0, 1]$, $\beta_P : P \to [0, 1]$, $\gamma_P : P \to [0, 1]$ and $\alpha_Q : P \times P \to [0, 1]$,*

$\beta_Q : P \times P \to [0,1]$, $\gamma_Q : P \times P \to [0,1]$ are one-to-one and onto such that the existence, uncertain and non-existence functions are distinct with respect to vertices and edges. Also,
$\alpha_Q(a_i, a_j) \leq [\alpha_P(a_i) \wedge \alpha_P(a_j)]$,
$\beta_Q(a_i, a_j) \leq [\beta_P(a_i) \wedge \beta_P(a_j)]$,
$\gamma_Q(a_i, a_j) \leq [\gamma_P(a_i) \vee \gamma_P(a_j)]$ *and*
$0 \leq \alpha_Q(a_i, a_j) + \beta_Q(a_i, a_j) + \gamma_Q(a_i, a_j) \leq 3$, *for every edge* (a_i, a_j).

Definition 2.3. *Let $Gr_1 = (P_1, Q_1)$ and $Gr_2 = (P_2, Q_2)$ be neutrosophic graphs (NeGs) of $Gr_1' = (\mathscr{V}_1, \mathscr{E}_1)$ and $Gr_2' = (\mathscr{V}_2, \mathscr{E}_2)$, respectively. Then union of NeGs $Gr_1 \cup Gr_2$ is mentioned as (P,Q) such that*

$$(i) \quad \alpha_P(a) = \begin{cases} \alpha_{P_1}(a) \quad if \ a \in \mathscr{V}_1 \ and \ a \notin \mathscr{V}_2, \\ \alpha_{P_2}(a) \quad if \ a \in \mathscr{V}_2 \ and \ a \notin \mathscr{V}_1, \\ (\alpha_{P_1}(a) \vee \alpha_{P_2}(a)) \ if \ a \in \mathscr{V}_1 \cap \mathscr{V}_2. \end{cases}$$

$$\beta_P(a) = \begin{cases} \beta_{P_1}(a) \quad if \ a \in \mathscr{V}_1 \ and \ a \notin \mathscr{V}_2, \\ \beta_{P_2}(a) \quad if \ a \in \mathscr{V}_2 \ and \ a \notin \mathscr{V}_1, \\ (\beta_{P_1}(a) \vee \beta_{P_2}(a)) \ if \ a \in \mathscr{V}_1 \cap \mathscr{V}_2. \end{cases}$$

$$\gamma_P(a) = \begin{cases} \gamma_{P_1}(a) \quad if \ a \in \mathscr{V}_1 \ and \ a \notin \mathscr{V}_2, \\ \gamma_{P_2}(a) \quad if \ a \in \mathscr{V}_2 \ and \ a \notin \mathscr{V}_1, \\ (\gamma_{P_1}(a) \wedge \gamma_{P_2}(a)) \ if \ a \in \mathscr{V}_1 \cap \mathscr{V}_2. \end{cases}$$

$$(ii) \quad \alpha_Q(ab) = \begin{cases} \alpha_{Q_1}(ab) \quad if \ ab \in \mathscr{E}_1 \ and \ ab \notin \mathscr{E}_2, \\ \alpha_{Q_2}(ab) \quad if \ ab \in \mathscr{E}_2 \ and \ ab \notin \mathscr{E}_1, \\ (\alpha_{Q_1}(ab) \vee \alpha_{Q_2}(ab)) \ if \ ab \in \mathscr{E}_1 \cap \mathscr{E}_2. \end{cases}$$

$$\beta_Q(ab) = \begin{cases} \beta_{Q_1}(ab) \quad if \ ab \in \mathscr{E}_1 \ and \ ab \notin \mathscr{E}_2, \\ \beta_{Q_2}(ab) \quad if \ ab \in \mathscr{E}_2 \ and \ ab \notin \mathscr{E}_1, \\ (\beta_{Q_1}(ab) \vee \alpha_{Q_2}(ab)) \ if \ ab \in \mathscr{E}_1 \cap \mathscr{E}_2. \end{cases}$$

$$\gamma_Q(ab) = \begin{cases} \gamma_{Q_1}(ab) \quad if \ ab \in \mathscr{E}_1 \ and \ ab \notin \mathscr{E}_2, \\ \gamma_{Q_2}(ab) \quad if \ ab \in \mathscr{E}_2 \ and \ ab \notin \mathscr{E}_1, \\ (\gamma_{Q_1}(ab) \wedge \gamma_{Q_2}(ab)) \ if \ ab \in \mathscr{E}_1 \cap \mathscr{E}_2. \end{cases}$$

