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VL-MULTIPLICATIVE TOTAL ECCENTRICITY INDEX AND VL-MULTIPLICATIVE HYPER TOTAL ECCENTRICITY INDEX OF SOME STANDARD GRAPHS

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Abstract: In this article, we study the VL-multiplicative total eccentricity index and VL-multiplicative hyper total eccentricity index of graphs. Also, established the results of these indices for complete graphs, path with n-vertices, cycle graphs, wheel graphs, star graphs and complete bipartite graphs.

Keywords and Phrases: Eccentricity, Total graph, Degree, Topological index.

2020 Mathematics Subject Classification: 05C35, 05C90, 05C12.

1. Introduction

Consider a finite graph G with n vertices and m edges. The vertex set of G is denoted as $V = V(G)$ and the edge set as $E = E(G)$. Let $e = uv$ denotes the edge connecting the vertices u and v. In such a case, u and v are said to be adjacent. In a simple graph G, the degree $deg_G(v)$ or briefly d_v of a vertex v is equal to the number of neighboring vertices. A vertex v is considered as well-connected if its degree is equal to $(n-1)$ meaning that it is adjacent to every other vertex in G. Let $d_G(e)$ or briefly d_e denote the degree of an edge in G. The eccentricity $e(v)$ of a vertex v is the distance to the vertex that is farthest from v. That is, $e(v) = max\{d(u,v); u \in V\}$.

The total graph $T(G)$ of G consists of all the vertices in $V = V(G) \bigcup E(G)$ where two vertices in $T(G)$ are adjacent if and only if they are adjacent edges or vertices in G or if one element is a vertex and the other is an edge in G that are incident with each other.

The eccentricity of an edge e or a vertex u in $T(G)$ is denoted by $e_{T(G)}(u)$ or $e_{T(G)}(e)$ respectively, [1].

Let K_n represent the complete graph with n vertices, $K_{(1,n)}$ represent the star graph with $(n+1)$ vertices, C_n represent the cycle graph on n vertices and $K_{(m,n)}$ represent the complete bipartite graph with $(m+n)$ vertices.

Several topological indices that rely on vertex eccentricity have already been in the focus of several investigations. The Topological indices are one of the mathematical models that can be defined by assigning a real number.

The total eccentricity index of graph G is defined as:

$$\xi(G) = \sum_{v \in V(G)} e_G(v).$$

In line with this measure, Dankelmann et al., [2], and Tang *et al.*, [11] conducted research on the mean eccentricity of graphs. Fathalikhani *et al.*, [4] conducted a study in which they examined the total eccentricity of several graph operations.

Results related Zagreb indices and found in [5]-[7], [9], [10]. Motivated from these, Deepika.T [3] introduced the Veerabhadraiah Lokesha (VL) index in the year 2021, defined as:

$$VL(G) = \frac{1}{2} \sum_{uv \in E(G)} [d_e + d_f + 4]$$

where $d_e = d_u + d_v - 2$ and $d_f = d_u \cdot d_v - 2$.
It is useful in QSPR and QSAR studies. Motivation from above, We define the VL-multiplicative total eccentricity index by

$$\xi(VL)T \prod(G) = \frac{1}{2} \prod_{ue} [e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]$$

and the VL-multiplicative hyper total eccentricity index by

$$H\xi(VL)T\prod(G) = \frac{1}{4}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]^2$$

where $e_{T(G)}u$ denotes the eccentricity of vertex u, $e_{T(G)}e$ denotes the eccentricity of edge e and ue means that the vertex u and edge e are incident in G.
In the forthcoming section, we obtain results of defined indices for standard graphs.

