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ON SECOND AND THIRD LEAP ZAGREB COINDICES OF SOME GRAPH OPERATIONS

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Abstract: Recently, the authors introduced the leap Zagreb coindices (LZCIs) of graphs. They presented many properties, so also, established upper and lower bounds for them. They, also in last work, studied and presented the general formulas for the first leap Zagreb coindex of some operations of a graph. In the present work, we investigate to continue in our work by computing the general formulas of the second and third LZCIs of union, cartesian product, composition, disjunction, symmetric difference of graphs.

Keywords and Phrases: Second-degrees, Topological indices, Coindices, Graph operations.

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1. Introduction

Throughout this paper, we assume that all graphs $\Gamma = (V, E)$ are simple. That is finite, have neither loops, nor multiple, nor directed edges. Let Γ be such a graph, the cardinality of the vertex set $V(\Gamma)$ and edge set $E(\Gamma)$, will be denoted by n and m, are called the order and size of a graph Γ, respectively. A distance from a vertex v to a vertex u, in Γ, denoted by $d(u, v)$ (or $d_\Gamma(u, v)$ if there is any confusion), and is the number of edges in a shortest path connecting them. the open second neighborhood of a vertex v in Γ is $N_2(v) = \{u \in V(\Gamma) : d(u, v) = 2\}$. The set of all second (2-distance) edges of a graph Γ denoted and defined as

$E_2(\Gamma) = \{\{u, v\} \subset V(\Gamma) : d(u, v) = 2\}$, as well as we denote by $\mu(\Gamma)$ (in short, μ) to the cardinality of $E_2(\Gamma)$. The second degree of v in Γ, denoted by $d_2(v/\Gamma)$ (or $d_2(v)$ if no misunderstanding), is the number of second neighbors of v.

The eccentricity of a vertex v in a graph Γ, is $e(v) = \max\{d(v, u) : u \in V(\Gamma)\}$, the diameter of Γ is $diam(\Gamma) = \max\{e(v) : v \in V(\Gamma)\}$ and the radius is $rad(\Gamma) = \min\{e(v) : v \in V(\Gamma)\}$. The induced subgraph $\langle H\rangle$ of Γ is a graph with vertex set $H \subseteq V(\Gamma)$ and consists of all edges in $E(\Gamma)$ that its both endpoints in H. An H-free graph is a graph have no induced subgraph isomorphic to H. The complement $\overline{\Gamma}$ of a graph Γ, is a graph with $V(\overline{\Gamma}) = V(\Gamma)$ and for any $u, v \in V(\overline{\Gamma})$, $uv \in E(\overline{\Gamma})$, if and only if $uv \notin E(\Gamma)$.
For any terminologies or notations not defined here, we refer the reader to [13].

A topological indices (TIs) of a graph are fixed parameters (invariants) that do not change for isomorphic graphs. The most important and studied among TIs the first and second Zagreb indices, which were defined in 1972, by Gutman and Trinajestic [12], [11], and defined as:

$$M_1(\Gamma) = \sum_{v\in V(\Gamma)} d^2(v) \quad \text{and} \quad M_2(\Gamma) = \sum_{uv\in E(\Gamma)} d(u)d(v).$$

For more details on these two indices and beyond, the reader is referred to the surveys [5, 9].

Analogously, the coindices of the Zagreb indices, were put forward by Ali Ashrafi et al. [2], as following:

$$\overline{M_1}(\Gamma) = \sum_{uv\notin E(\Gamma)} (d(u) + d(v)), \quad \text{and} \quad \overline{M_2}(\Gamma) = \sum_{uv\notin E(\Gamma)} d(u)d(v).$$

For more about Zagreb coindices, see [1, 2, 6, 9].

Naji et al. [19], introduced leap Zagreb indices (LZIs) of a graph Γ. They defined them as:

$$LM_1(\Gamma) = \sum_{v\in V(\Gamma)} d_2^2(v), LM_2(\Gamma) = \sum_{uv\in E(\Gamma)} d_2(u)d_2(v) \text{ and } LM_3(\Gamma) = \sum_{v\in V(\Gamma)} d(v)d_2(v).$$

These LZIs have various applications in chemistry. Surprisingly, the first one is very correlate with physical properties of chemical compounds, for instance, with the entropy, the boiling point, DHVAP, HVAP and the accentric factor [4].
For more properties and details on LZIs , the readers may be refer to [3, 4, 15-22].

Recently, Ferdose and Shivashankara [7], introduced the leap Zagreb coindices of a graph. They defined them as

$$\overline{L_1}(\Gamma) = \sum_{uv\notin E_2(\Gamma)} \Big(d_2(u) + d_2(v)\Big) \quad \text{and} \quad \overline{L_2}(\Gamma) = \sum_{uv\notin E(\Gamma)} \Big(d_2(u)d_2(v)\Big)$$

$$\overline{L_3}(\Gamma) = \sum_{uv \notin E(\Gamma)} \Big(d_2(u) + d_2(v) \Big).$$

They, also in [8], studied the first LZCI of many graph operations.
In this paper, the general formulas for $\overline{L_2}$ and $\overline{L_3}$ of some graph operations is presented.

The following results are useful and will be used in main our arguments through this study.

Theorem 1.1. [22, 23] *Let Γ be a connected graph with n vertices and m edges. Then*

$$d_2(v) \leq \left(\sum_{u \in N_1(v)} d_1(u) \right) - d_1(v).$$

Equality is holding if and only if Γ is a triangle- and quadrangle-free.

The following result directly follows from the above Theorem.