Definition 2.4. *Let $Gr_1 = (P_1, Q_1)$ and $Gr_2 = (P_2, Q_2)$ be NeGs of $Gr_1' = (\mathscr{V}_1, \mathscr{E}_1)$ and $Gr_2' = (\mathscr{V}_2, \mathscr{E}_2)$, respectively. Then join of NeGs $Gr_1 + Gr_2$ is mentioned as*

(P,Q) such that

$$(i)\quad \alpha_P(a)=\begin{cases}\alpha_{P_1}(a) \ \ if\ a\in \mathscr{V}_1 \ and\ a\notin \mathscr{V}_2,\\ \alpha_{P_2}(a) \ \ if\ a\in \mathscr{V}_2 \ and\ a\notin \mathscr{V}_1,\\ (\alpha_{P_1}(a)\vee \alpha_{P_2}(a))\ if\ a\in \mathscr{V}_1\cap \mathscr{V}_2\end{cases}$$

$$\beta_P(a)=\begin{cases}\beta_{P_1}(a) \ \ if\ a\in \mathscr{V}_1 \ and\ x\notin \mathscr{V}_2,\\ \beta_{P_2}(a) \ \ if\ a\in \mathscr{V}_2 \ and\ x\notin \mathscr{V}_1,\\ (\beta_{P_1}(a)\vee \beta_{P_2}(a))\ if\ a\in \mathscr{V}_1\cap \mathscr{V}_2\end{cases}$$

$$\gamma_P(a)=\begin{cases}\gamma_{P_1}(a) \ \ if\ a\in \mathscr{V}_1 \ and\ a\notin \mathscr{V}_2,\\ \gamma_{P_2}(a) \ \ if\ a\in \mathscr{V}_2 \ and\ a\notin \mathscr{V}_1,\\ (\gamma_{P_1}(a)\wedge \gamma_{P_2}(a))\ if\ a\in \mathscr{V}_1\cap \mathscr{V}_2\end{cases}$$

$$(ii)\quad \alpha_Q(ab)=\begin{cases}\alpha_{Q_1}(ab) \ \ if\ ab\in \mathscr{E}_1 \ and\ ab\notin \mathscr{E}_2,\\ \alpha_{Q_2}(ab) \ \ if\ ab\in \mathscr{E}_2 \ and\ ab\notin \mathscr{E}_1,\\ (\alpha_{Q_1}(ab)\vee \alpha_{Q_2}(ab))\ if\ ab\in \mathscr{E}_1\cap \mathscr{E}_2\\ (\alpha_{P_1}(a)\wedge \alpha_{P_2}(b))\ if\ ab\in \mathscr{E}'\end{cases}$$

$$\beta_Q(ab)=\begin{cases}\beta_{Q_1}(ab) \ \ if\ ab\in \mathscr{E}_1 \ and\ ab\notin \mathscr{E}_2,\\ \beta_{Q_2}(ab) \ \ if\ ab\in \mathscr{E}_2 \ and\ ab\notin \mathscr{E}_1,\\ (\beta_{Q_1}(ab)\vee \alpha_{Q_2}(ab))\ if\ ab\in \mathscr{E}_1\cap \mathscr{E}_2\\ (\beta_{P_1}(a)\wedge \beta_{P_2}(b))\ if\ ab\in \mathscr{E}'\end{cases}$$

$$\gamma_Q(ab)=\begin{cases}\gamma_{Q_1}(ab) \ \ if\ ab\in \mathscr{E}_1 \ and\ ab\notin \mathscr{E}_2,\\ \gamma_{Q_2}(ab) \ \ if\ ab\in \mathscr{E}_2 \ and\ ab\notin \mathscr{E}_1,\\ (\gamma_{Q_1}(ab)\wedge \gamma_{Q_2}(ab))\ if\ ab\in \mathscr{E}_1\cap \mathscr{E}_2\\ (\gamma_{P_1}(a)\vee \gamma_{P_2}(b))\ if\ ab\in \mathscr{E}'\end{cases}$$

where $\mathscr{E}'$ represents the lines which join the vertices of $(\alpha_{P_1},\beta_{P_1},\gamma_{P_1})$ & $(\alpha_{P_2},\beta_{P_2},\gamma_{P_2})$.

