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PRIMARY IDEALS IN $\Gamma-$SEMIRINGS

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Abstract: From an algebraic point of view, $\Gamma-$ semirings provide the most natural generalization of the theory of semirings. In this paper, we summarize the semiring theoretic results concerning the primary ideals and their radicals to non-commutative $\Gamma-$ semirings.

Keywords and Phrases: Noetherian $\Gamma-$semirings, prime ideal, semiprime ideal and primary ideal.

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1. Introduction

As a generalization of $\Gamma-$rings, the idea of $\Gamma-$ semiring was presented by Rao [7]. Later it was discovered that $\Gamma-$ semiring additionally gives an algebraic home to the set of rectangular matrices over a semiring. Dutta and Sardar [2] presented the thought of operator semiring of a $\Gamma-$ semiring in 2002 and by utilizing the connection between the operator semiring and the $\Gamma-$ semiring, they enriched the theory of $\Gamma-$ semiring and demonstrated the outcomes regarding prime ideals and prime radicals of a $\Gamma-$ semiring via its operator semirings which incorporates various characterizations of prime ideals and prime radicals.

The motivation of this paper is [8] where Sharma et.al received a substitute way to generalize primary ideals from commutative semirings to non-commutative semirings by replacing the role of elements with ideals. In this paper, we define primary ideals in terms of their ideals and prove some fundamental outcomes concerning the primary ideals and their radicals for noncommutative $\Gamma-$ semirings by making suitable changes in some of the results of [8].

2. Preliminaries

Recall from [7] that if $(R, +)$ and $(\Gamma, +)$ be two commutative semigroups then R is called a $\Gamma-$ semiring if there exists a mapping $R \times \Gamma \times R \to R$ denoted by $x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$ satisfying
(i) $x\alpha(y + z) = x\alpha y + x\alpha z$.
(ii) $(y + z)\alpha x = y\alpha x + z\alpha x$.
(iii) $x(\alpha + \beta)z = x\alpha z + x\beta z$.
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$.
Let A and B be semirings and let $R = Hom(A, B)$ and $\Gamma = Hom(B, A)$ denote the sets of homomorphisms from A to B and B to A respectively. Then R is a $\Gamma-$ semiring with operations of pointwise addition and composition of mappings. Further, let M be a $\Gamma-$ ring and R be the set of ideals of M. Define addition in the natural way and if $A, B \in R$, $\gamma \in \Gamma$, let $A\gamma B$ denote the ideal generated by $\{x\gamma y | x, y \in M\}$. Then R is a $\Gamma-$ semiring.

A $\Gamma-$ semiring R is said to be commutative if $x\gamma y = y\gamma x$ for all $x, y \in R$ and for all $\gamma \in \Gamma$. A $\Gamma-$ semiring R is said to have a zero element if $0\gamma x = 0 = x\gamma 0$ and $x + 0 = x = 0 + x$ for all $x \in R$ and $\gamma \in \Gamma$. R is said to have an identity element if there exists $\gamma \in \Gamma$ such that $1\gamma x = x = x\gamma 1$ for all $x \in R$. $\phi \neq I \subseteq R$ is said to be left (right) ideal of R if I is subsemigroup of $(R, +)$, $x\gamma y \in I$ $(y\gamma x \in I)$ for all $x \in R$, $y \in I$ and $\gamma \in \Gamma$. If I is both the left and right ideal of R then I is an ideal of R. $M \subset R$ is said to be the maximal ideal if there does not exist any other proper ideal of R containing M properly. An ideal P of R is $k-$ideal if $y \in P$, $x + y \in P$, $x \in R$ implies that $x \in P$. Let $H \neq \phi$. Then H is an $m-$system of R if $c\alpha_1 r\alpha_2 d \in H$, for any $c, d \in H$, $r \in R$ and $\alpha_1, \alpha_2 \in \Gamma$.

All through here, R will signify with '0' and '1' as zero and identity element except if in any case expressed.

3. Primary ideals in a non-commutative $\Gamma-$ semiring

Here, we introduce the idea of primary ideals in a $\Gamma-$semiring R which are characterized similarly as that of the prime just by replacing the role of elements with ideals. This methodology empowers us to demonstrate some basic results concerning the primary ideals and their radicals for a $\Gamma-$ semiring R.