Corollary 1.2. [22] *Let Γ be a connected graph with n vertices and m edges. Then*

$$\sum_{v \in V(\Gamma)} d_2(v) \leq M_1(\Gamma) - 2m,$$

and equality holds if and only if Γ is a $\{C_3, C_4\}$-free.

2. Main Results

For two graphs Γ_1 and Γ_2, the vertex set, edge and second-edge sets will be denoted by $V(\Gamma_1), V(\Gamma_2), E(\Gamma_1), E(\Gamma_2), E_2(\Gamma_1)$ and $E_2(\Gamma_2)$, respectively, and their cardinality by $n_1, n_2, m_1, m_2, \mu(\Gamma_1)$ and $\mu(\Gamma_2)$, respectively.

2.1. Union

Definition 2.1. [14] *The union $\Gamma_1 \cup \Gamma_2$ of graphs Γ_1 and Γ_2 is the graph whose vertex set $V(\Gamma_1 \cup \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$, and edge set $E(\Gamma_1 \cup \Gamma_2) = E(\Gamma_1) \cup E(\Gamma_2)$.*

Clearly that $|V(\Gamma_1 \cup \Gamma_2)| = n_1 + n_2$, $|E(\Gamma_1 \cup \Gamma_2)| = m_1 + m_2$, and $|E_2(\Gamma_1 \cup \Gamma_2)| = \mu_1 + \mu_2$, where $\mu = \mu(\Gamma) = |E_2(\Gamma)|$. So, the following result is straightforward,

Lemma 2.2. [17] *Let Γ_1 and Γ_2 be two disjoint connected graphs with n_1 and n_2 vertices. Then for each $v \in V(\Gamma_1 \cup \Gamma_2)$,*

$$d_2(v/(\Gamma_1 \cup \Gamma_2)) = \begin{cases} d_2(v/\Gamma_1), & \text{if } v \in V(\Gamma_1); \\ d_2(v/\Gamma_2), & \text{if } v \in V(\Gamma_2). \end{cases}$$

Theorem 2.3. *Let Γ_1 and Γ_2 be connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\overline{L_2}(\Gamma_1 \cup \Gamma_2) \leq \overline{L_2}(\Gamma_1) + \overline{L_2}(\Gamma_2) + M_1(\Gamma_1)M_1(\Gamma_2) - 2m_2 M_1(\Gamma_1) - 2m_1 M_1(\Gamma_2) + 4m_1 m_2.$$

Equality is holding if and only if Γ_1 *and* Γ_2 *are* (C_3, C_4)*-free graphs.*

Proof. From Lemma 2.2, if $u, v \in V(\Gamma_1 \cup \Gamma_2)$, then $uv \notin E(\Gamma_1 \cup \Gamma_2)$ if and only if $u, v \in V(\Gamma_1)$ and $uv \notin E(\Gamma_1)$, or $u, v \in V(\Gamma_2)$ and $uv \notin E(\Gamma_2)$, or $u \in V(\Gamma_1)$ and $v \in V(\Gamma_2)$. Thus $\overline{L_2}(\Gamma_1 \cup \Gamma_2)$ is equal to the sum of $\overline{L_2}(\Gamma_1)$ and $\overline{L_2}(\Gamma_2)$. In addition, the contributions which obtain from the missing edges between $V(\Gamma_1)$ and $V(\Gamma_2)$. Where there are $n_1 n_2$ of them. Then by this and Corollary 1.2, we obtain

$$\begin{aligned}
\overline{L_2}(\Gamma_1 \cup \Gamma_2) &= \sum_{uv \notin E(\Gamma_1 \cup \Gamma_2)} \Big(d_2(u/(\Gamma_1 \cup \Gamma_2)) d_2(v/(\Gamma_1 \cup \Gamma_2)) \Big) \\
&= \sum_{uv \notin E(\Gamma_1)} \Big(d_2(u/(\Gamma_1 \cup \Gamma_2)) d_2(v/(\Gamma_1 \cup \Gamma_2)) \Big) \\
&+ \sum_{uv \notin E(\Gamma_2)} \Big(d_2(u/(\Gamma_1 \cup \Gamma_2)) d_2(v/(\Gamma_1 \cup \Gamma_2)) \Big) \\
&+ \sum_{u \in V(\Gamma_1)} \sum_{v \in V(\Gamma_2)} \Big(d_2(u/(\Gamma_1 \cup \Gamma_2)) d_2(v/(\Gamma_1 \cup \Gamma_2)) \Big) \\
&= \sum_{uv \notin E(\Gamma_1)} \Big(d_2(u/\Gamma_1) d_2(v/\Gamma_1) \Big) + \sum_{uv \notin E(\Gamma_2)} \Big(d_2(u/\Gamma_2) d_2(v/\Gamma_2) \Big) \\
&+ \sum_{u \in V(\Gamma_1)} \sum_{v \in V(\Gamma_2)} \Big(d_2(u/\Gamma_1) d_2(v/\Gamma_2) \Big) \\
&= \overline{L_2}(\Gamma_1) + \overline{L_2}(\Gamma_2) + \sum_{u \in V(\Gamma)} \sum_{v \in V(\Gamma_2)} d_2(u/\Gamma_1) d_2(v/\Gamma_2) \\
&= \overline{L_2}(\Gamma_1) + \overline{L_2}(\Gamma_2) + \sum_{u \in V(\Gamma_1)} d_2(u/\Gamma_1) \sum_{v \in V(\Gamma_2)} d_2(v/\Gamma_2) \\
&\leq \overline{L_2}(\Gamma_1) + \overline{L_2}(\Gamma_2) + (M_1(\Gamma_1) - 2m_1)(M_1(\Gamma_2) - 2m_2) \\
&\leq \overline{L_2}(\Gamma_1) + \overline{L_2}(\Gamma_2) + M_1(\Gamma_2) M_1(\Gamma_1) - 2(m_2 M_1(\Gamma_1) + m_1 M_1(\Gamma_2)) + 4 m_1 m_2.
\end{aligned}$$