2. Results of VL related indices on few standard graphs

Theorem 2.1. *Let K_n be a complete graph with n vertices. Then*

$$(i) \quad \xi(VL)T\prod(K_n) = 32^{\frac{n(n-1)}{2}}.$$

$$(ii) \quad H\xi(VL)T\prod(K_n) = 1024^{\frac{n(n-1)}{2}}.$$

Proof. Let K_n be a complete graph with n vertices and $m = \frac{n(n-1)}{2}$ edges. Every edge of K_n is incident with exactly two vertices. Every vertex and edge has eccentricity 2 in $T(G)$.
Consider,

$$\textbf{(i)} \quad \xi(VL)T\prod(K_n) = \tfrac{1}{2}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]$$

$$= \tfrac{1}{2}\prod_{uv}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]$$

$$\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v \cdot e_{T(G)}e]]$$

$$= \tfrac{1}{2}\prod_{uv}[[2+2+2\cdot 2]\cdot[2+2+2\cdot 2]]$$

$$= 32^{\frac{n(n-1)}{2}}.$$

and

$$\textbf{(ii)} \quad H\xi(VL)T\prod(K_n) = \tfrac{1}{4}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]^2$$

$$= \tfrac{1}{4}\prod_{uv}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]^2$$

$$\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v \cdot e_{T(G)}e]^2]$$

$$= \tfrac{1}{4}\prod_{uv}[[2+2+2\cdot 2]^2\cdot[2+2+2\cdot 2]^2]$$

$$= 1024^{\frac{n(n-1)}{2}}.$$

Note 1: We observe that, $H\xi(VL)T\prod(K_n) = [\xi(VL)T\prod(K_n)]^2$.

Theorem 2.2. *Let P_n be a path with n vertices. Then*

$$(i)\ \xi(VL)T\prod(P_n) = \begin{cases} \prod_{k=0}^{m}\left[\left(n+k+\frac{n^2+2kn-1}{4}\right)\cdot\left(n+k+1+\frac{n^2+2kn+2k+1}{4}\right)\right] & \text{if } n \text{ is odd.} \\ 2\prod_{i=1}^{n}\left[2\left(2n+i+\frac{4n}{2}\right)^2\right] & \text{if } n \text{ is even.} \end{cases}$$

$$(ii)\ H\xi(VL)T\prod(P_n) = \begin{cases} \prod_{k=1}^{m}\left[\left(n+k+\frac{n^2+2kn-1}{4}\right)^2\cdot\left(n+k+\frac{n^2+4kn+4k^2+1}{4}\right)^2\right] & \text{if } n \text{ is odd.} \\ \frac{1}{4}\left[n+\frac{n^2}{4}\right]^4\cdot\prod_{k=1}^{m}\left[n+2k+\frac{n^2+4nk+4k^2}{4}\right]^4 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let P_n be a path with n vertices. Then P_n has $(n–1)$ edges. Every edge of P_n is incident with exactly two vertices. $T(P_n)$ has n point vertices and $(n–1)$ line vertices.

When n is odd:

Number of edges $e = uv$ in G	ξ of e in $T(G)$ $(e_T(G))$	ξ of end vertices $(e_T(u), e_T(v))$
2	$(\frac{n+1}{2})$	$((\frac{n-1}{2}), (\frac{n-1}{2})+1)$
2	$(\frac{n+3}{2})$	$((\frac{n+1}{2}), (\frac{n+3}{2}))$
2	$(\frac{n+5}{2})$	$((\frac{n+3}{2}), (\frac{n+5}{2}))$
......	......	....
2	$(n-1)$	$((n-2), (n-1))$

When n is even:

Number of edges $e = uv$ in G	ξ of e in T(G) $(e_T(G))$	ξ of end vertices $(e_T(u), e_T(v))$
1	$(\frac{n}{2})$	$((\frac{n}{2}), (\frac{n}{2}))$
2	$\frac{n}{2}+1$	$((\frac{n}{2}), (\frac{n}{2})+1)$
2	$(\frac{n}{2}+2)$	$((\frac{n}{2}+1), (\frac{n}{2}+2))$
......	......	....
2	$(n-1)$	$((n-2), (n-1))$

