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ON SPECIAL FUNCTION CHANGE OF m^{th}- ROOT FINSLER METRIC

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Abstract: In the present paper, we consider the special function change of $m^{th}-$ root Finsler metric. Firstly, we find the fundamental metric and then the necessary and sufficient condition under which the special function change of the $m^{th}-$ root Finsler metric is locally dually flat. Further, we prove that the special change of $m^{th}-$ root Finsler metric is locally projectively flat if and only if it is locally Minkowskian.

Keywords and Phrases: Locally dually flat metric, projectively flat metric, $m^{th}-$ root metric, Special function change, fundamental metric tensor.

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1. Introduction

Let M be an n-dimensional C^∞ manifold, TM its tangent bundle. Let $F = A^{\frac{1}{M}}$ be a Finsler metric on M, where A is given by $A = a_{i_1\, i_2 \ldots\ldots i_m}(x)\;\; y^{i_1}\, y^{i_2} \ldots\ldots y^{i_m}$ with $a_{i_1\, i_2 \ldots\ldots i_m}$ symmetric in all its indices ([13], [21], [4], [8]. Then F is called an $m^{th}-$ root Finsler metric. Suppose that A_{ij} is a positive definite tensor and A^{ij} denotes its inverse. For an $m^{th}-$ root Finsler metric [15], [17], put

$$A_i = \frac{\partial A}{\partial y^i}, \quad A_{ij} = \frac{\partial^2 A}{\partial y^i \partial y^j}, \quad A_{x^i} = \frac{\partial A}{\partial x^i}, \quad A_0 = A_{x^i} y^i. \tag{1}$$

Then the following hold :

$$l_i = \frac{1}{m} A_i F^{1-m}, \tag{2}$$

$$g_{ij} = \frac{A^{\frac{2-2m}{m}}}{m^2}[mAA_{ij} + (2-m)A_i A_j], \tag{3}$$

$$y^i A_i = mA, y^i A_{ij} = (m-1)A_j, y_i = \frac{1}{m} A^{\frac{2-m}{m}} A_i, \tag{4}$$

$$A^{ij} A_{jk} = \delta^i_k, A^{ij} A_i = \frac{1}{m-1} y^j, A_i A_j A^{ij} = \frac{m}{m-1} A. \tag{5}$$

Let (M, F) be a Finsler manifold [11]. For a 1- form $\beta(x, y) = b_i(x)y^i$ on M, we have a change of Finsler metric, defined by

$$F(x, y) \to F^*(x, y) = f(F, \beta), \tag{6}$$

where $f(F, \beta)$ is a positively homogenous function of F and β. This is called a $\beta-$ change of a metric. It is easy to see that , if $\|\beta\|_F = sup_{F(x,y)=1}|b_i(x)y^i| < 1$, then F^* is again a Finsler metric [9].

In this paper, we consider the recently new introduced metric in which special case of $\beta-$ change , namely,

$$F^*(x, y) = \sqrt{F\beta}, \tag{7}$$

which we call the Special function change. Gauree Shanker and Sruthy Asha Baby [19] has introduced the Kropina - Randers change of $m^{th}-$ root Finsler metric [18] in which the Randers and Kropina metrics are closely related to physical theories. [20]

The main purpose of the current paper is to investigate the special function change of an $m^{th}-$ root Finsler metric $F^* = A^{\frac{1}{2m}}\beta^{\frac{1}{2}}$. The paper is organised as follows:

First we find the fundamental metric tensor and its inverse for special function change of a Finsler space with $m^{th}-$ root metric (see Propositions 2.1 and 2.2). Next we prove that the special change of an $m^{th}-$ root Finsler metric is Locally projectively flat if and only if it is locally Minkowskian (see Proposition 3.1 and Theorem 3.2). Finally, we find the necessary and sufficient condition under which the special function change of an $m^{th}-$ root Finsler metric to be locally dually flat.