Theorem 2.4. *For connected graphs* Γ_1 *and* Γ_2 *of orders* n_1, n_2 *and size* m_1, m_2*, respectively.*

$$\overline{L_3}(\Gamma_1 \cup \Gamma_2) \leq \overline{L_3}(\Gamma_1) + \overline{L_3}(\Gamma_2) + n_2 M_1(\Gamma_1) + n_1 M_1(\Gamma_2) - 2(n_2 m_1 + n_1 m_2).$$

Equality is holding if and only if Γ_1 *and* Γ_2 *are* (C_3, C_4)*-free.*

Proof. Like the arguments as in the proof of Theorem 2.3, and since there are $n_1 n_2$ missing edges between $V(\Gamma_1)$ and $V(\Gamma_2)$ and by using Corollary 1.2. Then we obtain

$$
\begin{aligned}
\overline{L_3}(\Gamma_1 \cup \Gamma_2) &= \sum_{uv \notin E(\Gamma_1 \cup \Gamma_2)} \Big(d_2(u/(\Gamma_1 \cup \Gamma_2)) + d_2(v/(\Gamma_1 \cup \Gamma_2)) \Big) \\
&= \sum_{uv \notin E(\Gamma_1)} \Big(d_2(u/(\Gamma_1 \cup \Gamma_2)) + d_2(v/(\Gamma_1 \cup \Gamma_2)) \Big) \\
&+ \sum_{uv \notin E(\Gamma_2)} \Big(d_2(u/(\Gamma_1 \cup \Gamma_2)) + d_2(v/(\Gamma_1 \cup \Gamma_2)) \Big) \\
&+ \sum_{u \in V(\Gamma_1)} \sum_{v \in V(\Gamma_2)} \Big(d_2(u/\Gamma_1 \cup \Gamma_2) + d_2(v/\Gamma_1 \cup \Gamma_2) \Big) \\
\overline{L_3}(\Gamma_1 \cup \Gamma_2) &= \sum_{uv \notin E(\Gamma_1)} \Big(d_2(u/\Gamma_1) + d_2(v/\Gamma_1) \Big) + \sum_{uv \notin E(\Gamma_2)} \Big(d_2(u/\Gamma_2) + d_2(v/\Gamma_2) \Big) \\
&+ \sum_{u \in V(\Gamma_1)} \sum_{v \in V(\Gamma_2)} \Big(d_2(u/\Gamma_1) + d_2(v/\Gamma_2) \Big) \\
&= \overline{L_3}(\Gamma_1) + \overline{L_3}(\Gamma_2) + \sum_{u \in V(\Gamma)} \sum_{v \in V(\Gamma_2)} d_2(u/\Gamma_1) + d_2(v/\Gamma_2) \\
&= \overline{L_3}(\Gamma_1) + \overline{L_3}(\Gamma_2) + n_2 \sum_{u \in V(\Gamma_1)} d_2(u/\Gamma_1) + n_1 \sum_{v \in V(\Gamma_2)} d_2(v/\Gamma_2) \\
&\leq \overline{L_3}(\Gamma_1) + \overline{L_3}(\Gamma_2) + n_2(M_1(\Gamma_1) - 2m_1) + n_1(M_1(\Gamma_2) - 2m_2) \\
&\leq \overline{L_3}(\Gamma_1) + \overline{L_3}(\Gamma_2) + n_2 M_1(\Gamma_1) + n_1 M_1(\Gamma_2) - 2(n_2 m_1 + n_1 m_2).
\end{aligned}
$$

Theorem 2.5. *For $k \geq 2$, let $\Gamma_1, ..., \Gamma_k$ be connected graphs with n_i vertices and m_i edges, for every $i = 1, ..., k$. Then*

$$
\overline{L_2}(\bigcup_{i=1}^{k} \Gamma_i) \leq \sum_{i=1}^{k} \left(\overline{L_2}(\Gamma_i) + [M_1(\Gamma_i) - 2m_i] \sum_{i<j}^{k} [M_1(\Gamma_j) - 2m_j] \right). \tag{1}
$$

$$
\overline{L_3}(\bigcup_{i=1}^{k} \Gamma_i) \leq \sum_{i=1}^{k} \left(\overline{L_3}(\Gamma_i) + n_i \sum_{\substack{j=1 \\ j \neq i}}^{k} \left[M_1(\Gamma_j) - 2m_j \right] \right). \tag{2}
$$

Equalities hold if and only if Γ_i, for $i = 1, 2, ..., k$ is a (C_3, C_4)-free.

From Theorem 2.5, the following result holds.

Corollary 2.6. *Let F be a connected graph with n vertices, m edges and $diam(F) \geq 2$, and let $\Gamma = kF$, for $k \geq 2$. Then*

$$
\overline{L_2}(\Gamma) \leq k\overline{L_2}(F) + \frac{k(k-1)}{2}(M_1(F) - 2m)^2. \tag{3}
$$

$$\overline{L_3}(\Gamma) \leq k\overline{L_3}(F) + nk(k-1)(M_1(F) - 2m). \quad (4)$$

Equality is holding if and only if F is a (C_3, C_4)-free.

From Corollary 2.6, the following results follow, for every $n \geq 5$.