Definition 2.5. *The union and join of NeGs is also a NeG.*

Definition 2.6. *A triangular neutrosophic number with neutrosophic memberships is characterized as $A_{TrNN}=(d_1,d_2,d_3;e_1,e_2,e_3;f_1,f_2,f_3)$ and each neutrosophic* *membership are given as follows:*

$$\alpha_{TrNN}(a) = \begin{cases} \frac{a-d_1}{d_2-d_1}, & d_1 \leq a < d_2 \\ 1, & a = d_2 \\ \frac{d_3-a}{d_3-d_2}, & d_2 < a \leq d_3 \\ 0, & elsewhere \end{cases}$$

$$\beta_{TrNN}(a) = \begin{cases} \frac{e_2-a}{e_2-e_1}, & e_1 \leq a < e_2 \\ 0, & a = e_2 \\ \frac{a-e_2}{e_3-e_2}, & e_2 < a \leq e_3 \\ 1, & elsewhere \end{cases}$$

$$\gamma_{TrNN}(a) = \begin{cases} \frac{f_2-a}{f_2-f_1}, & f_1 \leq a < f_2 \\ 0, & a = f_2 \\ \frac{a-f_2}{f_3-f_2}, & f_2 < a \leq f_3 \\ 1, & elsewhere \end{cases}$$

where $0 \leq \alpha_{TrNN}(a) + \beta_{TrNN}(a) + \gamma_{TrNN}(a) \leq 3$*,* $a \in A_{TrNN}$

3. Triangular Neutrosophic Graceful Labeling Graphs

A triangular neutrosophic graceful labeling (TrNGL) graph is an extension that incorporates the neutrosophic graph principles, graceful behaviour and triangular fuzzy number. The feasibility is attained in final results since it marks up the imprecision efficiently, when compared to classical graceful labeling approaches. There are few relevant approaches in fuzzy and intuitionistic fuzzy graph scenario, as quoted in the introduction section. Comparatively, a TrNGL is far better in results than triangular fuzzy and intuitionistic fuzzy labeling, since the explicit segregation of uncertain cases is possible here. In this section, TrNGL graphs are acquired and some operations like union and join are checked with path graphs.

Definition 3.1. *A NeG* $Gr = (P, Q)$*, where* $\sigma = (\alpha_P, \beta_P, \gamma_P)$ *and* $\mu = (\alpha_Q, \beta_Q, \gamma_Q)$ *is said to satisfy graceful labeling if* $\alpha_P, \beta_P, \gamma_P : P \to [0, 1]$*,* $\alpha_Q, \beta_Q, \gamma_Q : P \times P \to [0, 1]$ *and all edges are distinct by each membership function when each edge label* $(\alpha_Q(a_i, a_j), \beta_Q(a_i, a_j), \gamma_Q(a_i, a_j)) \in Q(Gr)$ *is obtained by the absolute difference between the adjacent vertices in an NeG.*
(i.e.) $|(\alpha_P(a_i), \beta_P(a_i), \gamma_P(a_i)) - (\alpha_P(a_j), \beta_P(a_j), \gamma_P(a_j))|$*, then the NeG is known as neutrosophic graceful labeling.*

Definition 3.2. *A neutrosophic graph* $Gr = (P, Q)$ *is known to have triangular* *neutrosophic labeling, when the vertex labels are assigned using triangular neutrosophic number. A triangular neutrosophic labeling graph is termed as triangular neutrosophic graceful labeling (TrNGL), if it preserves the neutrosophic graceful labeling as per the definition 3.1.*

Theorem 3.3. *Every neutrosophic path graph P_r, $r \geq 2$ preserves TrNGL.*
Proof. Consider a path graph P_r with vertex membership $(\alpha_P, \beta_P, \gamma_P)$ and edge membership $(\alpha_Q, \beta_Q, \gamma_Q)$. Let $\alpha_P, \beta_P, \gamma_P : P \to TrNN$ be the one-one mappings such that,

$$\alpha_P(a_i) = \frac{(r-1)^2 + (i-1)^2}{10^n} \quad (1)$$

$$\beta_P(a_i) = \frac{r^2 + i^2}{10^n} \quad (2)$$

$$\gamma_P(a_i) = \frac{(r+1)^2 + (i+1)^2}{10^n} \quad (3)$$

$\forall a_i \in P$, $1 \leq i \leq r$.
Define the edge memberships $\alpha_Q, \beta_Q, \gamma_Q : P \times P \to TrNN$ such that,

