J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 12, No. 1 (2024), pp. 43-60

DOI: 10.56827/JRSMMS.2024.1201.3 **ISSN (Online): 2582-5461**

ISSN (Print): 2319-1023

ERROR BOUNDS OF AN ABSOLUTELY CONTINUOUS FUNCTIONS BY ORTHOGONAL PROJECTION OPERATOR USING EXTENDED PSEUDO-CHEBYSHEV WAVELET SERIES

Susheel Kumar and Gaurav Kumar Mishra

Department of Mathematics,
Tilak Dhari P. G. College,
Jaunpur - 222002, Uttar Pradesh, INDIA

E-mail : susheel22686@rediffmail.com, gm0092@gmail.com

(**Received:** *Dec. 10, 2024* **Accepted:** *Dec. 28, 2024* **Published:** *Dec. 30, 2024)*

Abstract: This paper introduces a novel computational strategy devised to address the challenges encountered in approximation theory. The strategy revolves around the utilization of extended pseudo-Chebyshev wavelet approximations, a concept pioneered by Lal et al. in 2022, which is grounded in the method of pseudo-Chebyshev wavelets approximation. The paper meticulously delineates the methodology, along with an evaluation of error for a specific function. To showcase the efficacy and efficiency of the extended pseudo-Chebyshev wavelet approximation approach, significant discoveries are exemplified through a practical instance. Furthermore, the paper establishes the error of a function associated with the class of absolutely continuous functions using extended pseudo-Chebyshev wavelets via orthogonal projection operators, thereby affirming these estimators as notably more precise and theoretically optimal within the domain of wavelet analysis.

Keywords and Phrases: Absolute Continuity, Wavelets, Extended pseudo Chebyshev wavelets, Orthogonal projection operators.

2020 Mathematics Subject Classification: 40A30, 42C15, 42A16, 42C40, 65T60, 65L10, 65L60, 65R20.

1. Introduction

Wavelets, a relatively recent development originating in the 1980s, have witnessed a significant expansion in their scope, captivating numerous researchers such as Morlet et al.[29], Daubechies [10], Chui [8-9], Strang [39], Strang and Ngyuen [40], Natanson [30], Meyer [27], Daubechies and Lagarias [11], Walter [43-44], Islam et al. [13], Razzaghi and Yousefi [32], Mohammadi [28], Lal et al. [21], Lal and Kumar [19], Malmir [22-25], Venkatesh [42], Keshavarz et al. [14], and many others spanning both pure and applied mathematics. In addition to harmonic theory and Fourier analysis, wavelets have been influenced by fractals and approximation theory, driving their ongoing evolution. Similarly, a multitude of researchers, including Rehman and Siddiqi [33], Strang [39], Lal and kumar [18-20], Bastin [1], Biazar [4], Babolian [2-3], Kumar [15], Kumar et al. [17] , S. Kumar, A. K. Awasthi, S. K. Mishra et al. [16], and others, have explored the applications of wavelet theory, highlighting their efficacy as powerful tools in science and technology.

Orthogonal functions play a pivotal role in addressing a diverse array of problems spanning from differential and integral equations to approximation theory and dynamical systems. This approach involves harnessing orthogonal functions to streamline the original problems by transforming them into truncated approximations using orthogonal functions. Among these functions, Chebyshev polynomials $T_m(t)$, where $m \geq 0$ and $0 \leq t \leq 1$, stand out as particularly effective numerically, as underscored in several references [5-6, 26, 34-35, 37]. In December 2018, Ricci introduced pseudo-Chebyshev functions of fractional degree [36], with Cesarano and Ricci further delving into their significant properties such as orthogonality and more in their investigation [7]. In 2022, Shyam Lal and Susheel Kumr et al. [21], devloped the concept of pseudo chebyshev wavelet with the help of pseudo-Chebyshev functions of fractional degree [36]. In this paper, we generalized the concept of the pseudo-Chebyshev wavelet technique, into the extended pseudoChebyshev wavelet method using orthogonal projection operators

Fractals, characterized by their bounded, continuous, and nowhere-differentiable functions, encompass phenomena such as Brownian trajectories, fractional Brownian motion, typical Feynman paths, complex Bernoulli spirals, and turbulent fluid motion. This revelation has ignited curiosity in investigating the approximation of functions under absolute continuity using extended pseudo-Chebyshev wavelets and their practical implications. Yet, as of now, there seems to be no effort directed towards examining the error associated with functions related to absolute continuity utilizing the orthogonal projection operator using extended pseudo-Chebyshev wavelets.