Case(i): When n is odd: Eccentricity of central vertex is $(\frac{n-1}{2})$, eccentricity of pendant vertices $(n-1)$, eccentricity of line vertices incident with central vertex is $(\frac{n+1}{2})$, eccentricity of line vertices incident with end vertices is $(n-1)$ in $T(G)$.

$$\textbf{(i) } \xi(VL)T\prod(P_n) = \frac{1}{2}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]$$

$$= \frac{1}{2}\prod_{uv\in E(G)}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]$$

$$\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v \cdot e_{T(G)}e]]$$

$$= \frac{1}{2}[2][(\frac{n-1}{2}) + (\frac{n+1}{2}) + (\frac{n-1}{2})\cdot(\frac{n+1}{2})]$$

$$\cdot\frac{1}{2}[2][(\frac{n-1}{2})+1) + (\frac{n+1}{2}) + (\frac{n-1}{2})+1)\cdot(\frac{n+1}{2})]$$

$$\cdot\tfrac{1}{2}[2][(\tfrac{n+1}{2})+(\tfrac{n+3}{2})+(\tfrac{n+1}{2})\cdot(\tfrac{n+3}{2})]\cdot\tfrac{1}{2}[2][(\tfrac{n+3}{2})+(\tfrac{n+3}{2})+(\tfrac{n+3}{2})\cdot(\tfrac{n+3}{2})]\cdot\ldots$$

$$\cdot\tfrac{1}{2}[2][(n-2)+(n-1)+(n-2)\cdot(n-1)+2(n-1)+(n-1)^2]$$

$$=\prod_{k=0}^{m}\left[\left(n+k+\frac{n^2+2kn-1}{4}\right)\cdot\ \left(n+k+1+\frac{n^2+2kn+2k+1}{4}\right)\right].$$

When n is even: Eccentricity of central vertex is $(\frac{n}{2})$ and $(\frac{n}{2})$, eccentricity of pendant vertices $(n-1)$,eccentricity of line vertices incident with central vertex is $(\frac{n}{2})$, eccentricity of line vertices incident with end vertices is $(n-1)$.

(i) $\xi(VL)T \quad \prod(P_n)=\frac{1}{2}\prod_{ue}[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]$

$$=\tfrac{1}{2}\prod_{uv\in E(G)}[[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]$$

$$\cdot[e_{T(G)}v+e_{T(G)}e+e_{T(G)}v\cdot e_{T(G)}e]]$$

$$=\tfrac{1}{2}[1][[(\tfrac{n}{2})+(\tfrac{n}{2})+(\tfrac{n}{2})\cdot(\tfrac{n}{2})]\cdot[(\tfrac{n}{2})+(\tfrac{n}{2})+(\tfrac{n}{2})\cdot(\tfrac{n}{2})]]$$

$$\cdot\tfrac{1}{2}[2][[(\tfrac{n}{2})+(\tfrac{n}{2}+1)+(\tfrac{n}{2})\cdot(\tfrac{n}{2}+1)]$$

$$\cdot[(\tfrac{n}{2}+1)+(\tfrac{n}{2}+1)+(\tfrac{n}{2}+1)\cdot(\tfrac{n}{2}+1)]]\cdot\ldots$$

$$\cdot\tfrac{1}{2}[2][(n-2)+(n-1)+(n-2)\cdot(n-1)+2(n-1)+(n-1)^2]$$

$$=2\prod_{i=1}^{n}\left[2\left(2n+i+\frac{4n}{2}\right)^2\right].$$

Case(ii): When n is odd: Eccentricity of central vertex is $(\frac{n-1}{2})$, eccentricity of pendant vertices $(n-1)$, eccentricity of line vertices incident with central vertex is $(\frac{n+1}{2})$, eccentricity of line vertices incident with end vertices is $(n-1)$ in $T(G)$.