$$\alpha_Q(a_i, a_{i+1}) = |\alpha_P(a_i) - \alpha_P(a_{i+1})|$$

$$\beta_Q(a_i, a_{i+1}) = |\beta_P(a_i) - \beta_P(a_{i+1})|$$

$$\gamma_Q(a_i, a_{i+1}) = |\gamma_P(a_i) - \gamma_P(a_{i+1})|$$

$\forall a_i \in P$, $1 \leq i \leq r$.
It is clear that, $\alpha_P(a_i) < \alpha_P(a_{i+1})$, $\beta_P(a_i) < \beta_P(a_{i+1})$, $\gamma_P(a_i) < \gamma_P(a_{i+1})$ for vertices and $\alpha_Q(a_i, a_{i+1}) < \alpha_Q(a_{i+1}, a_{i+2})$, $\beta_Q(a_i, a_{i+1}) < \beta_Q(a_{i+1}, a_{i+2})$, $\gamma_Q(a_i, a_{i+1}) < \gamma_Q(a_{i+1}, a_{i+2})$ for edges. The range of each membership of vertices and their total sum is sustained when value of n increased (or) by decrementing r & i value. As a result, each membership of vertices and edges of the path graph are distinct. Therefore, path graph with neutrosophic membership is a TrNGL graph.

Example 3.4. If we take r=3, i=1,2,3 and n=2 then path graph P_3 will be obtained as illustrated by figure. 1.

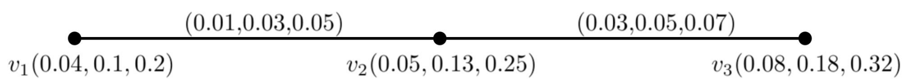

Figure 1: TrNGL P_3

Theorem 3.5. *The union of TrNGL path graphs is not always a triangular neutrosophic graceful labeling path graph.*

Proof. Let P_r and P_s be the triangular neutrosophic graceful labeling path graphs. Consider the graph shown in example 3.4 as P_s and the vertex & edge membership of P_r is defined as follows:

$$\alpha_{P_1}(a_i) = \frac{r^2+i^2}{10^n} \tag{4}$$

$$\beta_{P_1}(a_i) = \frac{(r+1)^2+(i+1)^2}{10^n} \tag{5}$$

$$\gamma_{P_1}(a_i) = \frac{(r+2)^2+(i+2)^2}{10^n} \tag{6}$$

$\forall a_i \in P$, $1 \leq i \leq r$.

Define the edge memberships $\alpha_{Q_1}, \beta_{Q_1}, \gamma_{Q_1} : P \times P \to TrNN$ such that,

$$\alpha_{Q_1}(a_i, a_{i+1}) = |\alpha_{P_1}(a_i) - \alpha_{P_1}(a_{i+1})|$$

$$\beta_{Q_1}(a_i, a_{i+1}) = |\beta_{P}(a_i) - \beta_{P_1}(a_{i+1})|$$

$$\gamma_{Q_1}(a_i, a_{i+1}) = |\gamma_{P}(a_i) - \gamma_{P_1}(a_{i+1})|$$

$\forall a_i \in P$, $1 \leq i \leq r$.

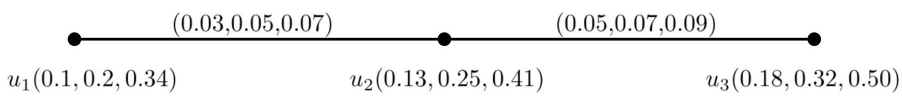

Figure 2: TrNGL P_r graph

The representation of union of P_r and P_s is as follows:

$P_r \cup P_s = \{(\alpha_{P_1}, \beta_{P_1}, \gamma_{P_1}) \cup (\alpha_{P_2}, \beta_{P_2}, \gamma_{P_2}), (\alpha_{Q_1}, \beta_{Q_1}, \gamma_{Q_1}) \cup (\alpha_{Q_2}, \beta_{Q_2}, \gamma_{Q_2})\}$. Here,

$$((\alpha_{P_1}, \beta_{P_1}, \gamma_{P_1}) \cup (\alpha_{P_2}, \beta_{P_2}, \gamma_{P_2}))(a_i, a_j) = \begin{cases} \{\alpha_{P_1}(a_i) \vee \alpha_{P_2}(a_j)\} = \alpha_{P_1}(a_i) \\ \{\beta_{P_1}(a_i) \vee \beta_{P_2}(a_j)\} = \beta_{P_1}(a_i) \\ \{\gamma_{P_1}(a_i) \wedge \gamma_{P_2}(a_j)\} = \gamma_{P_2}(a_j) \end{cases}$$

since by equations (1)-(6).