2. Fundamental Metric Tensor of Special Function Change of $m^{th}-$ root Finsler Metric

We consider the special function change of the $m^{th}-$ root metric [14] $F = A^{\frac{1}{m}}$ given by

$$F^* = \sqrt{F\beta} \tag{8}$$

The differentiation (8) with respect to y^i yields the normalized supporting element l_i given by

$$l_i^* = \frac{1}{2\sqrt{F\beta}}(Fb_i + \beta l_i)$$

In view of (2), we have

$$l_i^* = \frac{1}{2}\frac{A^{1/2m}}{\sqrt{\beta}}b_i + \frac{1}{2m}\sqrt{\beta}A^{\frac{1-2m}{2m}}A_i. \tag{9}$$

Differentiation of (9) with respect to y^j yields

$$\begin{aligned} l_{ij}^* = \frac{\sqrt{\beta}}{2m}[F^{\frac{1}{2}-m}A_{ij} + (\frac{1-2m}{2m})F^{\frac{1}{2}-2m}A_iA_j \\ +\frac{1}{2\beta}F^{\frac{1}{2}-m}(b_iA_j + b_jA_i) - \frac{m}{2\beta^2}F^{\frac{1}{2}}b_ib_j]. \end{aligned} \tag{10}$$

From (9) and (10), we obtain

$$h_{ij}^* = F^*l_{ij}^*$$

$$= \nu_1 A_{ij} + \nu_2 A_iA_j + \nu_3(A_jb_i + A_ib_j) + \nu_4 b_ib_j, \tag{11}$$

where

$$\nu_1 = \frac{\beta}{2m}F^{1-m}$$

$$\nu_2 = \frac{\beta(1-2m)}{4m^2}F^{1-2m}$$

$$\nu_3 = \frac{1}{4m} F^{1-m}$$
$$\nu_4 = -\frac{1}{4\beta} F$$

From (9) and (11), the fundamental metric tensor g^*_{ij} of Finsler space (M, F^*) is given by

$$g^*_{ij} = h^*_{ij} + l^*_i l^*_j$$

$$g^*_{ij} = \sigma_0 A_{ij} + \sigma_1 (b_i A_j + b_j A_i) + \sigma_2 A_i A_j, \tag{12}$$

where

$$\begin{aligned} \sigma_0 &= \frac{F^{2-m}}{m\beta}(\frac{F^{-1}\beta^2}{2}) \\ \sigma_1 &= \frac{1}{2m} F^{1-m} \\ \sigma_2 &= \frac{F^{\frac{1-2m}{2}}}{4m^2\beta}(\beta^2 + (1-2m)\beta^2 F^{\frac{1-2m}{2}}, \end{aligned} \tag{13}$$

Proposition 1. *For the special function transformed $m^{th}-$ root Finsler metric F^*, the fundamental tensor g^*_{ij} is given by equations (12) and (13).* [17]

3. Locally Projectively Flat Metric

A Finsler metric $F(x, y)$ on an open domain $U \subset R^n$ is said to be *locally projectively flat metric* if its geodesic coefficients G^i are in the following form:

$$G^i(x, y) = P(x, y)y^i, \tag{14}$$

where $P : TU = U \times R^n \to R$ is positively homogeneous with degree one , $P(x, \lambda y) = \lambda P(x, y), \lambda > 0$. We call $P(x, y)$ the projective factor of F.

Definition 1. *A Finsler metric $F = F(x, y)$ on an open subset $U \subset R^n$ is projectively flat [16] if and only if*

$$F_{x^k y^l} y^k - F_{x^l} = 0. \tag{15}$$

Now, we have the following lemma:

Lemma 1. *Suppose that the equation $\phi A^2 + \psi A + \Theta = 0$ holds, where ϕ, ψ, Θ are polynomials in y and $m > 2$. Then $\phi = \psi = \Theta = 0$.* [16]

Proposition 2. *Let $F = A^{\frac{1}{m}}$ be an $m^{th}-$ root Finsler metric on an open subset*

$U \subset R^n (n \geq 3)$, where A is irreducible. Suppose that $F^* = \sqrt{F\beta}$ is a special function change of F, where $\beta = b_i(x)y^i$. If F^* is projectively flat metric, then it reduces to a Berwald metric. [13]

Proof. Let $F^* = \sqrt{F\beta}$ be projectively flat metric. We have

$$F^*_{x^k} = \frac{1}{2m}\sqrt{\beta}A^{\frac{1-2m}{2m}}A_{x^k} + \frac{1}{2\sqrt{\beta}}A^{\frac{1}{2m}}\beta_{x^k},$$

$$\begin{aligned}F^*_{x^k y^l}y^k = \frac{\sqrt{\beta}}{2m}[A^{\frac{1-2m}{2m}}A_{0l} + (\frac{1-2m}{2m})A^{\frac{1-4m}{2m}}A_0A_l] \\ +\frac{1}{2\sqrt{\beta}}[\frac{1}{2m}A^{\frac{1-2m}{2m}}A_0\beta_l + A^{\frac{1}{2m}}\beta_{0l} + (\frac{1}{2m})A^{\frac{1-2m}{2m}}\beta_0A_l] \\ -\frac{1}{4\beta\sqrt{\beta}}A^{\frac{1}{2m}}\beta_0\beta_l,\end{aligned}$$