(ii) $H\xi(VL)T\prod(P_n) \quad =\frac{1}{4}\prod_{ue}[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]^2$

$$=\tfrac{1}{4}\prod_{uv}[[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]^2$$

$$\cdot[e_{T(G)}v+e_{T(G)}e+e_{T(G)}v\cdot e_{T(G)}e]^2]$$

$$=\tfrac{1}{4}[4][(\tfrac{n-1}{2})+(\tfrac{n+1}{2})+(\tfrac{n-1}{2})\cdot(\tfrac{n+1}{2})]^2$$

$$\cdot[(\tfrac{n+1}{2})+(\tfrac{n+1}{2})+(\tfrac{n+1}{2})\cdot(\tfrac{n+1}{2})]^2$$

$\cdot\frac{1}{4}[4][(\frac{n+1}{2})+(\frac{n+3}{2})+(\frac{n+1}{2})\cdot(\frac{n+3}{2})]^2\cdot[(\frac{n+3}{2})+(\frac{n+3}{2})+(\frac{n+3}{2})\cdot(\frac{n+3}{2})]^2\cdot\ldots$

$\cdot\frac{1}{4}[4][(n-2)+(n-1)+(n-2)\cdot(n-1)]^2\cdot[(n-1)+(n-1)+(n-1)\cdot(n-1)]^2$

$= \prod_{k=1}^{m}\left[\left(n+k+\frac{n^2+2kn-1}{4}\right)^2\cdot\ \left(n+k+\frac{n^2+4kn+4k^2+1}{4}\right)^2\right].$

When n is even: Eccentricity of the central vertex is $\frac{n}{2}$ and $\frac{n}{2}$, eccentricity of pendant vertices is $(n-1)$, eccentricity of line vertices incident with the central vertex is $\frac{n}{2}$, and eccentricity of line vertices incident with end vertices is $(n-1)$.

$$
\begin{aligned}
\textbf{(ii)}\quad H\xi(VL)T\prod(P_n) &= \tfrac{1}{4}\prod_{ue}[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]^2\\
&= \tfrac{1}{4}\prod_{uv}[[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]^2\\
&\cdot[e_{T(G)}v+e_{T(G)}e+e_{T(G)}v\cdot e_{T(G)}e]^2]\\
&= \tfrac{1}{4}[1][(\tfrac{n}{2})+(\tfrac{n}{2})+(\tfrac{n}{2})\cdot(\tfrac{n}{2})]^2\cdot[(\tfrac{n}{2})+(\tfrac{n}{2})+(\tfrac{n}{2})\cdot(\tfrac{n}{2})]^2\\
&\cdot\tfrac{1}{4}[4][(\tfrac{n}{2})+(\tfrac{n+2}{2})+(\tfrac{n}{2})\cdot(\tfrac{n+2}{2})]^2\\
&\cdot[(\tfrac{n+2}{2})+(\tfrac{n+2}{2})+(\tfrac{n+2}{2})\cdot(\tfrac{n+2}{2})]^2\cdot\ldots\\
&\cdot\tfrac{1}{4}(4)[(n-2)+(n-1)+(n-2)\cdot(n-1)]^2\\
&\cdot[(n-1)+(n-1)+(n-1)\cdot(n-1)]^2\\
&= \tfrac{1}{4}\left[n+\tfrac{n^2}{4}\right]^4\cdot\prod_{k=1}^{m}\left[n+2k+\tfrac{n^2+4nk+4k^2}{4}\right]^4.
\end{aligned}
$$

Theorem 2.3. *Let C_n be a cycle graph with $n\geq 4$ vertices. Then*

$$(i)\quad \xi(VL)T\prod(C_n)=\begin{cases}\frac{n(n^2+6n+5)^2}{32} & \text{if } n \text{ is odd.}\\ \frac{n(4n+n^2)^2}{32} & \text{if } n \text{ is even.}\end{cases}$$

$$(ii)\quad H\xi(VL)T\prod(C_n)=\begin{cases}\frac{n(n^2+2n+9)^4}{1024} & \text{if } n \text{ is odd.}\\ \frac{n[n^2+4n]^4}{1024} & \text{if } n \text{ is even.}\end{cases}$$

Proof. Let C_n be a cycle with n vertices. Then C_n has n edges. Every edge of C_n is incident with exactly two vertices. $T(G)$ has n point vertices and n line vertices.