Also,

$$((\alpha_{Q_1},\beta_{Q_1},\gamma_{Q_1})\cup(\alpha_{Q_2},\beta_{Q_2},\gamma_{Q_2}))[(a_i,a_{i+1})(a_j,a_{j+1})]$$
$$=\begin{cases}\{\alpha_{Q_1}(a_i,a_{i+1})\vee\alpha_{Q_2}(a_j,a_{j+1})\}\\ \{\beta_{Q_1}(a_i,a_{i+1})\vee\beta_{Q_2}(a_j,a_{j+1})\}\\ \{\gamma_{Q_1}(a_i,a_{i+1})\wedge\gamma_{Q_2}(a_j,a_{j+1})\}\end{cases}$$

$$(i.e),((\alpha_{Q_1},\beta_{Q_1},\gamma_{Q_1})\cup(\alpha_{Q_2},\beta_{Q_2},\gamma_{Q_2}))[(a_i,a_{i+1})(a_j,a_{j+1})]=\begin{cases}\alpha_{Q_1}(a_i,a_{i+1})\\ \beta_{Q_1}(a_i,a_{i+1})\\ \gamma_{Q_2}(a_j,a_{j+1})\end{cases}$$

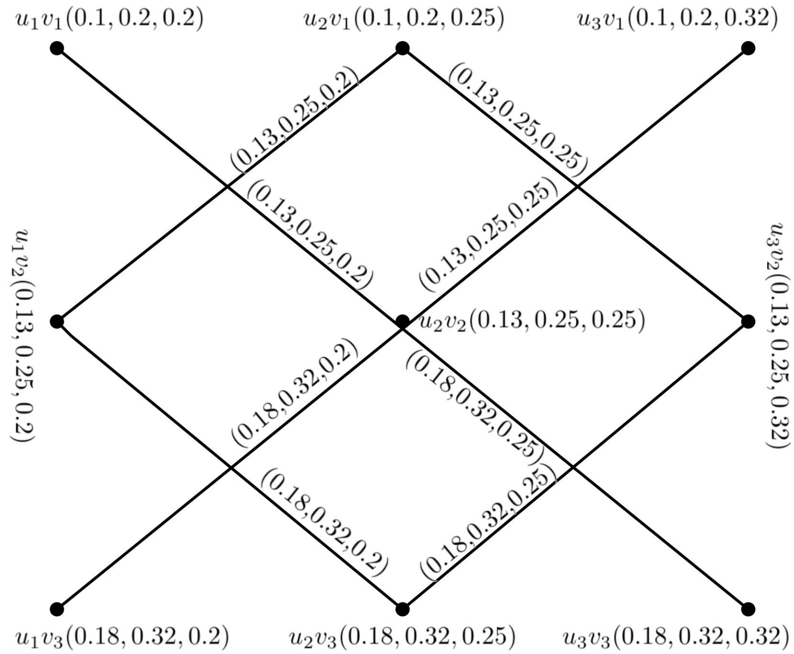

Figure 3: TrNGL $P_r \cup P_s$ graph

It's evident that, the vertex and edge labeling of $P_r \cup P_s$ are not distinct from the figure 3. Therefore, union of TrNGL path graphs is not always TrNGL path graph.

Theorem 3.6. *The join of TrNGL path graphs is not always a TrNGL path graph.*
Proof. Let P_r and P_s be the TrNGL path graphs. Consider the graph shown in example 3.4 as P_s and the vertex & edge membership of P_r is defined as the same in theorem 3.5.

