J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 12, No. 1 (2024), pp. 121-126

DOI: 10.56827/JRSMMS.2024.1201.10 **ISSN (Online): 2582-5461**

ISSN (Print): 2319-1023

A NOTE ON UNIQUENESS OF UNIFORM NORM PROPERTY IN THE BEURLING ALGEBRA $L^1(G_1 \times G_2, \omega)$

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(**Received:** *Jul. 29, 2023* **Accepted:** *Dec. 06, 2023* **Published:** *Dec. 30, 2024)*

Abstract: Let G_1 and G_2 be LCA groups with identities being e_1 and e_2, and let ω be a (Borel measurable) weight function on $G_1 \times G_2$. Let $\overline{\omega}(s,t) = \omega(s,e_2)\omega(e_1,t)$ $((s,t) \in G_1 \times G_2)$. Then $\overline{\omega}$ is also a weight function on $G_1 \times G_2$. In this small note, it is proved that the Beurling algebra $L^1(G_1 \times G_2, \omega)$ has unique uniform norm property iff $L^1(G_1 \times G_2, \overline{\omega})$ has the same property. This result is important because the above statement does not hold true for some properties.

Keywords and Phrases: LCA Group, Weight, Beurling Algebra, and Unique Uniform Norm Property (UUNP).

2020 Mathematics Subject Classification: Primary 46J05, Secondary 46J10.

1. Introduction

An algebra norm (not necessarily complete) $|\cdot|$ on an algebra $\mathcal{A}$ is a *uniform norm* if it satisfies the square property $|a^2| = |a|^2$ $(a \in A)$. For example, if $\mathcal{A}$ is semisimple and commutative, then the spectral radius $r_{\mathcal{A}}(\cdot)$ is a uniform norm on $\mathcal{A}$. By the spectral radius formula, we can show that any two equivalent uniform norms on $\mathcal{A}$ are identical. If $\mathcal{A}$ admits at least one uniform norm, then it must be commutative and semisimple. So throughout $\mathcal{A}$ is assumed to be semisimple and commutative. The $\mathcal{A}$ has *unique uniform norm property (UUNP)* if it admits exactly one uniform norm. This property was introduced and studied extensively by Bhatt and Dedania (see [2], [3], [5]). Dabhi and Dedania proved one surprising result that the $\mathcal{A}$ has either exactly one uniform norm (i.e. UUNP) or infinitely many [1]. Bhatt and Dedania asked the following question in [3], which is believed to be open: If $\mathcal{A}$ has UUNP, then does $\mathcal{A}$ have spectral extension property (SEP)?, i.e., every norm $|\cdot|$ on $\mathcal{A}$ satisfies the inequality $r_{\mathcal{A}}(a) \leq |a|$ $(a \in \mathcal{A})$?.

Consider LCA groups $(G_1, +)$ and $(G_2, +)$ with identities e_1 and e_2, respectively. Let ω be a (Borel measurable) weight function on $G_1 \times G_2$ (see [5]). Define

$$\begin{aligned} \omega_1(s) &= \omega(s, e_2) \ (s \in G_1); \\ \omega_2(t) &= \omega(e_1, t) \ (t \in G_2); \\ \overline{\omega}(s, t) &= \omega(s, e_2)\omega(e_1, t) = \omega_1(s)\omega_2(t) \ ((s, t) \in G_1 \times G_2). \end{aligned}$$

Clearly $\omega_1,\ \omega_2,$ and $\overline{\omega}$ are weight functions on $G_1,\ G_2,$ and $G_1 \times G_2$, respectively. Moreover, $\omega \leq \overline{\omega}$ on $G_1 \times G_2$. In general, they need not be equivalent; for example, take $\omega(m, n) = 1 + |m| + |n|$ or $e^m + e^n$ or $e^{|m|} + e^{|n|}$ on $\mathbb{Z}^2$. Since $\overline{\omega}$ is a product of single variable weight functions, it is a much simpler weight than the original two variable weight function ω. Also there are some Banach algebra properties which are not preserved between the Beurling algebras $L^1(G_1 \times G_2, \omega)$ and $L^1(G_1 \times G_2, \overline{\omega})$. However, here we shall prove by simple arguments that the algebra $L^1(G_1 \times G_2, \omega)$ has UUNP iff $L^1(G_1 \times G_2, \overline{\omega})$ has UUNP. Recently the two variable Beurling algebras are attracting attention of mathematicians (see [6], [7], [8], [9], [10]).