Case(i): When n is odd: Let $n = 2k+1$, each point vertex of $T(G)$ has eccentricity $\frac{n+1}{2} = k+1$, and each line vertex has eccentricity $\frac{n+1}{2} = k+1$.

(i) $$\xi(VL)T\prod(C_n) = \frac{1}{2}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]$$
$$= \frac{1}{2}\prod_{uv}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]$$
$$\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v \cdot e_{T(G)}e]]$$
$$= \frac{1}{2}\prod_{uv}[[(k+1) + (k+1) + (k+1)\cdot(k+1)]$$
$$\cdot[(k+1) + (k+1) + (k+1)\cdot(k+1)]]$$
$$= \frac{n(n^2+6n+5)^2}{32}.$$

When n is even: Let $n = 2k$. Each point vertex of $T(G)$ has eccentricity $\frac{n}{2}$ and each line vertex has eccentricity $\frac{n}{2}$.

(i) $$\xi(VL)T\prod(C_n) = \frac{1}{2}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]$$
$$= \frac{1}{2}\prod_{uv}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]$$
$$\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v \cdot e_{T(G)}e]]$$
$$= \frac{1}{2}\prod_{uv}[[(k) + (k) + (k)\cdot(k)] \cdot [(k) + (k) + (k)\cdot(k)]]$$
$$= \frac{n(4n+n^2)^2}{32}.$$

Case(ii): When n is odd: Let $n = 2k+1$, each point vertex of $T(G)$ has eccentricity $\frac{n+1}{2} = k+1$, and each line vertex has eccentricity $\frac{n+1}{2} = k+1$.

(ii) $$H\xi(VL)T\prod(C_n) = \frac{1}{4}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]^2$$
$$= \frac{1}{4}\prod_{uv}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]^2$$
$$\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v \cdot e_{T(G)}e]^2]$$
$$= \frac{1}{4}\prod_{uv}[[(k+1) + (k+1) + (k+1)\cdot(k+1)]^2$$
$$\cdot[(k+1) + (k+1) + (k+1)\cdot(k+1)]^2]$$
$$= \frac{n(n^2+2n+9)^4}{1024}.$$

When n is even: Let $n=2k$, each point vertex of T(G) has eccentricity $\frac{n}{2}$, and each line vertex has eccentricity $\frac{n}{2}$.

$$\textbf{(ii)}\ H\xi(VL)T\prod(C_n) = \frac{1}{4}\prod_{ue}[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]^2$$

$$= \frac{1}{4}\prod_{uv}[[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]^2$$