This research paper introduces a novel approximation approach devised to evaluate the error of functions associated with absolute continuity using the orthogonal projection operator technique. The methodology relies on extended pseudo-Chebyshev wavelet approximation, utilizing the extended pseudo-Chebyshev wavelet to compute approximations for functions falling within absolute continuity.

The structure of the paper is organized as follows: Section 2 provides the preliminary findings and results. Section 3 presents the main convergence theorems related to wavelet approximations, along with an algorithm for the extended pseudo-Chebyshev wavelets, which is applied to solve problem of approximation. Section 4 includes numerical examples that discuss the approximations and applications of extended pseudo-Chebyshev wavelets. Section 5 presents the discussions and conclusions. Finally, the references cited in this study are listed.

2. Definitions and Preliminaries

2.1. Wavelets and Extended Multiresolution Analysis

Wavelets: An element $\psi \in L^2(\mathbb{R})$ is termed a fundamental wavelet when it meets the condition of 'admissibility', expressed as:

$$0 \leq C_\psi = \int_{-\infty}^{\infty} \frac{\left|\hat{\psi}(\omega)\right|^2}{|\omega|} d\omega < \infty \quad [8].$$

Wavelets comprise a collection $\{\psi_{j,k}; j,k \in \mathbb{Z}\}$ of functions generated through translation and dilation of a single fundamental wavelet ψ, often referred to as the mother wavelet. If the dilation parameter a and the translation parameter b vary continuously, then the ensuing set of continuous wavelets is represented as:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right), a \neq 0, b \in \mathbb{R} \quad [10].$$

Extended Multiresolution Analysis: A family of closed subspaces W_n^μ of $L^2(\mathbb{R}), n \in \mathbb{Z}$ is termed an Extended Multiresolution Analysis if it adheres to the following criteria [12]:

(i) $W_n^\mu \subset W_{n+1}^\mu$,

(ii) if $\xi(x) \in W_n^\mu$ then $\xi(2x) \in W_{n+1}^\mu$,

(iii) $\xi(x) \in W_0^\mu$ if and only if $\xi(x+1) \in W_0^\mu$,

(iv) $\overline{\bigcup_{n=-\infty}^{\infty} W_n^\mu} = L^2(\mathbb{R})$ and $\bigcap_{n=-\infty}^{\infty} W_n^\mu = \{0\}$,

(v) $\exists$ a function ϕ such that the set $\{\phi(x-j); j \in \mathbb{Z}\}$ is a Riesz basis of W_0^{μ}.

Given that $\psi \in L^2(\mathbb{R})$, where$\psi_{j,k} := \mu^{\frac{j}{2}}\psi(\mu^j - k)$ and

$$V_j^{\mu} := clos \left\langle \psi_{j,k} : k \in \mathbb{Z} \right\rangle,$$

this sequences of closed subspaces provides a direct sum decomposition of $L^2(\mathbb{R})$ gives a direct sum decomposition of a signals ξ i.e. $\xi \in L^2(\mathbb{R})$, implying that every $\xi \in L^2(\mathbb{R})$ possesses a unique decomposition as:

$$\begin{aligned} \xi &= \sum_{n\in\mathbb{Z}} \xi_n = \cdots + \xi_{-2} + \xi_{-1} + \xi_0 + \xi_1 + \xi_2 + \cdots, \\ \text{where} \quad \xi_n &\in V_n^{\mu} \text{ for all } \; n \in \mathbb{Z}, \end{aligned}$$

described as

$$L^2(\mathbb{R}) = W_n^{\mu} \oplus_{i=n}^{\infty} V_i^{\mu}, \text{ where } W_n^{\mu} := \oplus_{m=-\infty}^{n-1} V_m^{\mu}.$$