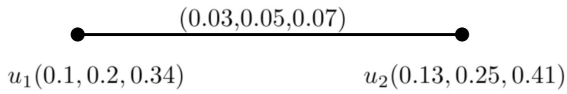

Figure 4: TrNGL P_r graph

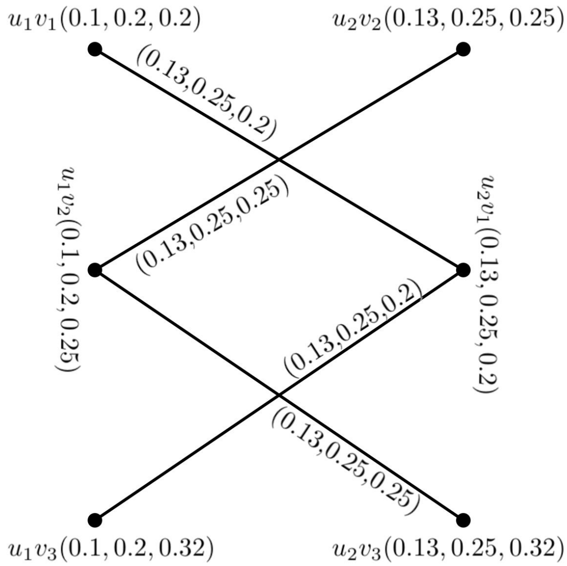

Figure 5: TrNGL $P_r + P_s$ graph

The representation of join of P_r and P_s is as follows:
$P_r + P_s = \{(\alpha_{P_1}, \beta_{P_1}, \gamma_{P_1}) + (\alpha_{P_2}, \beta_{P_2}, \gamma_{P_2}), (\alpha_{Q_1}, \beta_{Q_1}, \gamma_{Q_1}) + (\alpha_{Q_2}, \beta_{Q_2}, \gamma_{Q_2})\}$. Here,

$$((\alpha_{P_1}, \beta_{P_1}, \gamma_{P_1}) + (\alpha_{P_2}, \beta_{P_2}, \gamma_{P_2}))(a_i, a_j) = \begin{cases} \{\alpha_{P_1}(a_i) \vee \alpha_{P_2}(a_j)\} = \alpha_{P_1}(a_i) \\ \{\beta_{P_1}(a_i) \vee \beta_{P_2}(a_j)\} = \beta_{P_1}(a_i) \\ \{\gamma_{P_1}(a_i) \wedge \gamma_{P_2}(a_j)\} = \gamma_{P_2}(a_j) \end{cases}$$

since by equations (1)-(6).
Also,

$$((\alpha_{Q_1}, \beta_{Q_1}, \gamma_{Q_1}) + (\alpha_{Q_2}, \beta_{Q_2}, \gamma_{Q_2}))[(a_i, a_{i+1})(a_j, a_{j+1})]$$
$$= \begin{cases} \{\alpha_{Q_1}(a_i, a_{i+1}) \vee \alpha_{Q_2}(a_j, a_{j+1})\} \\ \{\beta_{Q_1}(a_i, a_{i+1}) \vee \beta_{Q_2}(a_j, a_{j+1})\} \\ \{\gamma_{Q_1}(a_i, a_{i+1}) \wedge \gamma_{Q_2}(a_j, a_{j+1})\} \end{cases}$$

$$(i.e), ((\alpha_{Q_1}, \beta_{Q_1}, \gamma_{Q_1}) + (\alpha_{Q_2}, \beta_{Q_2}, \gamma_{Q_2}))[(a_i, a_{i+1})(a_j, a_{j+1})] = \begin{cases} \alpha_{Q_1}(a_i, a_{i+1}) \\ \beta_{Q_1}(a_i, a_{i+1}) \\ \gamma_{Q_2}(a_j, a_{j+1}) \end{cases}$$

It's evident that, the vertex and edge labeling of $P_r + P_s$ are not distinct from the figure 5. Therefore, join of TrNGL path graphs is not always TrNGL path graph.

Theorem 3.7. *Every neutrosophic cycle graph $C_r, r \geq 3$ preserves TrNGL.*
Proof. Consider a cycle graph $C_r, r \geq 3$ with $(\alpha_P, \beta_P, \gamma_P)$ and $(\alpha_Q, \beta_Q, \gamma_Q)$ as neutrosophic membership of vertices & edges. Let $\alpha_P, \beta_P, \gamma_P : P \to TrNN$ be the one-one mappings such that, we get distinct edge values(using absolute difference of adjacent vertices) by assuming the same vertex labeling conditions (1),(2),(3) of theorem 3.3.