$$\cdot[e_{T(G)}v+e_{T(G)}e+e_{T(G)}v\cdot e_{T(G)}e]^2]$$

$$= \frac{1}{4}\prod_{uv}[[(k)+(k)+(k)\cdot(k)]^2\cdot[(k)+(k)+(k)\cdot(k)]^2]$$

$$= \frac{n[n^2+4n]^4}{1024}.$$

Theorem 2.4. *Let W_n be a wheel graph with $n\geq 5$ vertices. Then*

$$(i)\ \ \xi(VL)T\prod(W_n) = \frac{(19800)^n}{4}.$$

$$(ii)\ \ H\xi(VL)T\prod(W_n) = \frac{(19800)^{2n}}{16}.$$

Proof. Let W_n be a wheel with $n+1$ vertices. Then W_n has $2n$ edges. Every edge of W_n is incident with exactly two vertices. Let $W_n=K_1+C_n$. Let v be the central vertex of W_n, and $v_1,v_2,\ldots,v_n$ be the vertices of C_n. We have n edges of G which are incident with the central vertex and n edges on the cycle.
Let $E_1=\{$set of all edges incident with central vertex$\}$ and $E_2=\{$set of all edges on with cycle$\}$.
For $e_i=vv_i\in E_1(G)$ $e_T(e_i)=2$ and $e_T(v)=2$, $e_T(v_i)=3$. If $e_{ii+1}=v_iv_{i+1}\in E_2(G)$. Then $e_T(e_{ii+1})=3$, $e_T(v_i)=3$. Also, $|E_1|=|E_2|=n$.

$$\textbf{(i)}\ \xi(VL)T\prod(W_n) = \frac{1}{2}\prod_{ue}[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]$$

$$= \frac{1}{2}\prod_{uv}[[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]$$

$$\cdot[e_{T(G)}v+e_{T(G)}e+e_{T(G)}v\cdot e_{T(G)}e]]$$

$$= \frac{1}{2}\prod_{uv}[[2+2+2\cdot 2]\cdot[2+3+2\cdot 3]]\cdot\frac{1}{2}\prod_{uv}[[3+3+3\cdot 3]$$

$$\cdot[3+3+3\cdot 3]]$$

$$= \frac{(19800)^n}{4}.$$

and

(ii) $H\xi(VL)T\prod(W_n) = \frac{1}{4}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]^2$

$= \frac{1}{4}\prod_{uv}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]^2$

$\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v \cdot e_{T(G)}e]^2]$

$= \frac{1}{4}\prod_{uv\in E_1}[(2+2+2\cdot 2)^2 \cdot (2+3+2\cdot 3)]^2$

$\cdot\frac{1}{4}\prod_{uv\in E_2}[(3+3+3\cdot 3)^2 \cdot (3+3+3\cdot 3)^2]$

$= \frac{(19800)^{2n}}{16}.$

Note 2: We observe that, $H\xi(VL)T\prod(W_n) = [\xi(VL)T\prod(W_n)]^2$.

Theorem 2.5. *Let $K_{(1,n)}$ be a star graph. Then*

$$(i)\ \xi(VL)T\prod(K_{(1,n)}) = \frac{(40)^n}{2}.$$

$$(ii)\ H\xi(VL)T\prod(k_{(1,n)}) = \frac{(40)^{2n}}{4}.$$

Proof. Let $K_{(1,n)}$ be a star graph with $(n+1)$ vertices and n edges. Every edge of $K_{(1,n)}$ is incident with exactly two vertices. Let u be a central vertex. Eccentricity of u is one and all other point and line vertices are of eccentricity two in $T(G)$.

(i) $\xi(VL)T\prod(K_{(1,n)}) = \frac{1}{2}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]$

$= \frac{1}{2}\prod_{uv}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]$

$\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v \cdot e_{T(G)}e]]$

$= \frac{1}{2}\prod_{uv}[(1+2+1\cdot 2)\cdot(2+2+2\cdot 2)]$

$= \frac{(40)^n}{2}.$

and

(ii) $H\xi(VL)T\prod(K_{(1,n)}) = \frac{1}{4}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]^2$

$= \frac{1}{4}\prod_{uv\in E(G)}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]^2$
$\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v \cdot e_{T(G)}e]^2]$

$= \frac{1}{4}\prod_{uv\in E(G)}[(1+2+1\cdot 2)^2\cdot(2+2+2\cdot 2)^2]$

$= \frac{(40)^{2n}}{4}.$

Note 3: $H\xi(VL)T\prod K_{(1,n)} = [\xi(VL)T\prod K_{(1,n)}]^2$.

