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ON NEW THREE-TERM RECURRENCE RELATIONS FOR THE 3-j COEFFICIENT

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Abstract: Six new recurrence relations have been derived for the Clebsch-Gordan coefficient, also referred to as the Wigner 3-j coefficient. These are a consequence of the recurrence relations for the ${}_3F_2(\mathbf{a};\ \mathbf{b}; z)$ derived recently by Tamara Antonova, Roman Dmytryshyn and Serhii Sharyn(2021).

Keywords and Phrases: Generalized hypergeometric series, Angular momentum coupling coefficient, Clebsch-Gordan, or 3-j coefficient, recurrence relations.

2020 Mathematics Subject Classification: 33C20, 33C90.

1. Introduction

The well-known 3-j coefficient (or the Clebsch-Gordan coefficient), in Quantum Theory of Angular Momentum (QTAM) [8] is defined as:

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{m_1+m_2+m_3,0}\ (-1)^{j_1-j_2-m_3}\ \Delta(j_1j_2j_3) \times \prod_{i=1}^{3} [(j_i+m_i)!\ (j_i-m_i)!]^{1/2} \times \sum_t (-1)^t\ [\ t! \prod_{k=1}^{2} (t-\alpha_k)!\ \prod_{\ell=1}^{3} (\beta_\ell - t)!\ ]^{-1}. \quad (1)$$

where

$$t_{\min} \leq \ t \ \leq t_{\max}$$

$$t_{min} = \max(0, \alpha_1, \alpha_2), \qquad t_{max} = \ \min(\beta_1, \beta_2, \beta_3) \tag{2}$$

$$\begin{aligned} \alpha_1 = j_1 - j_3 + m_2 = (j_1 - m_1) - (j_3 + m_3), \\ \alpha_2 = j_2 - j_3 - m_1 = (j_2 + m_2) - (j_3 - m_3), \\ \beta_1 = j_1 - m_1, \qquad \beta_2 = j_2 + m_2, \qquad \beta_3 = j_1 + j_2 - j_3 \end{aligned} \tag{3}$$

and

$$\Delta(xyz) \ = \left[\frac{(-x+y+z)!(x-y+z)!(x+y-z)!}{(x+y+z+1)!}\right]^{1/2}. \tag{4}$$

2. The 3-j coefficient as a ${}_3F_2(1)$

The 3-j coefficient is defined as:

$$\begin{aligned} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta_{m_1+m_2+m_3,0} \prod_{i,k=1}^{3} \left[\frac{R_{ik}!}{(J+1)!}\right]^{1/2} (-1)^{j_1-j_2-m_3} \\ \times \sum_t (-1)^t \ [\ t! \ (j_1 - m_1 - t)! \ (j_2 + m_2 - t)! \\ \times (j_1 + j_2 - j_3 - t)!(t + j_3 - j_1 - m_2)! \\ \times (t + j_3 - j_2 + m_1)! \ ]^{-1} \end{aligned} \tag{5}$$

where

$$\Delta(j_1 j_2 j_3) \ = \left[\frac{(-j_1+j_2+j_3)!(j_1-j_2+j_3)!(j_1+j_2-j_3)!}{(J+1)!}\right]^{1/2} \tag{6}$$

and

$$J = j_1 + j_2 + j_3. \tag{7}$$

Rose (1955) [6] in an Appendix to this book, has shown that the 3-j coefficient can be written as a ${}_3F_2(1)$. Obviously, a ${}_3F_2(1)$ has at most 12-symmetries, due to 3! numerator parameter permutations, 2! denominator parameter permutations. However, there exist 72 symmetries for 3-j coefficient: after Regge [5] discovered six new symmetries on each of which the 12 'classical tetrahedral symmetries' can be superposed. Thus there are in all 72 symmetries for the 3-j coefficient. Since the given ${}_3F_2(1)$ of Rose accounts for only 12 of these 72 symmetries, KSR(1978) discovered the existence of a set of six ${}_3F_2(1)$s and showed that they are necessary and sufficient to account for all the 72 symmetries of the 3-j coefficient.