The set$\{\psi_{n,m};\ m \in \mathbb{Z}\}$ forms a Riesz basis of V_n^{μ}. Hence,

$$\xi = \sum_{n\in\mathbb{Z}} \langle \xi, \phi_{n,m} \rangle \phi_{n,m} + \sum_{k=m}^{\infty} \sum_{n=-\infty}^{\infty} \langle \xi, \psi_{n,k} \rangle \psi_{n,k} \quad [44].$$

2.2. Extended Pseudo Chebyshev Wavelets

An extended Pseudo Chebyshev wavelets is denoted by $\psi_{n,m}^{\mu} = \psi_{(k,n,m)}^{\mu}$, where $\mu \geq 2$, and given by

$$\psi_{n,m}^{\mu}(x) \; := \; \psi_{(k,n,m)}^{\mu}(x) = \begin{cases} \sqrt{\frac{4}{\pi}}\mu^{\frac{k-1}{2}} T_{m+1/2}(2\mu^{k-1}x - 2n + 1), \text{ for } \frac{n-1}{\mu^{k-1}} \leq x \leq \frac{n}{\mu^{k-1}}, \\ 0 \text{ otherwise, where } m \geq 0,\ n = 1, 2, 3, \cdots 2^{k-1} \text{ and } k \in \mathbb{N}. \end{cases}$$

It is remarkable to note that the set of extended pseudo Chebyshev wavelets $\left\{\psi_{n,m}^{\mu}\right\}$ is an orthonormal subset of $L_{\Omega}^2(\mathbb{R})$ with respect to the weight functions $\omega_{k,n}^{\mu}(x) = \omega(2\mu^{k-1}x - 2n + 1)$, where $\omega(x) = \frac{1}{\sqrt{1-x^2}}$, further detailed [21].

2.3. Orthogonal Projection Operators $P_n^{\mu}(f)$ using EPCW

An orthogonal projection operator $P_n^{\mu}(f)$ of a function $f \in L_{\Omega}^2(X)$ onto W_n^{μ} using EPCW, defined as [21]

$$P_n^{\mu}(f) \;=\; \sum_{m=0}^{\infty} \left\langle f, \psi_{n,m}^{\mu} \right\rangle_{\omega_{k,n}^{\mu}} \psi_{n,m}^{\mu}, \tag{1}$$

For each fixed, $n \in \mathbb{N}$, where $n = 1, 2, 3, \cdots, \mu^{k-1}$ & $\mu \geq 2$, the expansions

$$\begin{aligned}\left\langle P_n^\mu f, \psi_{n,m}^\mu \right\rangle &= \left\langle \sum_{m=0}^{\infty} \left\langle f, \psi_{n,m}^\mu \right\rangle_{\omega_{k,n}^\mu} \psi_{n,m}^\mu, \psi_{n,m}^\mu \right\rangle = \sum_{m=0}^{\infty} \left\langle f, \psi_{n,m}^\mu \right\rangle_{\omega_{k,n}^\mu} \left\langle \psi_{n,m}^\mu, \psi_{n,m}^\mu \right\rangle \\ &= \sum_{m=0}^{\infty} \left\langle f, \psi_{n,m}^\mu \right\rangle_{\omega_{k,n}^\mu} = \sum_{m=0}^{\infty} P_{k,n}^{m,\mu},\end{aligned}$$

are called n^{th} coefficients of an orthogonal projection operator for each fixed k, where $P_{k,n}^{m,\mu} = \int_\Omega f \psi_{n,m}^\mu \omega_{k,n}^\mu d\nu$ and each $n = 1, 2, 3, \cdots, \mu^{k-1}$, $k \in \mathbb{N}$, $\mu \geq 0$.