Theorem 2.6. *Let $K_{(m,n)}$ be a complete bipartite graph with $(2\le m\le n)$. Then*

$$(i)\ \ \xi(VL)T\prod(K_{(m,n)}) = \frac{(64)^{mn}}{2}.$$

$$(ii)\ H\xi(VL)T\prod(K_{(m,n)}) = \frac{(64)^{2mn}}{4}.$$

Proof. Let $K_{(m,n)}$ be a complete bipartite graph with $(m+n)$ vertices, mn edges and $|V_1| = m$, $|V_2| = n$, $V(K_{(m,n)}) = V_1\bigcup V_2$. Every edge of $K_{(m,n)}$ is incident with exactly two vertices. Every vertex of V_1 is incident with n edges and every vertex of V_2 is incident with m edges. Every point vertices and line vertices have eccentricity 2 in $T(G)$.

(i) $\xi(VL)T\prod(K_{(m,n)}) = \frac{1}{2}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u\cdot e_{T(G)}e]$

$= \frac{1}{2}\prod_{uv\in E(G)}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u\cdot e_{T(G)}e]$

$\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v\cdot e_{T(G)}e]]$

$= \frac{1}{2}\prod_{uv\in E(G)}[(2+2+2\cdot 2)\cdot(2+2+2\cdot 2)]$

$= \frac{(64)^{mn}}{2}.$

and

(ii)$H\xi(VL)T\prod(K_{(m,n)}) = \frac{1}{4}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u\cdot e_{T(G)}e]^2$

$= \frac{1}{4}\prod_{uv\in E(G)}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u\cdot e_{T(G)}e]^2$

$\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v\cdot e_{T(G)}e]^2]$

$= \frac{1}{4}\prod_{uv\in E(G)}[(2+2+2\cdot 2)^2\cdot(2+2+2\cdot 2)^2]$

$= \frac{(64)^{2mn}}{4}.$

Note 4 : We observe that, $H\xi(VL)T\prod K_{(m,n)} = [\xi(VL)T\prod K_{(m,n)}]^2$.

Corollary 2.1. *Let $K_{(n,n)}$ be a complete bipartite graph. Then*

$$(i)\ \ \xi(VL)T\prod(K_{(n,n)}) = \frac{(64)^{n^2}}{2}.$$

$$(ii)\ H\xi(VL)T\prod(K_{(n,n)}) = \frac{(64)^{2n^2}}{4}.$$

Proof. Put $m = n$ in the Theorem 2.6 we get a required result.

Theorem 2.7. *Let F_n be a fan graph. Then*

$$(i)\ \ \xi(VL)T\prod(F_n) = \frac{(88)^n(225)^{(n-1)}}{4}.$$

$$(ii)\ \ H\xi(VL)T\prod(F_n) = \frac{(88)^{2n}(225)^{2(n-1)}}{16}.$$

Proof. Let F_n be a fan graph with $(n+1)$ vertices and $(2n-1)$ edges, then $F_n = K_1 + P_n$. Let F_n be a fan graph with $p(n+1)$ vertices and $p(2n-1)$ edges. Then $F_n = K_1 + P_n$. Let v be the central vertex of F_n, and $v_1, v_2, \ldots, v_n$ be the vertices of P_n. We have n edges of G which are incident with the central vertex and $p(n-1)$ edges on the path.

Let $E_1(G) = \{$set of all edges incident with central vertex v$\}$ and $E_2(G) = \{$set of all edges on with Path $P_n\}$.