From (15), we get

$$\begin{aligned}\frac{1}{2m}\sqrt{\beta}A^{\frac{1-2m}{2m}}A_{x^l} + \frac{1}{2\sqrt{\beta}}A^{\frac{1}{2m}}\beta_{x^l} \\ -\frac{\sqrt{\beta}}{2m}[A^{\frac{1-2m}{2m}}A_{0l} + (\frac{1-2m}{2m})A^{\frac{1-4m}{2m}}A_0A_l] \\ -\frac{1}{2\sqrt{\beta}}[\frac{1}{2m}A^{\frac{1-2m}{2m}}A_0\beta_l + A^{\frac{1}{2m}}\beta_{0l} + (\frac{1}{2m})A^{\frac{1-2m}{2m}}\beta_0A_l] \\ +\frac{1}{4\beta\sqrt{\beta}}A^{\frac{1}{2m}}\beta_0\beta_l = 0\end{aligned}$$

which implies

$$\begin{aligned}\sqrt{\beta}A^{\frac{1-2m}{2m}}[\frac{1}{2m}(A_{x^l} - A_{0l} - (\frac{1-2m}{2m})A^{-1}A_0A_l) \\ -\frac{1}{4m\beta}(A_0\beta_l + \beta_0A_l) + \frac{1}{4\beta^2}A^{2m}(\beta_0\beta_l + 2\beta\beta_{x^l} - 2\beta\beta_{0l})] = 0\end{aligned}$$

Multiplying by A , we get

$$\begin{aligned}\sqrt{\beta}A^{\frac{1}{2m}}[\frac{1}{2m}(A_{x^l} - A_{0l} - (\frac{1-2m}{2m})A^{-1}A_0A_l) \\ -\frac{1}{4m\beta}(A_0\beta_l + \beta_0A_l) + \frac{1}{4\beta^2}A^{2m}(\beta_0\beta_l + 2\beta\beta_{x^l} - 2\beta\beta_{0l})] = 0\end{aligned}$$

which implies

$$[\frac{1}{2m}(A_{x^l} - A_{0l} - (\frac{1-2m}{2m})A^{-1}A_0A_l)$$
$$-\frac{1}{4m\beta}(A_0\beta_l + \beta_0 A_l) + \frac{1}{4\beta^2}A^{2m}(\beta_0\beta_l + 2\beta\beta_{x^l} - 2\beta\beta_{0l})] = 0$$

By Lemma 3.1, we have

$$2mA(A_{0l} - A_{x^l}) - (2m-1)A_0A_l = 0. \quad (16)$$

Then, irreducibility of A and $deg(A_l) = m - 1 < deg(A)$ implies that A_0 is divisible by A. This means that , there is a 1- form $\theta = \theta_i y^i$ on U such that the following holds:

$$A_0 = 2mA\theta. \quad (17)$$

Then $G^i = Py^i$, where $P = \theta$. Then F is a Berwald metric.

Theorem 1. *Let $F = A^{\frac{1}{m}}$ be an $m^{th}-$ root Finsler metric on an open subset $U \subset R^n(n \geq 3)$, where A is irreducible. suppose that $F^* = \sqrt{F\beta}$ is a special function change of F, where $\beta = b_i(x)y^i$. Then F^* is locally projectively flat if and only if it is locally Minkowskian.* [10]

Proof. By Proposition 3.1, if F is projectively flat, then it reduces to a Berwald metric. [9]

Now, if $n \neq 3$, then by Numata's theorem, every Berwald metric of non-zero scalar flag curvature K must be Riemannian. This contradicts with our assumption. Then $K = 0$, and in this case F reduces to a locally Minkowskian metric.

4. Locally Dually Flat Metric

The notion of dually flat Riemannian metrics was introduced by Amari and Nagaoka [2], [15] when they studied the information geometry on Riemannian manifold. In Finsler geometry, shen extended the notion of locally dually flatness for Finsler metrics. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry , which plays a very important role in studying flat Finsler information structure. Information geometry has been imerged from investigating the geometrical structure of the family of probability distributions.