2.4. Extended Pseudo Chebyshev Wavelet Series

A function $f \in L^2_\Omega(X)$ is expanded by EPCWS as [16]:

$$\begin{aligned}f &= \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{\infty} \left\langle f, \psi_{n,m}^\mu \right\rangle_{\omega_{k,n}^\mu} \psi_{n,m}^\mu, \\ &= \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{\infty} P_{k,n}^{m,\mu} \psi_{n,m}^\mu \quad \text{for each fixed } k \in \mathbb{N}, \qquad (2)\end{aligned}$$

where $P_{k,n}^{m,\mu}$ is said to be $(n, m)^{th}$ coefficients of an orthogonal projection operator for each fixed $k \in \mathbb{N}$ of a function f corresponding the orthonormal wavelets $\psi_{n,m}^\mu$ of the wavelet series.

If the wavelet series of any function f is truncated by an orthogonal projection operators

$$\left(P_{(\mu^{k-1},M)} f\right) = \sum_{n=1}^{\mu^{k-1}} \sum_{m=0}^{M} P_{k,n}^{m,\mu} \psi_{n,m}^\mu$$

then, we say that the wavelet series has sum s, for each point, if the sequences of functions $\left\{P_{(\mu^{k-1},M)} f(x)\right\}_{M=0}^{\infty}$, for each fixed k & μ, uniformly converges to s(x) i.e.,

$$s = \lim_{M \to \infty} P_{\mu^{k-1},M}(f) \quad \text{for each fixed } \mu \geq 2 \ \ \& \ \ k, \ \ \mu^{k-1} << \infty.$$

It is denoted by $f \approx s$ say wavelet approximation of function f. The approximation is called best wavelet approximation, if $s(x) = f(x)$ for each points in the domain of functions.

2.5. Error of Wavelet Approximation

The error function $E^{\mu}_{\mu^{k-1},M}(f)$ of pseudo Chebyshev wavelet approximation of a signals belonging to $f \in L^2_{\Omega}(X)$ by the orthogonal projection operators $P^{\mu}_{\mu^{k-1},M}(f)$ is defined as [45], and given by

$$E^{\mu}_{\mu^{k-1},M}(f)(x) = \inf_{M} \sum_{n=1}^{\mu^{k-1}} \sum_{m=M}^{\infty} e_{n,m}\psi^{\mu}_{n,m}(x) = \inf_{P_{\mu^{k-1},M}(f)} \left(f(x) - P^{\mu}_{\mu^{k-1},M}(f)(x) \right).$$

where M is a non negative integers and $k \in \mathbb{N}$, $\mu \geq 2$.

If the error function $E^{\mu}_{\mu^{k-1},M}(f)$ of a signal f is uniformly converges to zero function then $P^{\mu}_{\mu^{k-1},M}(f)$ is called the best wavelet approximation of a signal $f \in L^2_{\Omega}(X)$ [45].

2.6. Functions of Absolute Continuity

Let (X, ζ, ν) be the measurable space with non negative measure ν and Ω is a finite measurable set in X. Then a function $f \in (X, \zeta, \nu)$ is said to be function of absolute continuity or absolutely continuous, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$x_i, y_i \in \Omega \text{ such that } \sum_{i=1}^{n} |x_i - y_i| < \delta \leq \mu(\Omega) \Rightarrow \sum_{i=1}^{n} |f_{x_i} - f_{y_i}| < \epsilon, [31].$$

2.7. Remarks

(i) Every Lipschitz function is absolutely continuous.

(ii) An absolutely continuous function need not be Lipschitz function.

(iii) A differentiable function with bounded derivative is a function of class of absolutely continuous.

(iv) The class of absolutely continuous function is a real/complex linear space.

2.8. Auxiliary Lemmas

Lemma 1. *A function f is a class of functions of bounded variation if and only if $f = f_1 - f_2$ where f_1 and f_2 are non decreasing monotonic functions.* Jordan Decomposition Theorem: [31]

Lemma 2. *Let f be a finite Lebesgue integrable function on the measurable space Ω and g be a non negative non decreasing monotonic function on Ω. Then $\exists\ \alpha \in \Omega$ such that*

$$\int_{\Omega} fg d\nu \leq g(\alpha) \int_{\Omega} f d\nu. \text{ Generalized Mean Value Theorem: [31]}$$

Lemma 3. *Let f be a finite Lebesgue integrable function on the measurable space Ω and g be a non negative non increasing monotonic function on Ω. Then $\exists\ \beta \in \Omega$ such that*

$$\int_{\Omega} fg d\nu \ \geq \ g(\beta) \int_{\Omega} f d\nu \text{ Generalized Mean Value Theorem: [31]}$$

