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SOME INTEGRALS INVOLVING STRUVE AND MODIFIED STRUVE FUNCTIONS

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Abstract: In this paper, authors establish eight definite integral involving Struve function and modified Struve functions using basic properties of definite integrals and its techniques. Several closely-related results such as (for example) Generalized hypergeometric functions are also considered. These results provide some extensions in the scientific literature. Furthermore, these integrals play a significant role in the applied Mathematics.

Keywords and Phrases: Struve function, hypergeometric function.

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1. Introduction

Struve functions are explication of the heterogeneous Bessel's differential equation:

$$w^2\frac{d^2p}{dw^2}+w\frac{dp}{dw}+(w^2-\eta^2)p=\frac{4(\frac{w}{2})^{\eta+1}}{\sqrt{\pi}\,\Gamma(\eta+\frac{1}{2})} \tag{1.1}$$

and are defined as:

$$H_\eta(w) = \frac{2(\frac{w}{2})^\eta}{\Gamma(\eta+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\frac{\pi}{2}} sin(w\ cos\zeta) sin^{2\eta}(\zeta) d\zeta \tag{1.2}$$

Modified Struve functions are defined as:

$$L_\eta(w) = I_{-\eta}(w) - \frac{2(\frac{w}{2})^\eta}{\Gamma(\eta+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\infty} sin(w\mu)(1+\mu^2)^{\eta-\frac{1}{2}} d\mu \tag{1.3}$$

Generalized hypergeometric function is defines as

$${}_pF_q\left[\begin{array}{lcc} a_1, a_2, \cdots, a_p & ; & \\ & & z \\ b_1, b_2, \cdots, b_q & ; & \end{array}\right] = \sum_{m=0}^{\infty} \frac{(a_1)_m (a_2)_m \cdots (a_p)_m z^m}{(b_1)_m (b_2)_m \cdots (b_q)_m m!} \tag{1.4}$$

where the parameters $b_1, b_2, \cdots, b_q$ are positive integers.

2. Main Results

In this section, we establish eight definite integrals involving Struve and Modified Struve functions:

Theorem 1. *Each of the following assertion holds true:*

$$\int_0^1 t\ \ \log t\ H_0(at)\ dt = \frac{\pi\ H_0(a)\ \ - 2a}{\pi\ a^2} \tag{2.1}$$

$$\int_0^1 t\ \ \log t\ L_0(at)\ dt = \frac{2a - \pi\ L_0(a)}{\pi\ a^2} \tag{2.2}$$

$$\int_0^1 t^2\ \ \log t\ H_0(at)\ dt$$

$$= \frac{1}{2\pi a}\left[{}_3F_4\left(1, 1, 1; \frac{1}{2}, \frac{1}{2}, 2, 2; -\frac{a^2}{4}\right) - 1\right] \tag{2.3}$$

$$\int_0^1 t^2\ \ \log t\ L_0(at)\ dt$$

$$= -\frac{1}{2\pi a}\left[{}_3F_4\left(1, 1, 1; \frac{1}{2}, \frac{1}{2}, 2, 2; \frac{a^2}{4}\right) - 1\right] \tag{2.4}$$

$$\int_0^1 t^4\ \ \log t\ H_0(at)\ dt$$

$$= -\frac{1}{8\pi\ a^3}\left[4\ {}_3F_4\left(1,1,1;-\frac{1}{2},-\frac{1}{2},2,2;-\frac{a^2}{4}\right)+a^2-4\right] \tag{2.5}$$

$$\int_0^1 t^4\ \log t\ L_0(at)\ dt$$

$$= \frac{1}{8\pi\ a^3}\left[-4\ {}_3F_4\left(1,1,1;-\frac{1}{2},-\frac{1}{2},2,2;\frac{a^2}{4}\right)+a^2+4\right] \tag{2.6}$$

$$\int_0^1 t^6\ \log t\ H_0(at)\ dt$$

$$= \frac{1}{72\pi\ a^5}\left[324\ {}_3F_4\left(1,1,1;-\frac{3}{2},-\frac{3}{2},2,2;-\frac{a^2}{4}\right)-4a^4+9a^2-324\right] \tag{2.7}$$

and

$$\int_0^1 t^6\ \log t\ L_0(at)\ dt$$

$$= \frac{1}{72\pi\ a^5}\left[-324\ {}_3F_4\left(1,1,1;-\frac{3}{2},-\frac{3}{2},2,2;\frac{a^2}{4}\right)+4a^4+9a^2+324\right]. \tag{2.8}$$

provided that each member of the assertions (2.1) to (2.8) exists.