Definition 3.1. *An ideal P of a $\Gamma-$ semiring R is said to be prime ideal if $A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$, for any two ideals A and B of R.*

Definition 3.2. *An ideal* P *of a* $\Gamma-$ *semiring* R *is said to be semiprime if* $A\Gamma A \subseteq P$ *implies that* $A \subseteq P$*, for any ideal* A *of* R.

Definition 3.3. *For any ideal* P *of a* $\Gamma-$ *semiring* R*, the prime radical of* P *is defined as the intersection of all prime ideals of* R *containing* P*. It is denoted by* $r(P)$.

In [1], the authors considered a semiring R with 1 and for any ideal I of R define $r(I) = \{a \in R \mid a^n \in I$ for some positive integer $n\}$ and call it radical of I. The radical so defined has a nice property that if I is subtractive then $r(I)$ is subtractive([1], Proposition 2.19). But because of non-commutativity, we follow J.S. Golan [5] and have the following definition.

Definition 3.4. *For an ideal* A *of a* $\Gamma-$*semiring* R*, the radical of* A *is defined as* $r(A) = \{s \in R \mid$ *every* $m-$ *system containing* s *meets* A $\} \subseteq$ *radical* A*, where radical* A *is defined in [1]. In the special case when* R *is commutative* $\Gamma-$ *semiring, the inclusion "* $\subseteq$ *" above is equality.*

The following theorem is proved in [3], using "via operator semirings of a $\Gamma-$ semiring". As a consequence, we have prove the following result in terms of prime radicals for $\Gamma-$ semirings.

Theorem 3.5. *Let* A *and* B *be two ideals of a* $\Gamma-$ *semiring* R*. Then*
(i) if $A \subseteq B$ *then* $r(A) \subseteq r(B)$.
(ii) $r(r(A)) = A$.
(iii) $r(A + B) = r(r(A) + r(B))$*. (iv)* $r(A \cap B) = r(A) \cap r(B) = r(A\Gamma B)$.
Proof. (i) Let $r(A) = \bigcap_i\{P_i \mid P_i$ is prime, $A \subseteq P_i\}$ and $r(B) = \bigcap_i\{P'_i \mid P'_i$ is prime and $B \subseteq P'_i\}$, where P_i and P'_i are prime ideals of R. Now, $r(A) = \bigcap_i\{P_i \mid P_i$ is prime, $A \subseteq P_i\} \subseteq \bigcap_i\{P_i \mid P_i$ is prime and $B \subseteq P_i\} = r(B)$, since $A \subseteq B$.

The proof of (ii) and (iii) are obvious.
(iv) Since $A\cap B \subseteq A$ and $A\cap B \subseteq B$, by (i) we have $r(A\cap B) \subseteq r(A)\cap r(B)$. Further, $r(A) \cap r(B) = \left\{\bigcap_i\{P_i \mid P_i\ ,\ A \subseteq P_i\}\right\} \bigcap \left\{\bigcap_i\{Q_i \mid Q_i\ ,\ B \subseteq Q_i\}\right\} = \bigcap_i\{T_i \mid T_i$ either $A \subseteq T_i$ or $B \subseteq T_i\}$. Let P be any prime such that $P \supseteq A \cap B$. Then surely $A\Gamma B \subseteq A\cap B \subseteq P$ proves the reverse inclusion. Again, since $A\Gamma B \subseteq A\cap B$ implies that $r(A\Gamma B) \subseteq r(A\cap B) \subseteq r(A)\cap r(B)$. Furthermore, the primeness of P implies that $r(A) \cap r(B) \subseteq r(A\Gamma B)$, which completes the result.

The accompanying result, which is taken as a definition of the prime radical in [4] for weak primary decomposition of right Noetherian right k -$\Gamma-$ semiring, is likewise referenced in [5], however for completeness we state the following results, the proofs of which are easy and straight forward.

Theorem 3.6. *Let* A *be an ideal of a* $\Gamma-$ *semiring* R*. Then* $r(A)$ *equals the* *intersection of all the prime ideals of R containing A.*

Corollary 3.7. *The prime radical of an ideal A of R is an ideal.*

A primary ideal in the case of commutative rings, commutative semirings, and semigroups is defined in terms of elements. For example, in instance of semigroup S, let I be any ideal of S, then redical of I is $rad(I) = \{s \in S \mid ns \in I, n \in Z^+\}$. Additionally an appropriate ideal P of S is primary if $a + b \in P$ with $a, b \in S$ and $a \notin P$ implies that $b \in rad(P)$. For more details on primary ideals and radicals of semigroups, one can see [6]. But here we follow [8] and define the primary ideal of a $\Gamma-$ semiring (not necessarily commutative) regarding its ideals.