Lemma 4. *Let τ be an integer and a $f : [\tau, \infty) \to \mathbb{R}$ be a real valued monotonic decreasing function. Then*

$$\int_{\tau}^{\infty} f d\nu \leq \sum_{\tau}^{\infty} f(n) \leq f(\tau) + \int_{\tau}^{\infty} f d\nu. \text{ Cauchy Integral Test, see [38]}$$

3. Main results

In this section, two new theorems have been established in the following forms:

Theorem 1. *If X and Ω be a measurable space and f is a signal of absolute continuous , then the error function $E_{\mu^{k-1},M}(f)$ of the signal f by orthogonal projection operators using pseudo-Chebyshev wavelet series* (2) *is uniformly converges to null signal.*

Proof of Theorem 1. Since,

$$\begin{aligned}
\|E^{\mu}_{\mu^{k-1},M}(f)\|_2^2 &= \inf_{P^{\mu}_{\mu^{k-1},M}(f)} \int_{\Omega} \left| f(t) - P^{\mu}_{\mu^{k-1},M}(f)(t) \right|^2 d\nu = \inf_M \left\| \sum_{n=1}^{\mu^{k-1}} \sum_{m=M}^{\infty} e^{\mu}_{n,m} \psi^{\mu}_{n,m}(t) \right\|^2 \\
&= \inf_M \left\langle \sum_{n_1=1}^{\mu^{k-1}} \sum_{m_1=M}^{\infty} e^{\mu}_{n_1,m_1} \psi^{\mu}_{n_1,m_1}(t), \sum_{n_2=1}^{\mu^{k-1}} \sum_{m_2=M}^{\infty} e^{\mu}_{n_2,m_2} \psi^{\mu}_{n,m}(t) \right\rangle \\
&= \inf_M \sum_{n=1}^{\mu^{k-1}} \sum_{m=M}^{\infty} e^{\mu}_{n,m} \overline{e^{\mu}_{n,m}} \left\langle \psi^{\mu}_{n,m}(t), \psi^{\mu}_{n_2,m_2}(t) \right\rangle \\
&\quad \text{for same values of } n_i = n, m_j = m,\ i,j = 1,2 \\
&+ \inf_M \sum_{n_1=1}^{\mu^{k-1}} \sum_{m_1=M}^{\infty} e^{\mu}_{n_1,m_1} \overline{\sum_{n_2=1}^{\mu^{k-1}} \sum_{m_2=M}^{\infty} e^{\mu}_{n_2,m_2}} \left\langle \psi^{\mu}_{n_1,m_1}(t), \psi^{\mu}_{n_2,m_2}(t) \right\rangle \\
&\quad \text{for different values of } n_i, m_j, i,j = 1,2 \\
&= \inf_M \int_{\Omega_{n,k}} \sum_{n=1}^{\mu^{k-1}} \sum_{m=M}^{\infty} \left| e^{\mu}_{n,m} \right|^2 \left|\left| \psi^{\mu}_{n,m} \right|\right|^2 d\nu \\
&\quad \text{for same values of } n_i = n, m_j = m,\ i,j = 1,2
\end{aligned}$$

$$+ \inf_M \int_{\Omega_{n,k}} \sum_{n_1=1}^{\mu^{k-1}} \sum_{n_2=1}^{\mu^{k-1}} \sum_{m_1=M}^{\infty} \sum_{m_2=M}^{\infty} e^{\mu}_{n_1,m_1} \overline{e^{\mu}_{n_2,m_2}} \psi^{\mu}_{n_1,m_1} \omega^{\mu}_{k,n_1} \overline{\psi^{\mu}_{n_2,m_2} \omega^{\mu}_{k,n_2}} d\nu$$

$$= \inf_M \int\limits_{\Omega_{n,k}} \sum_{n=1}^{\mu^{k-1}} \sum_{m=M}^{\infty} \left| e^{\mu}_{n,m} \right|^2 + 0, = \inf_M \sum_{n=1}^{\mu^{k-1}} \sum_{m=M}^{\infty} \left| e^{\mu}_{n,m} \right|^2 \nu\left(\Omega_{n,k}\right).$$