2. Main Results

Lemma 2.1. *Let G be an LCA group and ω be a weight function on G. Then there exists a weight function $\widetilde{\omega}$ on G such that $\widetilde{\omega} \geq 1$ on G and $L^1(G, \omega)$ is isometrically algebra isomorphic to $L^1(G, \widetilde{\omega})$.*

Proof. By [Theorem 1, [4]], the algebra $L^1(G, \omega)$ is semisimple. By [Lemma 4.2, [3]], there exists a continuous group homomorphism $\alpha : (G, +) \longrightarrow (\mathbb{C} \setminus \{0\}, \times)$ such that $|\alpha(s)| \leq \omega(s)$ $(s \in G)$. Next define $\widetilde{\omega}(s) = \frac{\omega(s)}{|\alpha(s)|}$ $(s \in G)$. Clearly, $\widetilde{\omega}$ satisfies the hypothesis. Finally define $\varphi : L^1(G, \widetilde{\omega}) \longrightarrow L^1(G, \omega)$ as $\varphi(f) = f\alpha$. Then φ is an isometric, onto, algebra isomorphism.

Due to the above lemma, we will assume that $\omega(s) \geq 1$ $(s \in G)$.

Definition 2.2. [Definition 4.7.4, [11]] *A weight ω on G is non-quasi analytic if*

$$\sum_{n=-\infty}^{\infty} \frac{\ln \omega(ns)}{1 + n^2} < \infty \qquad (s \in G).$$

Theorem 2.3. *Let ω be a weight function on an LCA group G. Then, the following statements are equivalent.*

1. *ω is non-quasi analytic on G;*

2. *$L^1(G,\omega)$ has UUNP.*

Proof. By [4], the algebra $L^1(G,\omega)$ is semisimple. Now the above statement is exactly [Remark, P.182, [5]].

Following are our main results.

Theorem 2.4. *Let G_1 and G_2 be LCA groups and ω be a weight function on $G_1 \times G_2$. Then the following statements are equivalent.*

1. *$L^1(G_1 \times G_2, \omega)$ has UUNP;*

2. *ω is non-quasi analytic on $G_1 \times G_2$;*

3. *$L^1(G_1 \times G_2, \overline{\omega})$ has UUNP;*

4. *$\overline{\omega}$ is non-quasi analytic on $G_1 \times G_2$;*

5. *$L^1(G_1, \omega_1)$ and $L^1(G_2, \omega_2)$ have UUNP;*

6. *ω_1 and ω_2 are non-quasi analytic on G_1 and G_2, respectively.*

Proof. Since $\omega(s,t) \leq \overline{\omega}(s,t) = \omega_1(s)\omega_2(t)$ $((s,t) \in G_1 \times G_2)$, we have

$$\sum_{n=1}^{\infty} \frac{\ln \omega(ns,nt)}{1+n^2} \leq \sum_{n=1}^{\infty} \frac{\ln \overline{\omega}(ns,nt)}{1+n^2} = \sum_{n=1}^{\infty} \frac{\ln \omega_1(ns)}{1+n^2} + \sum_{n=1}^{\infty} \frac{\ln \omega_2(nt)}{1+n^2}$$

for each $(s,t) \in G_1 \times G_2$. Thus it follows that (6) $\iff$ (4) $\implies$ (2). On the other hand, for any $s \in G_1$ and $t \in G_2$, we have

$$\sum_{n=1}^{\infty} \frac{\ln \omega_1(ns)}{1+n^2} = \sum_{n=1}^{\infty} \frac{\ln \omega(ns,ne_2)}{1+n^2} \text{ and } \sum_{n=1}^{\infty} \frac{\ln \omega_2(nt)}{1+n^2} = \sum_{n=1}^{\infty} \frac{\ln \omega(ne_1,nt)}{1+n^2}.$$

Thus (2) $\implies$ (6). The rest follows from Theorem 2.3.

Next result is a bi-variate analogue of [Proposition 4.6, [3]].