Definition 3.8. *An ideal P of R is primary if for any two ideals A and B of R, $A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq r(P)$.*

Every prime ideal are primary ideal. We also recall.

Definition 3.9. *Let $M(\neq \phi)$ and S be two sets such that $M \subseteq S \subseteq R$. Then S is an $m-$system with respect to M if $a \in S, b \in M$, $r \in R$ we have $a\gamma r\beta b \in S$, $\beta, \gamma \in \Gamma$.*

Definition 3.10. *Let R be a $\Gamma-$ semiring and I be any ideal of R. Define $r(I) = \{x \in R \mid (x\alpha)^{n-1}x \in I$ for all $\alpha \in \Gamma$ and for some positive integer $n\}$ and call it radical of I.*

The radical so defined in a $\Gamma-$ semiring has also a nice property as in semirings that if I is a $k-$ ideal then $r(I)$ is also a $k-$ ideal. We now state the following three results referred from [3].

Proposition 3.11. [3] *Let S be $\Gamma-$ semiring and P be a proper ideal of S. If $x \in r(P)$ then $(x\gamma)^{n-1}x \in P$, for all $\gamma \in \Gamma$ and for some positive integer n, $((x\gamma)^0 x = x)$.*

Theorem 3.12. [3] *For a commutative $\Gamma-$ semiring S, $r(Q) = \{s \in S : (s\gamma)^{n-1}s \in Q$ for some positive integer n and for all $\gamma \in \Gamma\}$, where Q is a proper ideal of S.*

Theorem 3.13. [3] *For a proper ideal Q of a $\Gamma-$ semiring S, $r(Q) = \{s \in S \mid$ every m-system in S which contains s has a nonempty intersection with $Q\}$.*
Now, it is easy to verify the following semiring theoretic results for a $\Gamma-$ semiring.

Theorem 3.14. *For an ideal $P \subsetneq R$ the following statements are equivalent:*
(i) P is primary.
(ii) If $a, b \in R, (a)\Gamma(b) \subseteq P$ then either $a \in P$ or $b \in r(P)$, where $(a) = R\Gamma a\Gamma R$, $(b) = R\Gamma b\Gamma R$.
(iii) If $a\Gamma R\Gamma b \subseteq P$ then $a \in P$ or $b \in r(P)$.
(iv) P^c is an m-system with respect to $r(P)^c$, where $P^c = R\backslash P$ and $r(P)^c = R\backslash r(P$.

Theorem 3.15. *Let $A \subseteq R$. Then the following statements are equivalent:*
(i) A is semiprime.
(ii) $A = \bigcap_i \{P_i \mid P_i$ is prime, $A \subseteq P_i\}$.
(iii) $A = r(A)$.
Proof. The result follows from [c.f. [9], Theorem 3.20] and Theorem 3.5.
The following example from [8] is also held for Γ− semirings.

Example 3.16. [8] Let $R = \{\mathbb{N} \cup \{\infty\}, \oplus, \odot\}$, where $x \oplus y = max(x, y)$ and $x \odot y = min(x, y)$. Then R is a Γ− semiring with identity ∞. Let $A_t = \{0, 1, 2......, t\}$. Then A_t is semiprime. Thus, by Theorem 3.15, that $A_t = r(A_t$.

Corollary 3.17. *For any ideal $A \subseteq R$, $r(A)$ is the smallest semiprime ideal of R which contains A.*
Proof. It follows by using Theorem 3.5((i), (ii)) and Theorem 3.15.

Theorem 3.18. *Let R be a Γ−semiring and A, B be two ideals of R. Then B is prime radical of A and A is primary if and only if*
(i) $A \subseteq B$ and B is semiprime.
(ii) If $b \in B$, then every m-system containing b intersect A.
(iii) Let $a, b \in R, a\Gamma R\Gamma b \subseteq A$, then $a \in A$ or $b \in B$.
Proof. Suppose that all given conditions are satisfied. Let $a \notin A$ and $b \notin r(A)$, then by definition 3.3, there must be an m−system S containing b such that S does not intersect A. Therefore, by (*ii*), $b \notin B$ and hence by (*iii*), $a\Gamma R\Gamma b \nsubseteq A$. Thus, A is primary. Using Corollary 3.17, $B \subseteq r(A)$. Moreover, the semiprime character of B together with theorem 3.4(i) gives that $r(A) \subseteq r(B) = B$.
Converse follows from Corollary 3.17, Definition 3.3. and Theorem 3.14.

