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ON NEW FOUR-TERM RECURRENCE RELATIONS FOR THE 3-j COEFFICIENT

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Abstract: In our earlier article [9] "On new three-term recurrence relations for the 3-j coefficient", we derived six new three-term recurrence relations of fundamental importance in the Quantum Theory of Angular Momentum. In this article, we derive new four term recurrence relations for the 3-j coefficient, as a direct consequence of the recurrence relations for the ${}_3F_2(\mathbf{a};\ \mathbf{b};\ z)$ given in Tamara Antonova, Roman Dymtryshyn and Serhii Sharyn (2021) [1]. The derived 4-term recurrence relations for the 3-j coefficient are **new**.

Keywords and Phrases: Generalized hypergeometric series, Angular momentum coupling coefficient, Clebsch-Gordan, or 3-j coefficient, recurrence relations.

2020 Mathematics Subject Classification: 33C20, 33C90.

1. Introduction

It has been shown [7] that

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{m_1+m_2+m_3,0}(-1)^{j_1-j_2-m_3} \prod_{i,k=1}^{3} \left[\frac{R_{ik}!}{(J+1)!}\right]^{1/2} \\ \times (-1)^{\sigma(pqr)} \left[\Gamma(1-A, 1-B, 1-C, D, E)\right]^{-1} \\ \times\ {}_3F_2(A,\ B,\ C;\ D,\ E;\ 1)\end{aligned} \tag{1}$$

where,

$$A=-R_{2p},\quad B=-R_{3q},\quad C=-R_{1r},\quad D=1+R_{3r}-R_{2p}\quad E=1+R_{2r}-R_{3q}$$

and

$$\Gamma(x,y,\cdots)=\Gamma(x)\Gamma(y)\cdots$$

for all permutations of $(pqr)=(123)$, and

$$\sigma(pqr)=\begin{cases} R_{3p}-R_{2q}, & \text{for even permutations,}\\ R_{3p}-R_{2q}+J, & \text{for odd permutations.}\end{cases}$$

with $J=j_1+j_2+j_3$. The defining relation for the numerator and denominator parameters R_{ik}'s are the elements of the Regge(1959) 3×3 square symbol:

$$\|R_{ik}\|=\left\|\begin{matrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3\\ j_1-m_1 & j_2-m_2 & j_3-m_3\\ j_1+m_1 & j_2+m_2 & j_3+m_3\end{matrix}\right\| \tag{2}$$

2. The ${}_3F_2(1)$ and the 3-j coefficient

It has been shown (RS-KSR) that (1) can be inverted to write the ${}_3F_2(1)$ in terms of the 3-j coefficient as:

$$\begin{aligned}&{}_3F_2(A,\ B,\ C;\ D,\ E;\ 1)=(-1)^{D-E}\quad\Gamma(D,E)\\ &\times\left[\frac{\Gamma(1-A,1-B,1-C,s-1)}{\Gamma(D-A,D-B,D-C,E-A,E-B,E-C)}\right]^{1/2}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix},\end{aligned} \tag{3}$$

where $s=D+E-A-B-C$ is called the parameter excess.

In this article, we show that corresponding to the four-term recurrence relations for the ${}_3F_2(1)$, there exist two four-term recurrence relations for the 3-j coefficient, which are **new**.

From the following three-term recurrence relations ((5) and (6) of [1]):

$$\begin{aligned}&{}_3F_2(A,\ B,\ C;\ D,\ E;\ 1)={}_3F_2(A,\ B+1,\ C;\ D,\ E+1;\ 1)\\ &-\frac{(E-B)AC}{(E+1)DE}.\ {}_3F_2(A+1,\ B+1,\ C+1;\ D+1,\ E+2;\ 1)\end{aligned} \tag{4}$$

$$\begin{aligned}&{}_3F_2(A,\ B,\ C;\ D,\ E;\ 1)={}_3F_2(A,\ B,\ C+1;\ D+1,\ E;\ 1)\\ &-\frac{(D-C)AB}{(D+1)DE}.\ {}_3F_2(A+1,\ B+1,\ C+1;\ D+2,\ E+1;\ 1)\end{aligned} \tag{5}$$