- $M_1(K_n) = n(n-1)^2$, and $\overline{L_2}(K_n) = \overline{L_3}(K_n) = 0$
- $M_1(P_n) = 4n - 6$, and $\overline{L_2}(P_n) = 2(n^2 - 7n + 14)$, $\overline{L_3}(P_n) = 2n(n-5)$
- $M_1(C_n) = 4n$, and $\overline{L_2}(C_n) = \overline{L_3}(C_n) = 2n(n-3)$

Corollary 2.7. *For $p \geq 2$ and $n \geq 5$, we have*

1. $\overline{L_2}(pK_n) = \overline{L_3}(pK_n) = 0$,
2. $\overline{L_2}(pP_n) = 2p(n^2 - 7n + 14) + 2p(p-1)(n-2)^2$, $\overline{L_3}(pP_n) = 2np(np - 2p - 3)$
3. $\overline{L_2}(pC_n) = \overline{L_3}(pC_n) = 2np(np - 3)$

2.2. Join

Definition 2.8. [14] *For given graphs Γ_1 and Γ_2 with n_1, n_2, m_1 and m_2 orders and sizes, respectively. The join graph $\Gamma_1 + \Gamma_2$, is defined as the graph with $V(\Gamma_1 + \Gamma_2) = V(\Gamma_1) \cup V(\Gamma_2)$, and $E(\Gamma_1 + \Gamma_2) = E(\Gamma_1) \cup E(\Gamma_2) \cup \{uv : \forall u \in V(\Gamma_1)$ and $\forall v \in V(\Gamma_2)\}$.*

It is clear that $|V(\Gamma_1 + \Gamma_2)| = n_1 + n_2$ and $|E(\Gamma_1 + \Gamma_2)| = m_1 + m_2 + n_1 n_2$.

Lemma 2.9. [17] *Let Γ_1 and Γ_2 be two graphs with n_1 and n_2 vertices. Then*

$$d_2(v/(\Gamma_1 + \Gamma_2)) = \begin{cases} n_1 - 1 - d(v/\Gamma_1), & \text{if } v \in V(\Gamma_1); \\ n_2 - 1 - d(v/\Gamma_2), & \text{if } v \in V(\Gamma_2). \end{cases}$$

Theorem 2.10. *Let Γ_1 and Γ_2 be nontrivial graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\overline{L_2}(\Gamma_1 + \Gamma_2) = M_2(\overline{\Gamma_1}) + M_2(\overline{\Gamma_2}). \quad (5)$$

$$\overline{L_3}(\Gamma_1 + \Gamma_2) = M_1(\overline{\Gamma_1}) + M_1(\overline{\Gamma_2}). \quad (6)$$

Proof. Since for nontrivial graphs Γ_1 and Γ_2, the join graph $\Gamma_1 + \Gamma_2$ has diameter at most two. Hence, if $u, v \in V(\Gamma_1 + \Gamma_2)$, then $uv \notin E(\Gamma_1 + \Gamma_2)$, if and only if

$uv \notin E(\Gamma_1)$, or $uv \notin E(\Gamma_2)$. Then by Lemma 2.9,

$$\begin{aligned}
\overline{L_2}(\Gamma_1+\Gamma_2) &= \sum_{uv\notin E(\Gamma_1+\Gamma_2)} \Big(d_2(u/(\Gamma_1+\Gamma_2))d_2(v/(\Gamma_1+\Gamma_2))\Big) \\
&= \sum_{uv\notin E(\Gamma_1)} \Big(d_2(u/(\Gamma_1+\Gamma_2))d_2(v/(\Gamma_1+\Gamma_2))\Big) \\
&+ \sum_{uv\notin E(\Gamma_2)} \Big(d_2(u/(\Gamma_1+\Gamma_2))d_2(v/(\Gamma_1+\Gamma_2))\Big) \\
&= \sum_{uv\in E(\overline{\Gamma_1})} \Big(d(u/\overline{\Gamma_1})d(v/\overline{\Gamma_1})\Big) + \sum_{uv\in E(\overline{\Gamma_2})} \Big(d(u/\overline{\Gamma_2})d(v/\overline{\Gamma_2})\Big) \\
&= M_2(\overline{\Gamma_1}) + M_2(\overline{\Gamma_2}).
\end{aligned}$$

This completes the proof of second LZCI of join graph, for third LZCI, we have

$$\begin{aligned}
\overline{L_3}(\Gamma_1+\Gamma_2) &= \sum_{uv\notin E(\Gamma_1+\Gamma_2)} \Big(d_2(u/(\Gamma_1+\Gamma_2))+d_2(v/(\Gamma_1+\Gamma_2))\Big) \\
&= \sum_{uv\notin E(\Gamma_1)} \Big(d_2(u/(\Gamma_1+\Gamma_2))+d_2(v/(\Gamma_1+\Gamma_2))\Big) \\
&+ \sum_{uv\notin E(\Gamma_2)} \Big(d_2(u/(\Gamma_1+\Gamma_2))+d_2(v/(\Gamma_1+\Gamma_2))\Big) \\
&= \sum_{uv\in E(\overline{\Gamma_1})} \Big(d(u/\overline{\Gamma_1})+d(v/\overline{\Gamma_1})\Big) + \sum_{uv\in E(\overline{\Gamma_2})} \Big(d(u/\overline{\Gamma_2})+d(v/\overline{\Gamma_2})\Big) \\
&= M_1(\overline{\Gamma_1}) + M_1(\overline{\Gamma_2}).
\end{aligned}$$

From Theorem 2.10, the following generalization follows.