3. Proof.

In this section, we provide proofs for the assertions (2.1) to (2.8), as given below: We first prove our first assertion (2.1), by considering its left hand side, and using properties of definite integrals, we obtained:

$$\int_0^1 t\ \log t\ H_0(at)\ dt = \frac{1}{\pi a^2}\Big[\pi H_0(at)+at\big(\pi\ \log t\ H_1(at)-2\big)\Big]_0^1$$

$$= \frac{1}{\pi a^2}\Big[\big\{\pi H_0(a)+a(-2)\big\}-\big\{\pi H_0(0)\big\}\Big] = \frac{1}{\pi a^2}\left[\pi H_0(a)-2a\right]$$

$$= \frac{\pi\ H_0(a)\ -2a}{\pi\ a^2}$$

Hence, we arrived at right hand side on assertain (2.1).
This completes our demonstration of the first assertion (2.1).
Next, we prove our second assertion (2.2), by considering its left hand side, and further and using properties of definite integrals, we obtained:

$$\int_0^1 t\ \log t\ L_0(at)\ dt = \frac{1}{\pi a^2}\Big[-\pi L_0(at)+at\big(\pi\ \log t\ L_1(at)+2\big)\Big]_0^1$$

$$= \frac{1}{\pi a^2}\Big[-\big\{\pi L_0(a)+a(2)\big\}-\big\{\pi L_0(0)\big\}\Big] = \frac{1}{\pi a^2}\big[-\pi L_0(a)+2a\big]$$

$$= \frac{2a-\pi\ L_0(a)}{\pi\ a^2}.$$

Hence, we arrived at right hand side on assertain (2.2).
This completes our demonstration of the first assertion (2.2).
Next, we prove our second assertion (2.3), by considering its left hand side, and further and using properties of definite integrals, we obtained:

$$\int_0^1 t^2\ \log t\ H_0(at)\ dt$$

$$= \frac{1}{2\pi a}\Bigg[t^2\Bigg\{{}_3F_4\left(1,1,1;\frac{1}{2},\frac{1}{2},2,2;-\frac{a^2t^2}{4}\right)$$
$$-2\log t\ {}_2F_3\left(1,1;\frac{1}{2},\frac{1}{2},2;-\frac{a^2t^2}{4}\right)+2\log t-1\Bigg\}\Bigg]_0^1$$

$$= \frac{1}{2\pi a}\Bigg[\Bigg\{{}_3F_4\left(1,1,1;\frac{1}{2},\frac{1}{2},2,2;-\frac{a^2}{4}\right)-1\Bigg\}-\{0\}\Bigg]$$

$$= \frac{1}{2\pi a}\Bigg[{}_3F_4\left(1,1,1;\frac{1}{2},\frac{1}{2},2,2;-\frac{a^2}{4}\right)-1\Bigg].$$

Hence, we arrived at right hand side on assertain (2.3).
This completes our demonstration of the first assertion (2.3).
Next, we prove our second assertion (2.4), by considering its left hand side, and further and using properties of definite integrals, we obtained:

$$\int_0^1 t^2\ \log t\ L_0(at)\ dt$$

$$= -\frac{1}{2\pi a}\Bigg[t^2\Bigg\{{}_3F_4\left(1,1,1;\frac{1}{2},\frac{1}{2},2,2;\frac{a^2t^2}{4}\right)$$
$$-2\log t\ {}_2F_3\left(1,1;\frac{1}{2},\frac{1}{2},2;\frac{a^2t^2}{4}\right)+2\log t-1\Bigg\}\Bigg]_0^1$$

$$= \frac{1}{2\pi a}\Bigg[\Bigg\{{}_3F_4\left(1,1,1;\frac{1}{2},\frac{1}{2},2,2;\frac{a^2}{4}\right)-1\Bigg\}-\Big\{0\Big\}\Bigg]$$

$$= \frac{1}{2\pi a}\Bigg[{}_3F_4\left(1,1,1;\frac{1}{2},\frac{1}{2},2,2;\frac{a^2}{4}\right)-1\Bigg].$$