Antonova et.al. [1] show that the following four-term recurrence relation holds:

$$\begin{aligned}
&{}_3F_2(A,\ B,\ C;\ D,\ E;\ 1) = {}_3F_2(A,\ B+1,\ C+1;\ D+1,\ E+1;\ 1)\\
&-\frac{(D-C)AB}{(D+1)DE}.{}_3F_2(A+1,\ B+1,\ C+1;\ D+2,\ E+1;\ 1)\\
&-\frac{(E-B)(C+1)A}{(D+1)(E+1)E}.{}_3F_2(A+1,\ B+1,\ C+2;\ D+2,\ E+2;\ 1)
\end{aligned} \tag{6}$$

The other four-term recurrence relation derived from ((4), (7) of [1]) is:

$$\begin{aligned}
&{}_3F_2(A,\ B,\ C;\ D,\ E;\ 1) = {}_3F_2(A+1,\ B,\ C+1;\ D+1,\ E+1;\ 1)\\
&-\frac{(E-C)AB}{(E+1)DE}.{}_3F_2(A+1,\ B+1,\ C+1;\ D+1,\ E+2;\ 1)\\
&-\frac{(D-A)(C+1)B}{(D+1)(E+1)E}.{}_3F_2(A+1,\ B+1,\ C+2;\ D+2,\ E+2;\ 1)
\end{aligned} \tag{7}$$

In section 3, the recurrence relations used to derive the new four-term recurrence relations for 3-j coefficient are presented.

3. The Recurrence Relations for the 3-j coefficient

The recent article of Petreolle, Sokal and Zhu [4] [1]. To start with, the first four-term recurrence relation for the ${}_3F_2(1)$ is:

$$\begin{aligned}
&{}_3F_2(A,\ B,\ C;\ D,\ E;\ 1) = {}_3F_2(A,\ B+1,\ C+1;\ D+1,\ E+1;\ 1)\\
&-\frac{(D-C)AB}{(D+1)DE}.{}_3F_2(A+1,\ B+1,\ C+1;\ D+2,\ E+1;\ 1)\\
&-\frac{(E-B)(C+1)A}{(D+1)(E+1)E}.{}_3F_2(A+1,\ B+1,\ C+2;\ D+2,\ E+2;\ 1)
\end{aligned} \tag{8}$$

and the corresponding four-term recurrence relation for the 3-j coefficient is:

$$\begin{aligned}
&[(j_2+m_2)(j_1+j_2-j_3)]^{1/2}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}\\
&=\frac{(1-j_1+j_3-m_2)}{[(1+j_1-j_2+j_3)(1+j_3+m_3)]^{1/2}}\begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3+\frac{1}{2}\\ m_1 & m_2-\frac{1}{2} & m_3+\frac{1}{2}\end{pmatrix}\\
&+\frac{(j_2+m_2)[(j_1-m_1)(1+j_2-m_2)]^{1/2}}{[(1-j_1+j_2+j_3)(1+j_3+m_3)]^{1/2}}\\
&\times\begin{pmatrix} j_1-\frac{1}{2} & j_2 & j_3+\frac{1}{2}\\ m_1+\frac{1}{2} & m_2-1 & m_3+\frac{1}{2}\end{pmatrix}
\end{aligned} \tag{9}$$

$$+\frac{(1-j_1-j_2+j_3)(1-j_1+j_3-m_2)(1+j_3-m_3)(j_1-m_1)^{1/2}}{[(1-j_1+j_2+j_3)(1+j_3+m_3)(1+j_1-j_2+j_3)(1+j_3-m_3)]^{1/2}}$$
$$\times\frac{1}{(j_1+j_2-j_3-1)^{1/2}}\begin{pmatrix} j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_3+1 \\ m_1+\frac{1}{2} & m_2-\frac{1}{2} & m_3 \end{pmatrix}$$