Theorem 3.19. *Let A, B be ideals of a commutative Γ− semiring R for which*
(i) $A \subseteq B$.
(ii) If $b \in B$ then every m-system containing b intersect A.
(iii) B is maximal.
Then B is a prime radical of A and A is primary.
Proof. The maximality of B implies that B is prime (c.f. [9], Corollary 3.18), and hence B is semiprime. Let $a\Gamma R\Gamma b \subseteq A$ and $b \notin B$. Consider the ideal $R\Gamma b\Gamma R$ generated by b. Since $b \in R\Gamma b\Gamma R$ and $b \notin B$, $B + R\Gamma b\Gamma R \supsetneq B$. Hence, by maximality of B, $R = B + R\Gamma b\Gamma R$. Therefore, $1 = c + \sum_i r_i\gamma_i b\beta_i r'_i$, $r'_i \in R, c \in B, \gamma_i, \beta_i \in \Gamma$. This implies that $1 = c + l \in R\Gamma b\Gamma R, l = \sum_i r_i\gamma_i b\beta_i r'_i$. Since $c \in r(A)$, $(c\gamma)^{n-1}c \in A$ for some $n \geqslant 1$. Raising the above equality to the n^{th} power, $1 = (c\gamma)^{n-1}c + (l\beta)^{n-1}l + X(l)$, where $X(l)$ is the sum of the terms containing b and $(l\beta)^{n-1}l$, $X(l) \in R\Gamma b\Gamma R$ as $R\Gamma b\Gamma R$ is an ideal of R. Now,

$a = a\gamma 1 = a\gamma(c\gamma)^{n-1}c + a\gamma(l\beta)^{n-1}l + a\gamma X(l)$, where $a\gamma(c\gamma)^{n-1}c \in A$, $a\gamma(l\beta)^{n-1}l$ and $a\gamma X(l) \in R\Gamma b\Gamma R$. But $a\Gamma R\Gamma b \subseteq A$, So, by Theorem 3.18, $a \in A$.

Theorem 3.20. ***(i)*** *Let $P_1, P_2, ..., P_n$ be primary ideals of R and $r(P_i) = B(i = 1, 2, ..., n)$. Then*

(a) $P = \cap_i P_i$ is primary and $r(P) = B$.

(b) Let $P = P_1\Gamma P_2\Gamma ...\Gamma P_n$. If B is maximal then $r(P) = B$ and P is primary.

(c) Let $P = \sum_i^n P_i$. If B is maximal then $r(P) = B$ and P is primary.

(ii) *If A is primary, $B = r(A)$ and $U \nsubseteq A$, U is an ideal of R, then $A : U$ is primary and $r(A : U) = B$, where $A : U = \{r \in R | U\Gamma r \subseteq A\}$.*

Proof. **(i)(a)** It is sufficient to verify all the conditions of Theorem 3.18, for P, B. The first part of Theorem 3.18 is obvious for P and B. To prove theorem Theorem 3.18(ii), let $b \in B$ and S be an $m-$system containing b. Since $r(P_i) = B$ for all $(i = 1, 2, ..., n)$, S intersect each P_i. Let $d_1 \in S \cap P_1, d_2 \in S \cap P_2, ..., d_n \in S \cap P_n$. For $d_1, d_2, ..., d_n \in S$ there exist $r_1, r_2, ..., r_{n-1}$ such that $d = d_1\gamma_1 r_1\gamma_2 d_2....d_n \in S$, $\gamma_i \in \Gamma, i = 1, 2, ..., n$. $d \in P$ and hence S intersect each P. Now, to prove Theorem 3.18(iii), let $a, b \in R$ such that $a \notin P$ and $b \notin B$. Therefore, $a \notin P_i$ for some i. Thus, $a\Gamma R\Gamma b \nsubseteq P_i$ because P_i is primary and $r(P_i) = B$ and hence $a\Gamma R\Gamma b \nsubseteq P$.
(i)(b) and **(i)(c)** are obvious consequences of Theorem 3.19.
(ii) It again suffices to verify the three conditions of Theorem **??** for $A : U$ and B. To verify Theorem 3.18(i), To show $A : U \subseteq B$, we observe that $U\Gamma(A : U) \subseteq A$ and $U \nsubseteq A$. Therefore, $A : U \subseteq r(A) = B$ because A is primary. B being the prime radical of A is semiprime. This completes the first part of Theorem 3.18 for $A : U$ and B. To check Theorem 3.18(ii), let $b \in B$ and S be an $m-$system containing b. Then S intersects A as A is primary and $B = r(A)$ (c.f. Theorem 3.18, that is, there exists $d \in S \cap A$. Since $d \in A$, $U\Gamma d \subseteq A$ and so $d \in A : U$. This shows S intersect $A : U$. Finally, for Theorem 3.18(iii), let $a \notin A : U$ and $b \notin B$. The former implies that $c \in U$ exists such that $c\gamma A \notin A, \gamma \in \Gamma$. Therefore, for $c\gamma a \notin A$ and $b \notin B$, $r \in R$ we have $c\gamma a\beta r\alpha b \notin A$. Hence, $a\beta r\alpha b \notin A : U$, that is, $a\beta r\alpha b \notin A : U$, $\alpha, \beta, \gamma \in \Gamma$.

