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Exact Solutions of Einstein's Vacuum Field Equations

G.S. Dubey, A.K. Mishra, S.M.S. Rizvi, K.P.Mishra and V.K. Yadav
Department of Physics,
G.S.P.G. College Samodhpur Jaunpur (U.P.), India

Abstract: Some new solution of Einstein's vacuum field equations is investigated which is as a simple generalisation of Ozsvath-Schucking solution and explains its source of curvature in terms of some dimensional parameters.
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1. Introduction

The first solution of Einstein field equation was obtained by Schwarzschild and other important solution was investigated by Kerr. These two solutions have played important role for the study of black holes. Friedman solutions are very crucial for cosmology. Thus exact solutions of Einstein field equation have played very important roles in discussing the physical problems. The exact solutions of vacuum field equations are of vital importance.

$$R^{\alpha\beta} = 0 \tag{1}$$

We are trying to obtain a new solution of eq. (1) which provides the source of curvature of the O-S solution (1962). It is always possible to obtain the source of curvature in a vacuum solution in terms of dimensional parameters present in the solution or in its variant. These parameters come in the Riemann tensor indicating source. The presence of a singularity in a solution may be asserted by the divergence of the Kretschmann scalar K

$$K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \tag{2}$$

but the flatness of a solution maynot be asserted by the vanishing of K. For this one has to depend on the vanishing of the Riemann tensor. However this prescription of source in singularity does not work properly as there are solutions of equation (1), such as O-S solution (1962), Taub-NUT (1963) which are curved but singularity free. Therefore, we have to define the presence of source which may be applied properly to all the solutions of equation (1).

2. Some New Exact Vacuum Solution:

In order to obtain a solution of eq. (1), let us assume the angular momentum density J, representing a rotating spacetime such that for J=0, the solution reduces to the Minkowskian form. Let us consider two possible ways to obtain a parameter of the dimensions of space or time, or their inverses, such as

$$\ell = GJ/c^3, \tag{3}$$

$$m = GJ/c^3, \tag{4}$$

having the dimensions of the inverse of space and inverse of time respectively. Let us consider the spacetime rotating about X-axis. The corresponding metric reads

$$\begin{aligned} ds^2 &= (1 + a_0 \times^{b_1})c^2dt^2 - (1 + a_1 \times^{b_1})dx^2 - (1 + a_2 \times^{b_2})dy^2 \\ &-(1 + a_3 \times^{b_3})dx^2 + a_{02} \times^{b_{02}} c\ dt\ dy + a_{03} \times^{b_{03}} c\ dt\ dz \\ &-a_{12} \times^{b_{12}} dx\ dy - a_{23} \times^{b_{23}} dy\ dz - a_{13} \times^{b_{13}} dz\ dx, \end{aligned} \tag{5}$$

where $\chi = \ell x$ and a_i, b_i, a_{ij}, b_{ij} are constants. Let us select the metric (5) such that it reduces to the Minkowskian for $J = \ell = 0$. So, let us put

$$\left.\begin{array}{l} a_1 = a_2 = a_{12} = a_{13} = 0 \\ a_3 = -a_0 = 1/8, \\ a_{02} = a_{23} = 1, \\ a_{03} = \frac{1}{4}, \\ b_0 = b_3 = b_{03} = 2, \\ b_{02} = b_{23} = 1. \end{array}\right\} \tag{6}$$

In view of the above, an exact curved solution of eq. (1), one obtains

$$\begin{aligned} ds^2 &= \left(1 - \frac{\ell^2 x^2}{8}\right) c^2dt^2 - dx^2 - dy^2 - \left(1 + \frac{\ell^2 x^2}{8}\right) dz^2 \\ &+\ell x(c\ dt - dx)dy + \frac{\ell^2 x^2}{4} c\ dt\ dz. \end{aligned} \tag{7}$$

The nonvanishing components of the Christofel symbol read

$$\Gamma^x_{0y} = \Gamma^0_{xy} = \Gamma^x_{xy} = \Gamma^y_{xz} = -\Gamma^y_{0x} = -\Gamma^x_{yz} = \frac{\ell}{4}, \tag{8}$$

and nonvanishing Riemann tensor reads

$$R_{0x0x} = R_{0xxz} = R_{xzxz} = -R_{0y0y} = -R_{0yyz} = -R_{yzyz} = \frac{\ell^2}{16} \tag{9}$$

It is now obvious that the source of curvature be ℓ or J in the new solution given by eq. (7). The K = 0 even for $\ell \neq 0$. The solution (7) is singularity free and the source be here dimensional parameter ℓ.

Let us now show that O-S solution be a particular case of the new solution given by eq. (7). Ozsvath and Schucking (1962) obtained a stationary, geodesically complete ad singularity free solution of eq. (1) as

$$ds^2 = -(dx^1)^2 + 4x^0 dx' \, dx^3 - 2dx^2 dx^3 - 2(x^0)^2 (dx^3)^2 - (dx^0)^2 \tag{10}$$

Here K = 01 and with nonvanishing components of Riemann tensor $R_{3030} = -R_{1313} = 1$.

Let us execute the following coordinate transformation

$$x^1 = \bar{y}, \; x^2 = -c\bar{t} - \bar{z}, \; x^3 = c\bar{t} - \bar{z}, \; x^0 = \bar{x} \tag{11}$$

In view of eq. (11), the eq. (10) assumes the form

$$ds^2 = 2c^2(1 - \bar{x}^2) d\bar{t}^2 - d\bar{x}^2 - d\bar{y}^2 - 2(1 + \bar{x}^2) d\bar{z}^2$$
$$+ 4c\bar{x} d\bar{t} d\bar{y} + 4c\bar{x}^2 d\bar{t} d\bar{z} - 4\bar{x} d\bar{y} d\bar{z}. \tag{12}$$

Let us introduce further transformation as

$$\sqrt{2}\bar{t} = t, \; \sqrt{2}\bar{z} = z, \; \bar{x} = x, \; \bar{y} = y \tag{13}$$

In view of eq. (13), the eq. (12) assumes the form

$$ds^2 = (1 - x^2)c^2 dt^2 - dx^2 - dy^2 - (1 + x^2) dz^2 + 2\sqrt{2}x(cdt - dz)$$
$$dy + 2x^2 c \; dt \; dz, \tag{14}$$

which is a particular case of the new solution given by eq. (7) for $\ell = 2\sqrt{2}$. This also shows that O-S solution may be taken as a rotating spacetime.

3. Concluding Remarks:

We have obtained the source of curvature in vacuum solution showing that it is always possible to identify the source of curvature with some dimensional parameters. As the parameter vanish, so does the curvature. The new solution explains the source of curvature in O-S solution.

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