Hence, we arrived at right hand side on assertain (2.4).
This completes our demonstration of the first assertion (2.4).
Next, we prove our second assertion (2.5), by considering its left hand side, and further and using properties of definite integrals, we obtained:

$$\int_0^1 t^4 \ \log t \ H_0(at) \ dt = \frac{1}{8\pi a^3}\Bigg[t^2\Bigg\{-4 \ {}_3F_4\Bigg(1,1,1;-\frac{1}{2},-\frac{1}{2},2,2;-\frac{a^2t^2}{4}\Bigg)+$$

$$+8\log t \ {}_2F_3\Bigg(1,1;-\frac{1}{2},-\frac{1}{2},2;-\frac{a^2t^2}{4}\Bigg)-a^2t^2+4a^2t^2\log t-8\log t+4\Bigg\}\Bigg]_0^1$$

$$=\frac{1}{8\pi a^3}\Bigg[\Bigg\{-4 \ {}_3F_4\Bigg(1,1,1;-\frac{1}{2},-\frac{1}{2},2,2;-\frac{a^2}{4}\Bigg)-a^2+4\Bigg\}-\Bigg\{0\Bigg\}\Bigg]$$

$$=-\frac{1}{8\pi \ a^3}\Bigg[4 \ {}_3F_4\Bigg(1,1,1;-\frac{1}{2},-\frac{1}{2},2,2;-\frac{a^2}{4}\Bigg)+a^2-4\Bigg].$$

Hence, we arrived at right hand side on assertain (2.5).
This completes our demonstration of the first assertion (2.5).
Next, we prove our second assertion (2.6), by considering its left hand side, and further and using properties of definite integrals, we obtained:

$$\int_0^1 t^4 \ \log t \ L_0(at) \ dt = \frac{1}{8\pi a^3}\Bigg[t^2\Bigg\{-4 \ {}_3F_4\Bigg(1,1,1;-\frac{1}{2},-\frac{1}{2},2,2;\frac{a^2t^2}{4}\Bigg)+$$

$$+8\log t \ {}_2F_3\Bigg(1,1;-\frac{1}{2},-\frac{1}{2},2;\frac{a^2t^2}{4}\Bigg)+a^2t^2-4a^2t^2\log t-8\log t+4\Bigg\}\Bigg]_0^1$$

$$=\frac{1}{8\pi a^3}\Bigg[\Bigg\{-4 \ {}_3F_4\Bigg(1,1,1;-\frac{1}{2},-\frac{1}{2},2,2;\frac{a^2}{4}\Bigg)+a^2+4\Bigg\}-\Bigg\{0\Bigg\}\Bigg]$$

$$=\frac{1}{8\pi \ a^3}\Bigg[-4 \ {}_3F_4\Bigg(1,1,1;-\frac{1}{2},-\frac{1}{2},2,2;\frac{a^2}{4}\Bigg)+a^2+4\Bigg].$$

Hence, we arrived at right hand side on assertain (2.6).
This completes our demonstration of the first assertion (2.6).
Next, we prove our second assertion (2.7), by considering its left hand side, and further and using properties of definite integrals, we obtained:

$$\int_0^1 t^6 \ \log t \ H_0(at) \ dt = \frac{1}{72\pi a^5}\Bigg[t^2\Bigg\{324 \ {}_3F_4\Bigg(1,1,1;-\frac{3}{2},-\frac{3}{2},2,2;-\frac{a^2t^2}{4}\Bigg)$$

$$-648 \log t \; {}_2F_3\left(1, 1; -\frac{3}{2}, -\frac{3}{2}, 2; -\frac{a^2t^2}{4}\right)-$$

$$-4a^4t^4 + 24a^4t^4 \log t + 9a^2t^2 - 36a^2t^2 \log t + 648 \log t - 324\Bigg\}\Bigg]_0^1$$

$$= \frac{1}{72\pi a^5}\left[\left\{324 \; {}_3F_4\left(1, 1, 1; -\frac{3}{2}, -\frac{3}{2}, 2, 2; -\frac{a^2}{4}\right) - 4a^4 + 9a^2 - 324\right\} - \left\{0\right\}\right]$$

$$= \frac{1}{72\pi \; a^5}\left[324 \; {}_3F_4\left(1, 1, 1; -\frac{3}{2}, -\frac{3}{2}, 2, 2; -\frac{a^2}{4}\right) - 4a^4 + 9a^2 - 324\right].$$