Proposition 2.11. *For $k \geq 2$, let $\Gamma_1, ..., \Gamma_k$ be connected graphs with n_i vertices and m_i edges, respectively. Then*

$$\overline{L_2}(\sum_{i=1}^{k} \Gamma_i) = \sum_{i=1}^{k} M_2(\overline{\Gamma_i}). \tag{7}$$

$$\overline{L_3}(\sum_{i=1}^{k} \Gamma_i) = \sum_{i=1}^{k} M_1(\overline{\Gamma_i}). \tag{8}$$

By using the following facts, which is found in [10], and state that

$$M_2(\overline{\Gamma}) = \frac{1}{2}n(n-1)^3 + 2m^2 - 3m(n-1)^2 + \frac{2n-3}{2}M_1(\Gamma) - M_2(\Gamma) \quad (9)$$

$$M_1(\overline{\Gamma}) = M_1(\Gamma) + n(n-1)^2 - 4m(n-1). \quad (10)$$

The following results are straightforward.

Corollary 2.12. *For connected graphs $\Gamma_1, ..., \Gamma_k$, $k \geq 2$, with n_i vertices and m_i edges, for every $i = 1, 2, ..., k$.*

$$\overline{L_2}(\sum_{i=1}^{k}\Gamma_i) = \sum_{i=1}^{k}\Big(\frac{2n_i-3}{2}M_1(\Gamma_i) - M_2(\Gamma_i) + \frac{(n_i-1)^2}{2}\big[n_i(n_i-1) - 3m_i\big] + 2m_i^2\Big). \quad (11)$$

$$\overline{L_3}(\sum_{i=1}^{k}\Gamma_i) = \sum_{i=1}^{k}\Big(M_1(\Gamma_i) + n_i(n_i-1)^2 - 4m_i(n_i-1)\Big). \quad (12)$$

2.3. Cartesian product

Definition 2.13. [14] *The Cartesian product of two graphs Γ_1 and Γ_2, denoted by $\Gamma_1\Box\Gamma_2$, is a graph with $V(\Gamma_1\Box\Gamma_2) = V(\Gamma_1) \times V(\Gamma_2)$, such that if $u = (u_1, u_2)$ and $v = (v_1, v_2)$ vertices in $V(\Gamma_1\Box\Gamma_2)$, then $uv \in E(\Gamma_1\Box\Gamma_2)$, if and only if either $(u_1 = v_1$ and $u_2v_2 \in E(\Gamma_2))$ or $(u_2 = v_2$ and $u_1v_1 \in E(\Gamma_1))$.*

The Cartesian product of graphs is commutative and associative in operation. $|V(\Gamma_1\Box\Gamma_2)| = |V(\Gamma_1)||V(\Gamma_2)|$, and $d(u,v) = d_{\Gamma_1}(u_1,v_1) + d_{\Gamma_2}(u_2,v_2)$, for every $u, v \in V(\Gamma_1\Box\Gamma_2)$.

Lemma 2.14. [17] *For any connected graphs Γ_1 and Γ_2. If $(u,v) \in V(\Gamma_1\Box\Gamma_2)$, then*

$$d_2((u,v)/(\Gamma_1\Box\Gamma_2)) = d_2(u/\Gamma_1) + d_1(u/\Gamma_1)d_1(v/\Gamma_2) + d_2(v/\Gamma_2).$$

The following result is required to prove our main result,

Theorem 2.15. [18] *Let Γ_1 and Γ_2 be two nontrivial connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$L_3(\Gamma_1\Box\Gamma_2) = n_2L_3(\Gamma_1) + n_1L_3(\Gamma_2) + 2m_2(M_1(\Gamma_1) + 2\mu(\Gamma_1)) + 2m_1(M_1(\Gamma_2) + 2\mu(\Gamma_2)).$$

Theorem 2.16. *Let Γ_1 and Γ_2 be two nontrivial connected graphs with n_1, n_2 vertices and $\mu(\Gamma_1)$, $\mu(\Gamma_2)$ second edges, respectively. Then the third leap coindex of $(\Gamma_1\Box\Gamma_2)$ is given by*

$$\overline{L_3}(\Gamma_1\Box\Gamma_2) = M_1(\Gamma_1)\Big[2n_2(n_1n_2-1) - 2m_2\Big] + M_1(\Gamma_2)\Big[2n_1(n_1n_2-1) - 2m_1\Big]$$
$$- n_2L_3(\Gamma_1) - n_1L_3(\Gamma_2) - 4(n_1n_2-1)(n_1m_2+n_2m_1) + 4n_1n_2m_1m_2.$$

With equality if and only if Γ_1 *and* Γ_2 *are* (C_3, C_4)*-free graphs.*

Proof. From Lemma 2.14, Theorem 2.15 and by using Corollary 1.2 and applying the fact that state for any graph Γ, $\overline{L_3}(\Gamma) = (n-1)\sum_{v\in V(\Gamma)} d_2(v/\Gamma) - L_3(\Gamma)$, we obtain