If f is a absolutely continuous, then by Lemma 1 there are two non negative monotonically increasing functions f_1 and f_2 such that $f = f_1 - f_2$. Thus

$$\begin{aligned} e^{\mu}_{n,m}(f) &= \int_{\Omega} f \psi^{\mu}_{n,m} \omega^{\mu}_{k,n} d\nu = \int_{\Omega} (f_1 - f_2)\, \psi^{\mu}_{n,m} \omega^{\mu}_{k,n} d\nu, \\ &= \int_{\Omega} f_1 \psi^{\mu}_{n,m} \omega^{\mu}_{k,n} d\nu - \int_{\Omega} f_2 \psi^{\mu}_{n,m} \omega^{\mu}_{k,n} d\nu = e^{\mu}_{n,m}(f_1) - e^{\mu}_{n,m}(f_2). \end{aligned} \quad (3)$$

Now

$$\begin{aligned} e^{\mu}_{n,m}(f_1) &\le f_1\left(\frac{n}{\mu^{k-1}}\right) \int\limits_{\frac{n-1}{\mu^{k-1}}}^{\frac{n}{\mu^{k-1}}} \sqrt{\frac{4}{\pi}} \mu^{\frac{k-1}{2}} T_{m+\frac{1}{2}}(2\mu^{k-1}t - 2n + 1) \omega^{\mu}_{k,n}(t) dt, \text{ by Lemma 2,} \\ &= \sqrt{\frac{4}{\pi}} \frac{1}{2\mu^{\frac{k-1}{2}}} f\left(x_i\right) \int\limits_{-1}^{1} T_{m+\frac{1}{2}}(x) \omega(x) dx, \text{where } f(x_i) = \sup_{t \in \overline{\Omega}_{k,n}} f(t) \ge f_1\left(\frac{n}{2^{k-1}}\right) \\ &= \sqrt{\frac{4}{\pi}} \frac{1}{2\mu^{\frac{k-1}{2}}} f(x_i) \int\limits_{0}^{\pi} cos((m+1/2)x) dx, \\ &= \sqrt{\frac{4}{\pi}} \frac{1}{2\mu^{\frac{k-1}{2}}} \frac{(-1)^m}{m+1/2} f(x_i). \end{aligned}$$

Similarly,

$$e^{\mu}_{n,m}(f_2) = \int\limits_{\frac{n-1}{\mu^{k-1}}}^{\frac{n}{\mu^{k-1}}} f_2(t) \psi^{\mu}_{n,m}(t) \omega^{\mu}_{k,n}(t) dt$$

$$\ge f_2\left(\frac{n-1}{\mu^{k-1}}\right) \int\limits_{\frac{n-1}{\mu^{k-1}}}^{\frac{n}{\mu^{k-1}}} \sqrt{\frac{4}{\pi}} \mu^{\frac{k-1}{2}} T_{m+\frac{1}{2}}(2\mu^{k-1}t - 2n + 1) \omega^{\mu}_{k,n}(t) dt, \text{ by Lemma 3,}$$

$$\geq \quad f(y_i) \int_{\frac{n-1}{\mu^{k-1}}}^{\frac{n}{\mu^{k-1}}} \sqrt{\frac{4}{\pi}} \mu^{\frac{k-1}{2}} T_{m+\frac{1}{2}}(2\mu^{k-1}t - 2n + 1)\omega^{\mu}_{k,n}(t)dt,$$

$$\text{where } f(y_i) = \inf_{t\in\overline{\Omega}_{k,n}} f(t) \leq f\left(\frac{n-1}{2^{k-1}}\right)$$

$$= \quad \sqrt{\frac{4}{\pi}} \frac{1}{2\mu^{\frac{k-1}{2}}} \frac{(-1)^m}{m+1/2} f(y_i).$$