For $e_i = vv_i \in E_1(G)$, $e_T(e_i) = 2$ and $e_T(v) = 2$, $e_T(v_i) = 3$. If $e_{ii+1} = v_iv_{i+1} \in E_2(G)$. Then $e_T(e_{ii+1}) = 3$, $e_T(v_i) = 3$.

$$\begin{aligned}
\textbf{(i)}\qquad \xi(VL)T\prod(F_n) &= \tfrac{1}{2}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e] \\
&= \tfrac{1}{2}\prod_{uv\in E(G)}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e] \\
&\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v \cdot e_{T(G)}e]] \\
&= \tfrac{1}{2}\prod_{e\in E_1}[(2+2+2\cdot 2)\cdot(3+2+3\cdot 2)] \\
&\cdot\tfrac{1}{2}\prod_{e\in E_2}[[3+3+3\cdot 3]\cdot(3+3+3\cdot 3)] \\
&= \tfrac{(88)^n(225)^{(n-1)}}{4}.
\end{aligned}$$

and

$$\begin{aligned}
\textbf{(ii)}H\xi(VL)T\prod(F_n) &= \tfrac{1}{4}\prod_{ue}[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]^2 \\
&= \tfrac{1}{4}\prod_{uv}[[e_{T(G)}u + e_{T(G)}e + e_{T(G)}u \cdot e_{T(G)}e]^2 \\
&\cdot[e_{T(G)}v + e_{T(G)}e + e_{T(G)}v \cdot e_{T(G)}e]^2]
\end{aligned}$$

$$= \frac{1}{4}\prod_{uv\in E_1}[(2+2+2\cdot 2)^2\cdot(3+2+3\cdot 2)^2]$$

$$\cdot\frac{1}{4}\prod_{uv\in E_2}[(3+3+3\cdot 3)^2\cdot(3+3+3\cdot 3)^2]$$

$$= \frac{(88)^{2n}(225)^{2(n-1)}}{16}.$$

Note 5: We observe that, $H\xi(VL)T\prod(F_n) = [\xi(VL)T\prod(F_n)]^2$.

Theorem 2.8. *Let K_{2n} be a complete graph with $2n$ vertices and F is a 1-Factor of K_{2n} . Then*

$$(i)\quad \xi(VL)T\prod(K_{2n}-F) = 64n(n-1).$$

$$(ii)\quad H\xi(VL)T\prod(K_{2n}-F) = 2048n(n-1).$$

Proof. Let K_{2n} be a complete graph with $2n$ vertices and $[\frac{2n(2n-1)}{2}] = 2n^2-n$ edges. F is a 1-factor of K_{2n}. $K_{2n}-F$ has $= 2n^2-2n$ edges. Every point vertices and line vertices have eccentricity two in $T(K_{2n}-F)$.

$$\textbf{(i)}\quad \xi(VL)T\prod(K_{2n}-F) = \frac{1}{2}\prod_{ue}[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]$$

$$= \frac{1}{2}\prod_{uv\in E(G)}[[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]$$

$$\cdot[e_{T(G)}v+e_{T(G)}e+e_{T(G)}v\cdot e_{T(G)}e]]$$

$$= \frac{1}{2}\prod_{uv\in E(G)}[(2+2+2\cdot 2)\cdot(2+2+2\cdot 2)]$$

$$= 64n(n-1).$$

and

$$\textbf{(ii)}\quad H\xi(VL)T\prod(K_{2n}-F) = \frac{1}{4}\prod_{ue}[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]^2$$

$$= \frac{1}{4}\prod_{uv}[[e_{T(G)}u+e_{T(G)}e+e_{T(G)}u\cdot e_{T(G)}e]^2$$

$$\cdot[e_{T(G)}v+e_{T(G)}e+e_{T(G)}v\cdot e_{T(G)}e]^2]$$

$$= \frac{1}{4}\prod_{uv}[(2+2+2\cdot 2)^2\cdot(2+2+2\cdot 2)^2]$$

$$= 2048n(n-1).$$

Note 6: We observe that, $H\xi(VL)T\prod(K_{2n}-F) = 32\ \xi(VL)T\prod(K_{2n}-F)$.

3. Conclusions

In this study, we found the results on VL-multiplicative total eccentricity index and VL- multiplicative hyper total eccentricity index on specific standard graphs. The $\xi\ (VL)$ - index might be regarded as a valuable tool in computational domains.

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