$$\begin{aligned}
&\overline{L_3}(\Gamma_1\Box\Gamma_2) = (n_1n_2-1)\sum_{(u,v)\in V(\Gamma_1\Box\Gamma_2)}\Big(d_2((u,v)/(\Gamma_1\Box\Gamma_2))\Big) - L_3(\Gamma_1\Box\Gamma_2)\\
&= (n_1n_2-1)\sum_{u\in V(\Gamma_1)}\sum_{v\in V(\Gamma_2)}\Big(d_2((u,v)/(\Gamma_1\Box\Gamma_2))\Big) - L_3(\Gamma_1\Box\Gamma_2)\\
&= (n_1n_2-1)\sum_{u\in V(\Gamma_1)}\sum_{v\in V(\Gamma_2)}\Big[d_2(u/\Gamma_1) + d_1(u/\Gamma_1)d_1(v/\Gamma_2) + d_2(v/\Gamma_2))\Big] - L_3(\Gamma_1\Box\Gamma_2)\\
&= (n_1n_2-1)\Big[n_2\sum_{u\in V(\Gamma_1)} d_2(u/\Gamma_1) + 2m_1m_2 + n_1\sum_{u\in V(\Gamma_1)} d_2(v/\Gamma_2)\Big]\\
&- \Big[n_2L_3(\Gamma_1) + 2m_2(M_1(\Gamma_1) + 2\mu(\Gamma_1)) + n_1L_3(\Gamma_2) + 2m_1(M_1(\Gamma_2) + 2\mu(\Gamma_2))\Big]\\
&= (n_1n_2-1)\Big[2n_2\mu(\Gamma_1) + 4m_1m_2 + 2n_1\mu(\Gamma_2)\Big]\\
&\quad - \Big[n_2L_3(\Gamma_1) + 2m_2(M_1(\Gamma_1) + 2\mu(\Gamma_1)) + n_1L_3(\Gamma_2) + 2m_1(M_1(\Gamma_2) + 2\mu(\Gamma_2))\Big]\\
&= 2\mu(\Gamma_1)\big[2n_2(n_1n_2-1) - m_2\big] - n_2L_3(\Gamma_1) - 2m_2M_1(\Gamma_1)\\
&+ 2\mu(\Gamma_2)\big[2n_1(n_1n_2-1) - m_1\big] - n_1L_3(\Gamma_2) - 2m_1M_1(\Gamma_2) + 4m_1m_2(n_1n_2-1)\\
&\le (M_1(\Gamma_1) - 2m)\big[2n_2(n_1n_2-1) - m_2\big] - n_2L_3(\Gamma_1) - 2m_2M_1(\Gamma_1) + (M_1(\Gamma_2) - 2m_2)\\
&\quad \big[2n_1(n_1n_2-1) - m_1\big] - n_1L_3(\Gamma_2) - 2m_1M_1(\Gamma_2) + 4m_1m_2(n_1n_2-1)\\
&= M_1(\Gamma_1)\Big[2n_2(n_1n_2-1) - 2m_2\Big] - 4m_1n_2(n-1n_2-1) + 2m_1m_2 - n_2L_3(\Gamma_1)\\
&\quad + M_1(\Gamma_2)\Big[2n_1(n_1n_2-1) - 2m_1\Big] - 4m_2n_1(n-1n_2-1) + 2m_1m_2 - n_1L_3(\Gamma_2)\\
&\quad + 4m_1m_2(n_1n_2-1)\\
&= M_1(\Gamma_1)\Big[2n_2(n_1n_2-1) - 2m_2\Big] - n_2L_3(\Gamma_1) + M_1(\Gamma_2)\Big[2n_1(n_1n_2-1) - 2m_1\Big]\\
&- n_1L_3(\Gamma_2) - 4(n_1n_2-1)(n_1m_2 + n_2m_1) + 4n_1n_2m_1m_2.
\end{aligned}$$

As an application of this result, we list explicit formulae for the third leap Zagreb coindex for the cartesian product of two complete graphs with p and q vertices and the rectangular grid $P_p\Box P_q$, the C_4-nanotube $P_p\Box C_q$, and the C_4-nanotorus $C_p\Box C_q$, respectively. From Theorem 2.16, by plugging in the expressions the following values the next result follows.

- $M_1(K_p) = p(p-1)^2$, and $L_3(K_p) = 0$,

- $M_1(P_p) = 4p - 6$, and $L_3(P_p) = 2(2p - 5)$,
- $M_1(C_p) = 4p$, and $L_3(C_p) = 4p$.

Observation 2.17. *For the integers number $p, q \geq 5$, the following results holds:*

- $\overline{L_3}(K_p \square K_q) = 4\binom{pq}{2}\Big[\binom{p-1}{2} + \binom{q-1}{2}\Big] + 2pq\Big[\binom{pq}{2} - \binom{p+q}{2}\Big] + 2pq(p + q - 1)$.
- $\overline{L_3}(P_p \square P_q) = 8pq(pq - 4) - 2(p + q)(4pq - 13) - 24$.
- $\overline{L_3}(P_p \square C_q) = 4q(p - 1)(3pq - 8)$.
- $\overline{L_3}(C_p \square C_q) = 4pq(3pq - 8)$.

2.4. Composition

Definition 2.18. [14] *The composition of two graphs Γ_1 and Γ_2 with disjoint vertex sets and edge sets, denote by $\Gamma_1[\Gamma_2]$, is a graph on vertex set $V(\Gamma_1[\Gamma_2]) = V(\Gamma_1) \times V(\Gamma_2)$ in which two vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever $u_1u_2 \in E(\Gamma_1)$ or $u_1 = u_2$ and $v_1v_2 \in V(\Gamma_2)$.*

The composition of graphs is not commutative and $|E(\Gamma_1[\Gamma_2])| = n_1m_2 + n_2^2m_1$.