Proposition 2.5. *Let H be a subgroup of $\mathbb{Q}^2$ containing $\mathbb{Z}^2$. Let ω be a weight function on H. Then $l^1(H,\omega)$ has UUNP iff $l^1(\mathbb{Z}^2,\omega)$ has UUNP.*

Proof. The necessary condition is obvious by Theorem 2.3. For the sufficient condition, it is enough to show that $\overline{\omega}$ is non-quasi analytic on H due to Theorem 2.4. Let $s = (p_1/q_1, p_2/q_2) \in \mathbb{Q}_+^2$ with $(p_1,q_1) = (p_2,q_2) = 1$. By the Division

Algorithm, for any $n \in \mathbb{N}$, there exist unique $k_1, k_2, i_1, i_2 \in \mathbb{N} \cup \{0\}$ such that $n = k_1 q_1 + i_1, n = k_2 q_2 + i_2$, $0 \leq i_1 < q_1$, and $0 \leq i_2 < q_2$. Now

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\ln \overline{\omega}(ns)}{1+n^2} &= \sum_{n=1}^{\infty} \frac{\ln \overline{\omega}(\frac{np_1}{q_1}, \frac{np_2}{q_2})}{1+n^2} \\
&= \sum_{n=1}^{\infty} \frac{\ln \omega_1(\frac{np_1}{q_1})}{1+n^2} + \sum_{n=1}^{\infty} \frac{\ln \omega_2(\frac{np_2}{q_2})}{1+n^2} \\
&= \sum_{i_1=0}^{q_1-1} \left(\sum_{k_1=0}^{\infty} \frac{\ln \omega_1((k_1 q_1 + i_1)p_1/q_1)}{1+(k_1 q_1 + i_1)^2} \right) \\
&\quad + \sum_{i_2=0}^{q_2-1} \left(\sum_{k_2=0}^{\infty} \frac{\ln \omega_2((k_2 q_2 + i_2)p_2/q_2)}{1+(k_2 q_2 + i_2)^2} \right) \\
&\leq \sum_{i_1=0}^{q_1-1} \left(\sum_{k_1=0}^{\infty} \frac{\ln \omega_1(k_1 p_1) + \ln \omega_2(i_1 p_1/q_1)}{1+(k_1 q_1 + i_1)^2} \right) \\
&\quad + \sum_{i_2=0}^{q_2-1} \left(\sum_{k_2=0}^{\infty} \frac{\ln \omega_2(k_2 p_2) + \ln \omega_2(i_2 p_2/q_2)}{1+(k_2 q_2 + i_2)^2} \right) \\
&= \sum_{i_1=0}^{q_1-1} \sum_{k_1=0}^{\infty} \frac{\ln \omega_1(k_1 p_1)}{1+(k_1 q_1 + i_1)^2} + \sum_{i_1=0}^{q_1-1} \sum_{k_1=0}^{\infty} \frac{\ln \omega_1(i_1 p_1/q_1)}{1+(k_1 q_1 + i_1)^2} \\
&\quad + \sum_{i_2=0}^{q_2-1} \sum_{k_2=0}^{\infty} \frac{\ln \omega_2(k_2 p_2)}{1+(k_2 q_2 + i_2)^2} + \sum_{i_2=0}^{q_2-1} \sum_{k_2=0}^{\infty} \frac{\ln \omega_2(i_2 p_2/q_2)}{1+(k_2 q_2 + i_2)^2} \\
&\leq \sum_{i_1=0}^{q_1-1} \sum_{k_1=0}^{\infty} \frac{\ln \omega_1(k_1 p_1)}{1+k_1^2} + \sum_{i_1=0}^{q_1-1} \ln \omega_1 \left(\frac{i_1 p_1}{q_1} \right) \sum_{k_1=0}^{\infty} \frac{1}{1+k_1^2} \\
&\quad + \sum_{i_2=0}^{q_2-1} \sum_{k_2=0}^{\infty} \frac{\ln \omega_2(k_2 p_2)}{1+k_2^2} + \sum_{i_2=0}^{q_2-1} \ln \omega_2 \left(\frac{i_2 p_2}{q_2} \right) \sum_{k_2=0}^{\infty} \frac{1}{1+k_2^2} \\
&= q_1 \sum_{k_1=0}^{\infty} \frac{\ln \omega(k_1(p_1, 0))}{1+k_1^2} + \left(1 + \frac{\pi^2}{6}\right) \sum_{i_1=0}^{q_1-1} \ln \omega_1 \left(\frac{i_1 p_1}{q_1} \right) \\
&\quad + q_2 \sum_{k_2=0}^{\infty} \frac{\ln \omega(k_2(0, p_2))}{1+k_2^2} + \left(1 + \frac{\pi^2}{6}\right) \sum_{i_2=0}^{q_2-1} \ln \omega_2 \left(\frac{i_2 p_2}{q_2} \right) \\
&< \infty.
\end{aligned}$$