Definition 2. *A Finsler metric $F = F(x, y)$ on a manifold M^n is said to be locally dually flat, if at any point there is a standard coordinate system (x^i, y^i) in TM such that*

$$[F^2]_{x^k y^l} y^k = 2[F^2]_{x^l}. \quad (18)$$

Theorem 2. *Let $F = A^{\frac{1}{m}}$ be an $m^{th}-$ root Finsler metric on an open subset $U \subset R^n$, where A is irreducible. suppose that $F^* = \sqrt{F\beta}$ is a special function change of F, where $\beta = b_i(x)y^i$. Then F^* is locally dually flat if and only if there exists a 1- form $\theta = \theta_i(x)y^i$ on U such that the following holds:*
where $\beta_{0l} = \beta_{x^k y^l}y^k$, $\beta_{x^l} = (b_i)_{x^i}y^i$, $\beta_0 = \beta_{x^i}y^i$ and $\beta_{0l} = (b_l)_0)$. [14]

Proof. Let $F^* = \sqrt{F\beta}$ be a locally dually flat Finsler metric. We have

$$F^{*2} = A^{\frac{1}{m}}\beta$$

$$[F^{*2}]_{x^k} = A^{\frac{1}{m}}\beta_{x^k} + \frac{\beta}{m}A^{\frac{1-m}{m}}A_{x^k},$$

$$[F^{*2}]_{x^k y^l}y^k = \beta[\frac{1}{m}A^{\frac{1-m}{m}}A_{0l} + \frac{1}{m}(\frac{1-m}{m})A^{\frac{1-2m}{m}}A_0A_l] + [\frac{1}{m}A^{\frac{1-m}{m}}A_0\beta_l + \frac{1}{m}A^{\frac{1-m}{m}}\beta_0A_l] + A^{\frac{1}{m}}\beta_{0l}$$

From (18), we get

$$2[A^{\frac{1}{m}}\beta_{x^l} + \frac{\beta}{m}A^{\frac{1-m}{m}}A_{x^l}] = \beta[\frac{1}{m}A^{\frac{1-m}{m}}A_{0l} + \frac{1}{m}(\frac{1-m}{m})A^{\frac{1-2m}{m}}A_0A_l] + [\frac{1}{m}A^{\frac{1-m}{m}}A_0\beta_l + \frac{1}{m}A^{\frac{1-m}{m}}\beta_0A_l] + A^{\frac{1}{m}}\beta_{0l}, \tag{19}$$

Multiplying (19) by $A^{\frac{1+2m}{m}}$, we get

$$A^{\frac{2+m}{m}}[\frac{2\beta}{m}A_{x^l} - \frac{\beta}{m}A_{0l} - \frac{1}{m}A_0\beta_l - \frac{1}{m}\beta_0A_l] - \frac{\beta}{m}(\frac{1-m}{m})A^{\frac{2}{m}}A_0A_l + A^{\frac{2+2m}{m}}[2\beta_{x^l} - \beta_{0l}] = 0. \tag{20}$$

Simplifying (20), we get

$$\frac{\beta A^{\frac{2+m}{m}}}{m}[2A_{x^l} - A_{0l} - \frac{1}{\beta}A_0\beta_l - \frac{1}{\beta}\beta_0A_l] - \frac{\beta}{m}(\frac{1-m}{m})A^{\frac{2}{m}}A_0A_l + A^{\frac{2+2m}{m}}[2\beta_{x^l} - \beta_{0l}] = 0. \tag{21}$$

By Lemma 3.1 , we have

$$2\beta_{x^l} = \beta_{0l}, \quad 2A_{x^l} = A_{0l} + \frac{1}{\beta}A_0\beta_l + \frac{1}{\beta}\beta_0A_l. \tag{22}$$

which implies

$$\begin{aligned} 2\beta_{x^l} &= \beta_{0l}, \\ 2\beta A_{x^l} &= \beta A_{0l} + A_0\beta_l + \beta_0 A_l. \end{aligned} \tag{23}$$

Equation (23) implies that there exists a 1- form $\theta = \theta_i(x)y^i$ on U such that

$$\beta_0 = \theta\beta. \tag{24}$$

From (24), we get

$$\beta_{0l} = \theta\beta_l + \beta\theta_l - \beta_{x^l}. \tag{25}$$

Substituting (25) and (24) into (22) yields

$$3\beta_{x^l} = \theta\beta_l + \beta\theta_l. \tag{26}$$

The converse is a direct computation. This completes the proof.

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