Lemma 2.19. [17] *For any two graphs Γ_1 of order n_1 and Γ_2 of order n_2, such that $V(\Gamma_1) \cap V(\Gamma_2) = \phi$. Then for every $(u, v) \in V(\Gamma_1[\Gamma_2])$,*

$$d_2((u, v)/(\Gamma_1[\Gamma_2])) = n_2 d_2(u/\Gamma_1) + d_1(v/\overline{\Gamma_2}).$$

We need the following result to show our main result,

Theorem 2.20. *For nontrivial connected graphs Γ_1 and Γ_2 with n_1, n_2 vertices and m_1, m_2 edges, respectively.*
$L_3(\Gamma_1[\Gamma_2]) = n_2^3 L_3(\Gamma_1) - n_1 M_1(\Gamma_2) - 4n_2m_1m_2 + 2(n_2 - 1)(n_1m_2 + n_2^2m_1) + 2n_2m_2\mu(\Gamma_1)$.
Where, $\mu(\Gamma_1) = \sum_{v \in V(\Gamma_1)} d_2(v/\Gamma_1)$.

Theorem 2.21. *Let Γ_1 and Γ_2 be two nontrivial connected graphs with n_1, n_2 vertices and $\mu(\Gamma_1)$, $\mu(\Gamma_2)$ second edges, respectively. Then*

$$\begin{aligned}\overline{L_3}(\Gamma_1[\Gamma_2]) \leq{}& n_2 M_1(\Gamma_1)\Big[n_2(n_1n_2 - 1) - m_2\Big] - n_2^3 L_3(\Gamma_1) + n_1 M_1(\Gamma_2) - (n_1n_2 - 1) \\ & \Big[n_2(n_1 - 2m_1) - n_1\Big] - 2(n_1m_2 + 2n_2^2m_1)\Big[n_2(n_1 + 1) - 2)\Big] + 8n_2m_1m_2.\end{aligned}$$

The equality holds if and only if both the graphs Γ_1 and Γ_2 are (C_3, C_4)-free.
Proof. From Lemma 2.19, Theorem 2.20 and by using the fact that state for a

graph Γ,
$\overline{L_3}(\Gamma) = (n-1)\sum_{v\in V(\Gamma)} d_2(v/\Gamma) - L_3(\Gamma)$. Thus

$$\begin{aligned}
\overline{L_3}(\Gamma_1[\Gamma_2]) &= (n_1n_2-1)\sum_{(u,v)\in V(\Gamma_1[\Gamma_2])} d_2((u,v)/(\Gamma_1[\Gamma_2])) - L_3(\Gamma_1[\Gamma_2])\\
&= (n_1n_2-1)\sum_{u\in V(\Gamma_1)}\sum_{v\in V(\Gamma_2)}[d_2(u/\Gamma_1)-(n_2-1)-d(v/\Gamma_2)] - L_3(\Gamma_1[\Gamma_2])\\
&= (n_1n_2-1)\Big[2n_2^2\mu(\Gamma_1)+n_1n_2(n_2-1)-2n_1m_2\Big]\\
&\quad - \Big[n_2^3L_3(\Gamma_1)-n_1M_1(\Gamma_2)-4n_2m_1m_2+2(n_2-1)(n_1m_2+n_2^2m_1)+2n_2m_2\mu(\Gamma_1)\Big]\\
&= 2n_2\mu(\Gamma_1)\Big[n_2(n_1n_2-1)-m_2\Big]+n_1n_2(n_2-1)(n_1n_2-1)\\
&\quad -2n_1m_2(n_1n_2-1)-n_2^3L_3(\Gamma_1)+n_1M_1(\Gamma_2)+4n_2m_1m_2-2(n_2-1)(n_1m_2+n_2^2m_2)\\
&\le n_2(M_1(\Gamma_1)-2m_1)\Big[n_2(n_1n_2-1)-m_2\Big]+n_1n_2(n_2-1)(n_1n_2-1)\\
&\quad -2n_1m_2(n_1n_2-1)-n_2^3L_3(\Gamma_1)+n_1M_1(\Gamma_2)+4n_2m_1m_2-2(n_2-1)(n_1m_2+n_2^2m_2)\\
&= n_2M_1(\Gamma_1)\Big[n_2(n_1n_2-1)-m_2\Big]-4n_2^2m_1(n_1n_2-1)+4n_2m_1m_2+n_1n_2(n_2-1)(n_1n_2-1)\\
&\quad -2n_1m_2(n_1n_2-1)-n_2^3L_3(\Gamma_1)+n_1M_1(\Gamma_2)+4n_2m_1m_2-2(n_2-1)(n_1m_2+n_2^2m_2)\\
&= n_2M_1(\Gamma_1)\Big[n_2(n_1n_2-1)-m_2\Big]-n_2^3L_3(\Gamma_1)+n_1M_1(\Gamma_2)-2(n_1n_2-1)(n_1m_2+2n_2^2m_1)\\
&\quad +n_1n_2(n_2-1)(n_1n_2-1)+8n_2m_1m_2-2(n_2-1)(n_1m_2+n_2^2m_2)\\
&= n_2M_1(\Gamma_1)\Big[n_2(n_1n_2-1)-m_2\Big]-n_2^3L_3(\Gamma_1)+n_1M_1(\Gamma_2)-2n_2^2m_1(n_1n_2-1)\\
&\quad -2(n_1m_2+2n_2^2m_1)\big[(n_1n_2-1)+(n_2-1)\big]+n_1n_2(n_2-1)(n_1n_2-1)+8n_2m_1m_2\\
&= n_2M_1(\Gamma_1)\Big[n_2(n_1n_2-1)-m_2\Big]-n_2^3L_3(\Gamma_1)-(n_1n_2-1)\big[n_1n_2(n_2-1)-2n_2^2m_1\big]\\
&\quad -2(n_1m_2+2n_2^2m_1)\big[(n_1n_2-1)+(n_2-1)\big]+8n_2m_1m_2+n_1M_1(\Gamma_2)\\
&= n_2M_1(\Gamma_1)\Big[n_2(n_1n_2-1)-m_2\Big]-n_2^3L_3(\Gamma_1)-(n_1n_2-1)\big[n_2(n_1-2m_1)-n_1\big]\\
&\quad -2(n_1m_2+2n_2^2m_1)\big[n_2(n_1+1)-2)\big]+8n_2m_1m_2+n_1M_1(\Gamma_2).
\end{aligned}$$