It has been shown [9] that

$$\begin{aligned}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= \delta_{m_1+m_2+m_3,0}(-1)^{j_1-j_2-m_3} \prod_{i,k=1}^{3} \left[\frac{R_{ik}!}{(J+1)!}\right]^{1/2} \\ &\times (-1)^{\sigma(pqr)} \; [\Gamma(1-A, 1-B, 1-C, D, E)]^{-1} \\ &\times \; {}_3F_2(A,\; B,\; C;\; D,\; E;\; 1)\end{aligned} \tag{8}$$

where

$A = -R_{2p}, \quad B = -R_{3q}, \quad C = -R_{1r}, \quad D = 1 + R_{3r} - R_{2p}, \quad E = 1 + R_{2r} - R_{3q}$

and

$$\Gamma(x, y, \cdots) = \Gamma(x)\Gamma(y)\cdots$$

for all permutations of $(pqr) = (123)$, and

$$\sigma(pqr) = \begin{cases} R_{3p} - R_{2q}, & \text{for even permutations,} \\ R_{3p} - R_{2q} + J, & \text{for odd permutations.} \end{cases}$$

with $J = j_1 + j_2 + j_3$. The defining relation for the numerator and denominator parameters R_{ik}'s are the elements of the Regge (1959) 3×3 square symbol :

$$\|R_{ik}\| = \left\| \begin{matrix} -j_1 + j_2 + j_3 & j_1 - j_2 + j_3 & j_1 + j_2 - j_3 \\ j_1 - m_1 & j_2 - m_2 & j_3 - m_3 \\ j_1 + m_1 & j_2 + m_2 & j_3 + m_3 \end{matrix} \right\| \tag{9}$$

It has been noted that (8) represents a set of six terminating ${}_3F_2(1)$ of the Van Der Waerden (1932) form. This set has been shown by one us (KSR) [9] to be necessary and sufficient to account for the 72 symmetries of that 3-j coefficient.

3. The ${}_3F_2(1)$ and the 3-j coefficient

It has been shown (RS-KSR) that (8) can be inverted to write the ${}_3F_2(1)$ in terms of the 3-j coefficient as:

$$\begin{aligned}{}_3F_2(A,\; B,\; C;\; D,\; E;\; 1) &= (-1)^{D-E} \quad \Gamma(D, E) \\ \times \left[\frac{\Gamma(1-A, 1-B, 1-C, s-1)}{\Gamma(D-A, D-B, D-C, E-A, E-B, E-C)}\right]^{1/2} &\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix},\end{aligned} \tag{10}$$

where $s = D + E - A - B - C$ is called the parameter excess.

In this article, we show that corresponding to the each of six three-term recurrence relations for the ${}_3F_2(1)$, there exist six three term recurrence relations for the

3-j coefficient each of which constitutes a unique pair of relations. Given below are these six pairs of recurrence relations, which are **new**.

4. Recurrence Relations for the 3-j coefficient

In a recent article, Petreolle, Sokal and Zhu, derive (Lemma 14.1) [4] three-term contiguous recurrence relations for the ${}_rF_s$. From this Lemma, we specialize the case for three-term recurrence relations for the ${}_3F_2(1)$:

To start with, the first three-term recurrence relation for the ${}_3F_2(1)$ is:

$$\begin{aligned} {}_3F_2(A,\ B,\ C;\ D,\ E;\ 1) &= {}_3F_2(A+1,\ B,\ C;\ D+1,\ E;\ 1) \\ &- \frac{(D-A)BC}{(D+1)DE}\ {}_3F_2(A+1,\ B+1,\ C+1;\ D+2,\ E+1;\ 1) \end{aligned} \tag{11}$$

and the corresponding three-term recurrence relation for the 3-j coefficient is:

$$\begin{aligned} &[(j_1-m_1)(1-j_1+j_2+j_3)(1+j_2-m_2)]^{1/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\ &+ (1-j_1+j_3-m_2)(j_1-j_2+j_3)^{1/2} \begin{pmatrix} j_1-\frac{1}{2} & j_2+\frac{1}{2} & j_3 \\ m_1+\frac{1}{2} & m_2-\frac{1}{2} & m_3 \end{pmatrix} \\ &= [(1+j_3+m_3)(j_2+m_2)(j_1+j_2-j_3)]^{1/2} \begin{pmatrix} j_1-\frac{1}{2} & j_2 & j_3+\frac{1}{2} \\ m_1+\frac{1}{2} & m_2-1 & m_3+\frac{1}{2} \end{pmatrix} \end{aligned} \tag{12}$$

Corresponding to the second three-term recurrence relation for the ${}_3F_2(1)$:

$$\begin{aligned} {}_3F_2(A,\ B,\ C;\ D,\ E;\ 1) &= {}_3F_2(A+1,\ B,\ C;\ D,\ E+1;\ 1) \\ &- \frac{(E-A)BC}{(E+1)DE}\ {}_3F_2(A+1,\ B+1,\ C+1;\ D+1,\ E+2;\ 1) \end{aligned} \tag{13}$$

we have the second three-term recurrence relation for the 3-j coefficient as:

$$\begin{aligned} &[(j_1-m_1)(1+j_1+m_1)(1+j_3-m_3)]^{1/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \\ &+ (1-j_2+j_3+m_1)(j_3+m_3)^{1/2}. \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1+1 & m_2 & m_3-1 \end{pmatrix} \\ &= [(1+j_1-j_2+j_3)(j_2+m_2)(j_1+j_2-j_3)]^{1/2}. \begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3+\frac{1}{2} \\ m_1+1 & m_2-\frac{1}{2} & m_3-\frac{1}{2} \end{pmatrix} \end{aligned} \tag{14}$$

Corresponding to the third three-term recurrence relation for the ${}_3F_2(1)$:

$$\begin{aligned} {}_3F_2(A,\ B,\ C;\ D,\ E;\ 1) &= {}_3F_2(A,\ B+1,\ C;\ D+1,\ E;\ 1) \\ &- \frac{(D-B)AC}{(D+1)DE}\ {}_3F_2(A+1,\ B+1,\ C+1;\ D+2,\ E+1;\ 1) \end{aligned} \tag{15}$$

we have the third recurrence relation for the 3-j coefficient as:

$$[(j_2+m_2)(1+j_2-m_2)(1+j_3+m_3)]^{1/2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + (1-j_1+j_3-m_2)(j_3-m_3)^{1/2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2-1 & m_3+1 \end{pmatrix} = [(1-j_1+j_2+j_3)(j_1-m_1)(j_1+j_2-j_3)]^{1/2}\begin{pmatrix} j_1-\frac{1}{2} & j_2 & j_3+\frac{1}{2} \\ m_1+\frac{1}{2} & m_2-1 & m_3+\frac{1}{2} \end{pmatrix} \quad (16)$$

Corresponding to the fourth three-term recurrence relation for the ${}_3F_2(1)$:

$$ {}_3F_2(A,\ B,\ C;\ D,\ E;\ 1) = {}_3F_2(A,\ B+1,\ C;\ D,\ E+1;\ 1) - \frac{(E-B)AC}{(E+1)DE}\ {}_3F_2(A+1,\ B+1,\ C+1;\ D+1,\ E+2;\ 1) \quad (17)$$

we have the fourth recurrence relation for the 3-j coefficient as:

$$[(j_2+m_2)(1+j_1-j_2+j_3)(1+j_1+m_1)]^{1/2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + (1-j_2+j_3+m_1)(-j_1+j_2+j_3)^{1/2}\begin{pmatrix} j_1+\frac{1}{2} & j_2-\frac{1}{2} & j_3 \\ m_1+\frac{1}{2} & m_2-\frac{1}{2} & m_3 \end{pmatrix} = [(1+j_3-m_3)(j_1-m_1)(j_1+j_2-j_3)]^{1/2}\begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3+\frac{1}{2} \\ m_1+1 & m_2-\frac{1}{2} & m_3-\frac{1}{2} \end{pmatrix} \quad (18)$$