Corollary 3.21. *Let M be a maximal ideal of a $\Gamma-$ semiring R. Then $(M\Gamma)^{n-1}M$ is primary and $r((M\Gamma)^{n-1}M) = M$.*
Proof. Since M is maximal, therefore by Theorem 3.20 (i)(b) and Theorem 3.15, $r((M\Gamma)^{n-1}M) = r(M\Gamma M\Gamma....\Gamma M) = r(M) = M$ and so $(M\Gamma)^{n-1}M$ is primary.

Definition 3.22. *A $\Gamma-$ semiring R is left Noetherian if and only if it satisfies the ascending chain conditions on left ideals. Similarly, we can define right Noetherian.*

Theorem 3.23. *Let R be a Noetherian $\Gamma-$ semiring.*
(i) Let $r(A) = B$, for any two ideals A, B of R. Then for $n \geq 1$, $(B\Gamma)^{n-1}B \subseteq A$.
(ii) If A is primary, then $r(A)$ is prime.
Proof. (i) R being Noetherian, so we have $M \subset R$, maximal with respect to $(M\Gamma)^{n-1}M \subseteq A$ for some $n \geq 1$. Clearly M is semiprime. Moreover, $M \subseteq B$ since $(M\Gamma)^{n-1}M \subseteq A \subseteq B$ and $B = r(A)$ However, B being smallest prime ideal containing A (c.f. Corollary 3.17), it follows that $M = B$.

(ii) On contrary, suppose that $r(A)$ is not prime. Then $B \nsubseteq r(A), C \nsubseteq r(A)$ and $B\Gamma C \subseteq r(A)$, B, C are ideals of R. By (i), for $n \geq 1$ we have, $(r(A)\Gamma)^{n-1}r(A) \subseteq A$. Thus, $B\Gamma C \subseteq r(A)$ implies that $((B\Gamma C)\Gamma)^{n-1}(B\Gamma C) \subseteq A$. Since $C \nsubseteq r(A)$, and A is primary so $(B\Gamma C)\Gamma(B\Gamma C)\Gamma...\Gamma B \subseteq A$. Again, since $B \nsubseteq r(A)$, and A is primary implies that $(B\Gamma C)\Gamma(B\Gamma C)\Gamma\frac{(n-1)times}{}(B\Gamma C) \subseteq A$. Repeating the process we finally get $B \subseteq A$. But this is a contradiction. Hence, $r(A)$ is prime.

Definition 3.24. *Let R_1 and R_2 be two $\Gamma-$ semirings. Then $f : R_1 \to R_2$ be a $\Gamma-$ homomorphism if $f(x+y) = f(x) + f(y)$ and $f(x\gamma y) = f(x)\gamma f(y)$ for all $x, y \in R_1$ and $\gamma \in \Gamma$.*

Let R_1 and R_2 be two $\Gamma-$ semirings and $T : R_1 \to R_2$ be an onto $\Gamma-$ homomorphism. Let $K_T = \{a \in R_1|$ there exist $b, c \in R_1$ such that $a = b + c$ and $T(b) = T(c)\}$. Then K_T is an ideal of R_1 containing $KerT$, where $KerT = \{a \in R_1 \mid T(a) = 0\}$.