2.5. Disjunction

Definition 2.22. [14] *The disjunction graph $\Gamma_1\vee\Gamma_2$ of two graphs Γ_1 and Γ_2 with disjoint vertex and edge sets is a graph with $V(\Gamma_1\vee\Gamma_2)=V(\Gamma_1)\times V(\Gamma_2)$ and if two vertices $u=(u_1,u_2)$, $v=(v_1,v_2)$ in $V(\Gamma_1\vee\Gamma_2)$, then $uv\in E(\Gamma_1\vee\Gamma_2)$, whenever $u_1v_1\in E(\Gamma_1)$ or $u_2v_2\in E(\Gamma_2)$.*

The disjunction operation $\Gamma_1\vee\Gamma_2$ is commutative, $diam(\Gamma_1\vee\Gamma_2)\le 2$ and

$|E(\Gamma_1 \vee \Gamma_2) = n_1^2 m_2 + n_2^2 m_1 - 2m_1 m_2$.

Lemma 2.23. [17] *Let Γ_1 and Γ_2 be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then*

1. $d_1((u,v)/(\Gamma_1 \vee \Gamma_2)) = n_2 d_1(u/\Gamma_1) + n_1 d_1(v/\Gamma_2) - d_1(u/\Gamma_1)d_1(v/\Gamma_2)$

2. $d_2((u,v)/(\Gamma_1 \vee \Gamma_2)) = (n_1 n_2 - 1) - n_2 d_1(u/\Gamma_1) - n_1 d_1(v/\Gamma_2) + d_1(u/\Gamma_1)d_1(v/\Gamma_2)$.

Since the diameter of $\Gamma_1 \vee \Gamma_2$ is at most two, bring in the mind that, "if $diam(\Gamma) \leq 2$, then $\overline{L_3}(\Gamma) = L_1(\Gamma)$", (see Theorem 4.3, in [7]), and by using Theorem 2.18 in [18], the following expression of third leap coindex of $\Gamma_1 \vee \Gamma_2$ is straightforward.

Theorem 2.24. *If one of the graphs Γ_1 and Γ_2 is not complete with n_1, n_2, m_1 and m_2 orders and sizes, respectively, then*

$$\begin{aligned}\overline{L_3}(\Gamma_1 \vee \Gamma_2) = &(n_1 n_2^2 - 4n_2 m_2)M_1(\Gamma_1) + (n_2 n_1^2 - 4n_1 m_1)M_1(\Gamma_2) + M_1(\Gamma_1)M_1(\Gamma_2)\\ &+ 8n_1 n_2 m_2 + n_1 n_2 (n_1 n_2 - 1)^2 - 4(n_1 n_2 - 1)(n_2^2 m_1 + n_1^2 m_2 - 2m_1 m_2).\end{aligned}$$

2.6. Symmetric difference

Definition 2.25. [14] *The Symmetric difference $\Gamma_1 \oplus \Gamma_2$ of two graphs Γ_1 and Γ_2 with disjoint vertex sets and edge sets is the graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ in which (u_1, v_1) is adjacent with (u_2, v_2) whenever u_1 is adjacent with u_2 in Γ_1 or v_1 is adjacent with v_2 in Γ_2 but not both.*

The Symmetric difference is commutative, with $|V(\Gamma_1 \oplus \Gamma_2)| = n_1 n_2$ vertices, $diam(\Gamma_1 \oplus \Gamma_2) \leq 2$ and $|E(\Gamma_1 \oplus \Gamma_2)| = n_1^2 m_2 + n_2^2 m_1 - 4m_1 m_2$ edges.

Lemma 2.26. *Let Γ_1 and Γ_2 be distinct graphs of n_1 and n_2 orders and m_1 and m_2 sizes, respectively. Then*

1. $d_1((u,v)/(\Gamma_1 \oplus \Gamma_2)) = n_2 d_1(u/\Gamma_1) + n_1 d_1(v/\Gamma_2) - 2d_1(u/\Gamma_1)d_1(v/\Gamma_2)$

2. $d_2((u,v)/(\Gamma_1 \oplus \Gamma_2)) = (n_1 n_2 - 1) - n_2 d_1(u/\Gamma_1) - n_1 d_1(v/\Gamma_2) + 2d_1(u/\Gamma_1)d_1(v/\Gamma_2)$.

By similar arguments as in Theorem 2.26, the following expression of third leap coindex of $\Gamma_1 \oplus \Gamma_2$ is straightforward.

Theorem 2.27. *Let Γ_1 and Γ_2 be two graphs, such that one of them is not complete with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then*

$$\begin{aligned}\overline{L_3}(\Gamma_1 \oplus \Gamma_2) = &(n_1 n_2^2 - 8n_2 m_2)M_1(\Gamma_1) + (n_2 n_1^2 - 8n_1 m_1)M_1(\Gamma_2) + 4M_1(\Gamma_1)M_1(\Gamma_2)\\ &+ 8n_1 n_2 m_1 m_2 + n_1 n_2((n_1 n_2 - 1)^2 - 4(n_1 n_2 - 1)(n_2^2 m_1 + n_1^2 m2 - 4m_1 m_2).\end{aligned}$$

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