because ω is non-quasi analytic on $\mathbb{Z}^2$ due to assumption. Similarly, we can show that $\sum_{n=1}^{\infty} \frac{\ln \overline{\omega}(ns)}{1+n^2} < \infty$ when $s \in \mathbb{Q}_-^2$ or $\mathbb{Q}_- \times \mathbb{Q}_+$ or $\mathbb{Q}_+ \times \mathbb{Q}_-$. Thus $l^1(H,\omega)$ has UUNP due to Theorem 2.4.

Example 2.6. Here we give examples of weights with different properties.
(1) Let $H = \mathbb{Z} \times (\mathbb{Z} + \lambda\mathbb{Z})$ for some positive irrational number λ. Then $\mathbb{Z}^2 \subset H$ and H is a subgroup of $\mathbb{R}^2$. Define $\omega(m, n + \lambda k) = e^{|k|}$. Then $l^1(\mathbb{Z}^2,\omega)$ has UUNP but $l^1(H,\omega)$ does not have UUNP. Thus Proposition 2.5 fails because H is not a subgroup $\mathbb{Q}^2$.
(2) Let $\omega(s,t) = 1 + |s| + e^{|t|}$ $(s,t \in \mathbb{R})$. Then $L^1(\mathbb{R}^2,\omega)$ does not have UUNP because the weight function $\omega_2(t) = 2 + e^{|t|}$ associated with ω is not a non-quasi-analytic on $\mathbb{R}$.
(3) Let $\omega(s,t) = 1 + |s| + |t|$ $(s,t \in \mathbb{R})$. Then $L^1(\mathbb{R}^2,\omega)$ has UUNP because the weight functions $\omega_1(s) = \omega_2(s) = 2 + |s|$ are non-quasi-analytic on $\mathbb{R}$.
(4) Let $\omega(m,n) = e^{|m|} + e^{|n|}$ $(m,n \in \mathbb{Z})$. Then $\omega_1(m) = \omega_2(m) = 1 + e^{|m|}$. The Gel'fand space of $l^1(\mathbb{Z}^2,\overline{\omega})$ is homeomorphic to the product of two closed annuli $\Gamma(\frac{1}{e}, e)$ in $\mathbb{C}$ with center zero and radii are $\frac{1}{e}$ and e [Theorem 3.7, [9]]. However, by [Example 2, [9]] the Gel'fand space of $l^1(\mathbb{Z}^2,\omega)$ is homeomorphic to $\Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \times \Gamma(\frac{1}{\sqrt{e}}, \sqrt{e}) \cup [\cup_{i=1}^4 (\mathbb{T}_i \cup \mathbb{T}_i^{op})]$, where

$$\begin{aligned}
\mathbb{T}_1 &= \{(z,w) : 1 \le |z| \le \sqrt{e} \le |w| \le e \text{ and } |z||w| \le e\} \\
\mathbb{T}_2 &= \{(z,w) : \frac{1}{\sqrt{e}} \le |z| \le 1, \sqrt{e} \le |w| \le e \text{ and } |z|^{-1}|w| \le e\}, \\
\mathbb{T}_3 &= \{(z,w) : \frac{1}{e} \le |z| \le \frac{1}{\sqrt{e}}, 1 \le |w| \le \sqrt{e} \text{ and } |z|^{-1}|w| \le e\}, \\
\mathbb{T}_4 &= \{(z,w) : \frac{1}{e} \le |z| \le \frac{1}{\sqrt{e}} \le |w| \le 1 \text{ and } |z||w| \ge \frac{1}{e}\}, \\
\mathbb{T}_i^{op} &= \{(w,z) : (z,w) \in \mathbb{T}_i\} \ (i = 1,2,3,4).
\end{aligned}$$

Remark 2.7. *It follows from Theorem 4.1 and the diagram on Page-212 in [3] that the UUNP, SEP, Regularity, and Weak regularity are equivalent in the Beurling algebra $L^1(G,\omega)$. Hence these properties also hold in the algebras $L^1(G_1 \times G_2,\omega)$ and $L^1(G_1 \times G_2,\overline{\omega})$ simultaneously.*

Acknowledgement

The corresponding author is thankful to CSIR-HRDG, New Delhi, for providing Senior Research Fellowship.

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