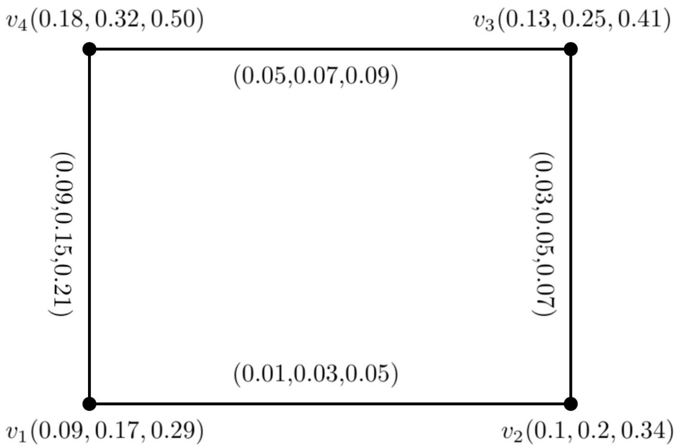

Figure 6: TrNGL C_4 graph

The range of each membership of vertices and their total sum is sustained when value of n increased (or) by decrementing r & i value in conditions (1),(2) and (3). As a result, each vertex and edge membership of the cycle graph are distinct. Therefore, a cycle graph with neutrosophic membership is a TrNGL graph.

Theorem 3.8. *Every neutrosophic star $K_{1,r}, r \geq 1$ graph preserves triangular neutrosophic graceful labeling.*
Proof. Let $K_{1,r}, r \geq 1$ be a star graph with vertex membership $(\alpha_P, \beta_P, \gamma_P)$

and edge membership $(\alpha_Q, \beta_Q, \gamma_Q)$. Let $\alpha_P, \beta_P, \gamma_P : V \rightarrow TrNN$ be the one-one mappings such that, we get distinct edge values(using absolute difference of adjacent vertices) by assuming the same vertex labeling conditions (1),(2),(3) of theorem 3.3.

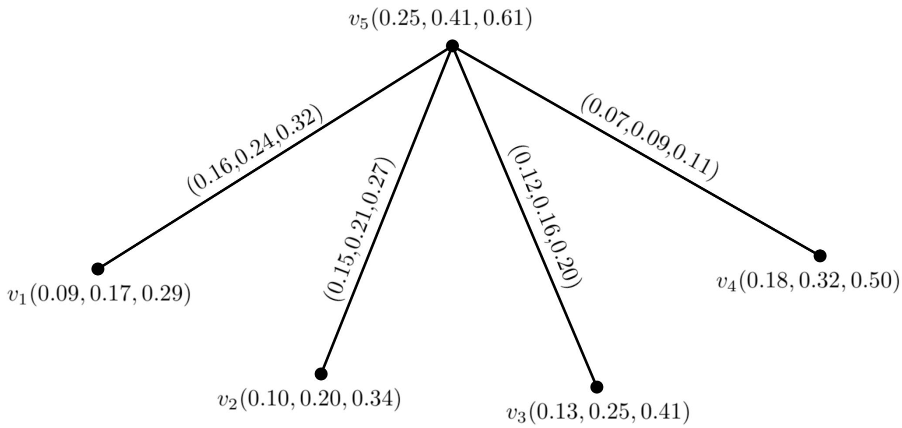

Figure 7: TrNGL $K_{1,4}$ graph

The range of each membership of vertices and their total sum is sustained when value of n increased (or) by decrementing r & i value in conditions (1),(2) and (3). As a result, each vertex and edge membership of the star graph are distinct. Therefore, a neutrosophic star graph is a TrNGL graph.

Corollary 3.9. *Every neutrosophic fan graph $F_{1,n}$, $n \geq 2$ admits TrNGL.*
Proof. In figure 7, if the pendent vertices of star graph is linked using edges then we obtain fan graph $F_{1,4}$, where all edges are distinct and thereby satisfies the TrNGL condition.

4. Application

In conventional fuzzy graphical approach, there are no discussions about algorithm and applications that involves triangular fuzzy and intuitionistic fuzzy labeling. In this section, some applications of triangular neutrosophic graceful labeling (TrNGL) graphs are given, that works systematic than prior classical approaches of fuzzy and intuitionistic fuzzy case.