Corresponding to the second four-term recurrence relation for the ${}_3F_2(1)$:

$$\begin{aligned} &{}_3F_2(A,\ B,\ C;\ D,\ E;\ 1) = {}_3F_2(A+1,\ B,\ C+1;\ D+1,\ E+1;\ 1) \\ &-\frac{(E-C)AB}{(E+1)DE}{}_3F_2(A+1,\ B+1,\ C+1;\ D+1,\ E+2;\ 1) \\ &-\frac{(D-A)(C+1)B}{(D+1)(E+1)E}{}_3F_2(A+1,\ B+1,\ C+2;\ D+2,\ E+2;\ 1) \end{aligned} \tag{10}$$

we have the second four-term recurrence relation for the 3-j coefficient as:

$$\begin{aligned} &[(j_1-m_1)(j_1+j_2-j_3)]^{1/2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\ &= \frac{(1-j_1+j_3-m_2)(1-j_2+j_3+m_1)}{[(1-j_1+j_2+j_3)(1+j_3-m_3)]^{1/2}}\begin{pmatrix} j_1-\frac{1}{2} & j_2 & j_3+\frac{1}{2} \\ m_1+\frac{1}{2} & m_2 & m_3-\frac{1}{2} \end{pmatrix} \\ &+\frac{(j_1-m_1)(2-j_2+j_3+m_1)[(1+j_1+m_1)(j_2+m_2)]^{1/2}}{(2-j_1+j_3-m_2)[(1+j_1-j_2+j_3)(1+j_3-m_3)]^{1/2}} \\ &\times\begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3+\frac{1}{2} \\ m_1+1 & m_2-\frac{1}{2} & m_3-\frac{1}{2} \end{pmatrix} \\ &+\frac{(1-j_1-j_2+j_3)(1-j_1+j_3-m_2)[(j_2+m_2)(1+j_3+m_3)]^{1/2}}{[(1-j_1+j_2+j_3)(1+j_1-j_2+j_3)(j_1+j_2-j_3-1)(1+j_3-m_3)]^{1/2}} \\ &\times\begin{pmatrix} j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_3+1 \\ m_1+\frac{1}{2} & m_2-\frac{1}{2} & m_3 \end{pmatrix}. \end{aligned} \tag{11}$$

4. Numerical verification

A numerical verification of the new recurrence relations has been done for the two new recurrence relations. To illustrate the methodology adopted, shown below are the details for the first recurrence relation (9), for

$$j_1=2, j_2=2, j_3=1, m_1=1, m_2=-1, m_3=0.$$

Using the tables of Rotenberg et.al. [5], for the lhs and the rhs of (9) the value obtained is $-\frac{1}{\sqrt{10}}$. This is also the value for the lhs and the rhs of (11).

5. Conclusion

In this article, we have derived two new recurrence relations for the 3-j coefficient, which are a consequence of the existing six recurrence relations for the ${}_3F_2(1)$ hypergeometric function. Such recurrence relations are of significance and relevance in numerical computations of matrix elements of tensor-operators which arise in Atomic, Molecular and Nuclear Physics studies.

It is to be noted that we have derived from the two three-term recurrence relations (11) and (21) for the ${}_3F_2(1)$ in [9], the new four-term recurrence relation (9) for the 3-j coefficient.

And from the two three-term recurrence relations (17) and (19) for the ${}_3F_2(1)$ in [9] another new four-term recurrence relation (11) for the 3-j coefficient has been derived.

However, from the two three-term recurrence relations (13) and (15) for the ${}_3F_2(1)$ in [9], no new recurrence relation can be derived for the 3-j coefficient.

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