Corresponding to the fifth three-term recurrence relation for the ${}_3F_2(1)$:

$$ {}_3F_2(A,\ B,\ C;\ D,\ E;\ 1) = {}_3F_2(A,\ B,\ C+1;\ D+1,\ E;\ 1) - \frac{(D-C)AB}{(D+1)DE}\ {}_3F_2(A+1,\ B+1,\ C+1;\ D+2,\ E+1;\ 1) \quad (19)$$

we have the fifth recurrence relation for the 3-j coefficient as:

$$[(j_1+j_2-j_3)(1+j_3+m_3)(1-j_1+j_2+j_3)]^{1/2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + (1-j_1+j_3-m_2)(j_1+m_1)^{1/2}\begin{pmatrix} j_1-\frac{1}{2} & j_2 & j_3+\frac{1}{2} \\ m_1-\frac{1}{2} & m_2 & m_3+\frac{1}{2} \end{pmatrix} = [(1+j_2-m_2)(j_1-m_1)(j_2+m_2)]^{1/2}.\begin{pmatrix} j_1-\frac{1}{2} & j_2 & j_3+\frac{1}{2} \\ m_1+\frac{1}{2} & m_2-1 & m_3+\frac{1}{2} \end{pmatrix} \quad (20)$$

Corresponding to the sixth three-term recurrence relation for the ${}_3F_2(1)$:

$$\begin{aligned}&{}_3F_2(A,\ B,\ C;\ D,\ E;\ 1) = {}_3F_2(A,\ B,\ C+1;\ D,\ E+1;\ 1)\\ &-\frac{(E-C)AB}{(E+1)DE}.\ {}_3F_2(A+1,\ B+1,\ C+1;\ D+1,\ E+2;\ 1)\end{aligned} \tag{21}$$

we have the sixth three-term recurrence relation for the 3-j coefficient as:

$$\begin{aligned}&[(j_1+j_2-j_3)(1+j_3-m_3)(1+j_1-j_2+j_3)]^{1/2}\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}\\ &+(1-j_2+j_3+m_1)(j_2-m_2)^{1/2}\begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3+\frac{1}{2}\\ m_1 & m_2+\frac{1}{2} & m_3-\frac{1}{2}\end{pmatrix}\\ &=[(1+j_1+m_1)(j_1-m_1)(j_2+m_2)]^{1/2}\begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3+\frac{1}{2}\\ m_1+1 & m_2-\frac{1}{2} & m_3-\frac{1}{2}\end{pmatrix}\end{aligned} \tag{22}$$

5. Numerical verification

A numerical verification of the new recurrence relations has been done for all the six relations. To illustrate the methodology adopted, shown below are the details for the first recurrence relation (12), for

$$j_1 = 2, j_2 = 2, j_3 = 1, m_1 = 1, m_2 = -1, m_3 = 0.$$

Using the tables, Rotenberg et.al. [7], obtained for the lhs and the rhs of (12) the value: $-\frac{3}{\sqrt{15}}$. For these values of j_i, m_i, it has been found that for the second, third and fourth recurrence relations also the value is the same: $-\frac{3}{\sqrt{15}}$. However, for the fifth and sixth recurrence relations the value is different and it is: $-\frac{2}{\sqrt{10}}$.

6. Conclusion

In this article, we have derived six **new** recurrence relations for the 3-j coefficient, which are a direct consequence of the existing six recurrence relations for the ${}_3F_2(1)$ hypergeometric functions. Such recurrence relations are of significance and relevance in numerical computations of matrix elements of tensor-operators in Atomic, Molecular and Nuclear Physics studies.

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