(i) **Frieght Railway Transportation**:

Freight rail transport is very much helpful in transporting large volume of goods over far distances. It is reliable, cost efficient and environment friendly, when compared to road transport. Here, we consider the vertices of TrNGL graph as railway stations, where goods are picked up and the edges as the pathway between stations, where some goods will be delivered and collected at certain substations. The vertex membership functions are assumed to be amount of registered goods settled in compartments(truth membership), incomplete information or damaged goods that are kept in pending to transport (indeterminacy) and the withdrawn goods by the customers at last time(false membership). The edge memberships are taken as delivered goods at substations (truth membership), goods that are on hold to deliver since it was known to be damaged after import(indeterminacy) and the goods that are cancelled to deliver after import(false membership). Through this approach, we can segregate the events happened during boarding and arriving of goods train. The number of goods delivered, kept on hold and cancelled to deliver between stations can be easily highlighted since there are separate defined membership functions for each vertex and edge.

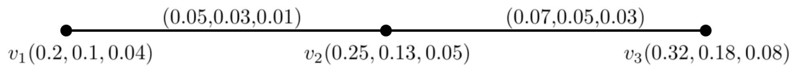

Figure 8: Goods Transport System

(ii) **System communication**:

There are various communication methods like networking protocols, inter-process communication, remote procedure calls etc., through which one system can communicate with other systems. It is time efficient to share the information and messages instantly through systems. Here, we take the vertices as sys-

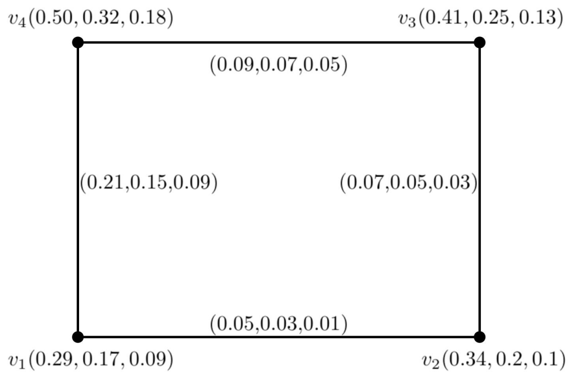

Figure 9: Information Transfer Mechanism

tems and the edges to be the communication network between them. The vertex memberships are considered as the total information sent and received by systems(truth membership), information not yet sent or received because of network issues(indeterminacy) and the information not sent or received because of cancellation(false membership). The edge memberships are assumed as the total number of information transferred(truth), information on hold to transfer(indeterminacy) and the information cancelled during transfer(false membership). Through this neutrosophic approach, one can count the events in specific manner and render the balance service. The passage of information through systems and the issues occured can be learned individually.

Algorithm

There are some algorithmic approaches [3, 4, 5] in normal neutrosophic graphical environment, but there is no algorithm stated before for neutrosophic graphs involving graceful labeling and triangular neutrosophic number (TrNN). This is a special case that can't be analysed with the prevailing neutrosophic designs, since it deals with range based uncertainty. The neutrosophic graphs is appropriate in case of simple applications. But when complex model is given, there is a need to include some specifications. This algorithm doesn't include any complexity of time, because it specifically concentrates on each edge membership values attained through adjacent vertices. It is time efficient since the individual results(edge memberships) are directly obtained, which is used to decide the final output. The following algorithm is framed for the triangular neutrosophic graceful labeling (TrNGL) graphs:

Step-1: Input the neutrosophic membership values for all vertices, based on the definition of TrNN.

Step-2: Calculate the neutrosophic membership values of all edges using the expression $|(\alpha_P(a_i), \beta_P(a_i), \gamma_P(a_i)) - (\alpha_P(a_j), \beta_P(a_j), \gamma_P(a_j))|$, where a_i and a_j represents the adjacent vertices of a TrNGL graph.

Step-3: Analyse the individual responses (edge value) recorded between the adjacent vertices.

Step-4: Find the overall output by analysing the function of segregated edge membership values and its associated vertices.

5. Conclusion

In this manuscript, a latest finding by applying triangular neutrosophic number(TrNN) to neutrosophic graceful labeled graphs is discussed widely with some operations. As a result, triangular neutrosophic graceful labeling graphs are obtained. The methodology of labeling the graphs gracefully using TrNN is quite useful to apply on certain special cases, where some limitation is involved or required. In addition, an application which discovers the usage of TrNN in neutrosophic graceful labeled graphs is demonstrated. Our future work is to explore the other graph labeling concepts of crisp and fuzzy type to neutrosophic background.

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