Hence, we arrived at right hand side on assertain (2.7).
This completes our demonstration of the first assertion (2.7).
Next, we prove our second assertion (2.8), by considering its left hand side, and further and using properties of definite integrals, we obtained:

$$\int_0^1 t^6 \;\log t \; L_0(at) \; dt = \frac{1}{72\pi a^5}\Bigg[t^2\Bigg\{324 \; {}_3F_4\left(1, 1, 1; -\frac{3}{2}, -\frac{3}{2}, 2, 2; \frac{a^2t^2}{4}\right)$$

$$+648 \log t \; {}_2F_3\left(1, 1; -\frac{3}{2}, -\frac{3}{2}, 2; \frac{a^2t^2}{4}\right)-$$

$$+4a^4t^4 - 24a^4t^4 \log t + 9a^2t^2 - 36a^2t^2 \log t - 648 \log t + 324\Bigg\}\Bigg]_0^1$$

$$= \frac{1}{72\pi a^5}\left[\left\{324 \; {}_3F_4\left(1, 1, 1; -\frac{3}{2}, -\frac{3}{2}, 2, 2; \frac{a^2}{4}\right) + 4a^4 + 9a^2 + 324\right\} - \left\{0\right\}\right]$$

$$= \frac{1}{72\pi \; a^5}\left[324 \; {}_3F_4\left(1, 1, 1; -\frac{3}{2}, -\frac{3}{2}, 2, 2; \frac{a^2}{4}\right) + 4a^4 + 9a^2 + 324\right].$$

This completes our demonstration of assertion (2.8).
This obviously completes our proof of Theorem 1.

4. Concluding Remarks and Observations

In our present investigation, we have made use of the Gamma function as well as the hypergeometric and the generalized hypergeometric functions with a view developing several definite integrals involving Struve and Modified Struve functions, respectively. The numerical approximation of these definite integrals and the corresponding hypergeometric functions are also presented. The results derived in this article are believed to be new and would extend and unify those that are available in the scientific literature.

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References

[1] Abramowitz, M. and Stegun, I. A., Handbook of mathematical functions with formulas, graphs, and mathematical tables, National Bureau of Standards, 1970.

[2] Brychkov, Y. A., Handbook of special functions: derivatives, integrals,series and other formulas, CRC Press,Taylor & Francis Group, London, U. K, 2008.

[3] Chaudhary, M. P., Certain Aspects of Special Functions and Integral Operators, LAMBERT Academic Publishing, Germany, 2014.

[4] Chaudhary, M. P., A simple solution of some integrals given by Srinivasa Ramanujan, Resonance: J. Sci. Education (publication of Indian Academy of Science, Bangalore), 13(9) (2008), 882-884.

[5] Chaudhary, M. P., Uddin, Salah, Ouedraogo, H. W. and Arjika, Sama, Certain definite integral in association with complete elliptic integral, Math. Education, LVI(4) (2022), 1-8.

[6] Chaudhary, M. P. and Abubakar, U. M., Fractional calculus of the extended Bessel-Wright function and its applications to fractional kinetic equations, Journal of Fractional Calculus and Applications, 14(2) (2023), 1-17.

[7] Gauss, C. F., Disquisitiones generales circa seriem infinitam ... , Comm. soc. reg. sci. Gott. rec., 2 (1813), 123-162.

[8] Kaurangini, M. L., Chaudhary, M. P., Abubakar, U. M., Kiymaz, I. O. and Ata, E., On some applications of the new extended hypergeometric function, Journal of Contemporary Applied Mathematics, 14(1) (2024), 28-45.

[9] Kaurangini, M. L., Chaudhary, M. P., Abubakar, U. M., Kiymaz, I. O. and Ata, E., Fractional calculus and fractional kinetic equation involving extended hypergeometric function, J. of Ramanujan Society of Mathematics and Mathematical Sciences, 11(1) (2023), 159-174.

[10] Prudnikov, A. P., Brychkov, Yu. A. and Marichev, O. I., Integral and series Vol 3: more special functions, Nauka, Moscow, 2003.

[11] Shpot, M. A., Chaudhary, M. P. and Paris, R. B., Integrals of products of Hurwitz zeta functions and the Casimir effect in ϕ_4 field theories, J. Classical Analysis, 9(2) (2016), 99-115.

[12] Srivastava, H. M., Uddin, Salah and Chaudhary, M. P., Some definite integrals involving elliptic integrals in association with hypergeometric functions, Filomat, 38(11) (2024), 3719-3730.

[13] Uddin, Salah and Chaudhary, M. P., Some definite integrals associated with hypergeometric functions, Honam Mathematical J., 45(4) (2023), 682-688.

[14] Uddin, Salah and Chaudhary, M. P., Some elliptical integrals associated with hypergeometric functions, J. of Ramanujan Society of Mathematics and Mathematical Sciences, 11(1) (2023), 107-114.