Finally, we have

Theorem 3.25. *Let $T : R_1 \to R_2$ be an onto homomorphism of two $\Gamma-$ semirings R_1 and R_2. Let $A \subseteq R_1$ for which both A and $r(A)$ are $k-$ideals and $K_T \subseteq A$. Then A is primary if and only if $T(A)$ is primary when this is so, $r(T(A)) = T(r(A))$.*
Proof. Since $T(A) \subseteq T(r(A))$ as $A \subseteq r(A)$. First we show that $T(r(A))$ is semiprime, let $a \in R_2$ such that $a \notin T(r(A))$. Then no preimage c of a is in $r(A)$. By semiprimeness of $r(A)$, we have $r_1 \in R_1$, $\gamma \in \Gamma$ for which $c\gamma r_1\gamma c \notin r(A)$. If x is any pre image of $T(c\gamma r_1\gamma c) = T(c)\Gamma T(r_1)\Gamma T(c) = a\Gamma T(r_1)\Gamma a$, then $T(c\gamma r_1\gamma c) = T(x)$ implies that $x + c\gamma r_1\gamma c \in K_T \subseteq A \subseteq r(A)$. Therefore $x \notin r(A)$, for otherwise $k-$ideal character of $r(A)$ gives the contradiction $c\gamma r_1\gamma c \in r(A)$. Consequently $a\Gamma T(r_1)\Gamma a \notin T(r(A))$. Thus $T(r(A))$ is semiprime and we are through with the verification of condition (i) of Theorem 3.18. To verify the condition (ii) of Theorem 3.18, let $b \in T(r(A))$, S_2 is an m-system in R_2 containing b. Let $S = T^{-1}(S_2) \subseteq R_1$, then S is an m-system in R_1. Since $b \in T(r(A))$, there exists $a \in r(A)$ so $T(a) = b$. Moreover, $b \in S_2$ implies that $c \in S$ gives that $T(c) = b$. Thus, $T(a) = T(c)$ implying $a + c \in K_T \subseteq A \subseteq r(A)$. Since $a + c \in r(A)$ and $r(A)$ is $k-$ideal, we get

$c \in r(A)$. Thus, every $m-$system containing c, in particular S intersects A non-trivially. That is, there exists $d \in S \cap A$ and hence $T(d) \in S_2 \cap T(A)$ and we have verified condition (ii) of theorem 3.18. Finally, for condition (iii) of theorem 3.18, let $a \notin T(A)$ and $b \notin T(r(A))$. Then obviously no pre-image l of a is in A and no pre-image m of b is in $r(A)$. Therefore, by primary character of A, $r_1 \in R_1$ we have $l\gamma r_1 \gamma m \notin A$. If y is any pre image of $T(l\gamma r_1\gamma m) = T(l)\Gamma T(r_1)\Gamma T(m) = a\Gamma T(r_1)\Gamma b$, then $T(l\gamma r_1\gamma m) = T(y)$ implies that $y + l\gamma r_1\gamma m \in K_T \subseteq A$. Therefore $y \notin A$, for otherwise $k-$ideal character of A gives the contradiction $l\gamma r_1\gamma m \in A$, for all $\gamma \in \Gamma$. Consequently $a\Gamma T(r_1)\Gamma b \notin T(A)$. Thus $T(A)$ is primary.

For the converse, let $T(A)$ be primary and $r(T(A)) = T(r(A)$. To show A is primary, let $a \notin A$ and $b \notin r(A)$, $a, b \in R_1$. Then as above using the $k-$ideal character of A together with $K_T \subseteq A$, we get $T(a) \notin T(r(A))$. By a similar argument $T(b) \notin T(r(A))$ as $r(A)$ is also $k-$ideal. Since $T(A)$ is primary and $r(T(A)) = T(r(A))$, we have $r_2 \in R_2$ for which $T(a)\Gamma r_2\Gamma T(b) \notin T(A)$. As T is onto, so $r_1 \in R_1$ gives that $T(r_1) = r_2$. Therefore, for all $\gamma \in \Gamma$, $T(a\gamma r_1\gamma b) = T(a)\Gamma T(r_1)\Gamma T(b) = T(a)\Gamma r_2\Gamma T(b) \notin T(A)$. Thus, we have shown that $a\gamma r_1\gamma b \notin A$, for all $\gamma \in \Gamma$, that is $a\Gamma R_1\Gamma b \nsubseteq A$, proving that A is primary.

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