By eq(3), we have

$$\begin{aligned} e^{\mu}_{n,m}(f) &\leq \sqrt{\frac{4}{\pi}} \frac{1}{2\mu^{\frac{k-1}{2}}} \frac{(-1)^m}{m+1/2} \left(f(x_i) - f(y_i)\right) \text{ where } x_i, y_i \in \overline{\Omega}_{k,n}, \\ &\leq \sqrt{\frac{4}{\pi}} \frac{1}{2\mu^{\frac{k-1}{2}}} \frac{(-1)^m}{m+1/2} \sum_{i=1}^{n} \left|f(x_i) - f(y_i)\right|, \text{where } x_i, y_i \in \Omega, \qquad (4) \\ &\quad \text{for each fixed } \ k \in \mathbb{N} \text{ and } n = 1, 2, 3, \cdots, \mu^{k-1}, \ \mu \geq 2. \end{aligned}$$

Now,

$$\begin{aligned} \|E_{\mu^{k-1},M}(f)\|_2^2 &= \inf_M \sum_{n=1}^{\mu^{k-1}} \sum_{m=M}^{\infty} \left|e^{\mu}_{n,m}\right|^2 \nu\left(\Omega_{n,k}\right), \\ &\leq \inf_M \sum_{n=1}^{\mu^{k-1}} \sum_{m=M}^{\infty} \left|\sqrt{\frac{4}{\pi}} \frac{1}{2\mu^{\frac{k-1}{2}}} \frac{(-1)^m}{m+1/2} \sum_{i=1}^{n} |f(x_i) - f(y_i)|\right|^2 \nu(\Omega_{n,k}), \\ &\leq \inf_M \sum_{n=1}^{\mu^{k-1}} \sum_{m=M}^{\infty} \left|\sqrt{\frac{4}{\pi}} \frac{1}{2\mu^{\frac{k-1}{2}}} \frac{(-1)^m}{m+1/2} \sum_{i=1}^{n} |f(x_i) - f(y_i)|\right|^2 \frac{1}{\mu^{k-1}} \\ &= \frac{1}{\pi} \sum_{n=1}^{\mu^{k-1}} \frac{1}{\mu^{(k-1)}} \left|\sum_{i=1}^{n} |f(x_i) - f(y_i)|\right|^2 \inf_M \sum_{m=M}^{\infty} \frac{1}{(m+1/2)^2} \frac{1}{\mu^{k-1}} \\ &= \frac{1}{\pi} \frac{1}{\mu^{(k-1)}} \left|\sum_{i=1}^{n} |f(x_i) - f(y_i)|\right|^2 \inf_M \sum_{m=M}^{\infty} \frac{1}{(m+1/2)^2}. \end{aligned}$$

Then,

$$\|E_{\mu^{k-1},M}(f)\|_2^2 \leq \frac{1}{\pi\mu^{k-1}} \left|\sum_{i=1}^{n} |f(x_i) - f(y_i)|\right|^2 \left(\frac{1}{(M+1/2)^2} + \int_{M}^{\infty} \frac{dx}{(x+1/2)^2}\right),$$

by Lemma 4

$$= \frac{1}{\pi\mu^{k-1}}\left|\sum_{i=1}^{n}|f(x_i)-f(y_i)|\right|^2\left(\frac{1}{(M+1/2)^2}+\frac{1}{(M+1/2)}\right). \tag{5}$$

By the eq(5) and definition of absolute continuity, $0<\frac{\epsilon'}{\sqrt{\mu^{(k-1)/2}\ \pi\ \ (M+1/2)}}<\epsilon,$ for each fixed $k\in\mathbb{N}$ & $M\geq 0$,

$$\begin{aligned} x_i,y_i\ \in\Omega_{k,n},\ \sum_{i=1}^{n}|x_i-y_i|<\delta\leq\mu(\Omega_{k,n}) &\Rightarrow \sum_{i=1}^{n}|f(x_i)-f(y_i)|<\epsilon' \\ &\Rightarrow \|E_{2^{k-1},M}(f)\|_2^2<\frac{4\ \epsilon'^2}{\pi}\frac{1}{\mu^{k-1}\ \ (M+1/2)} \\ &\Rightarrow \|E_{\mu^{k-1},M}(f)\|_2<\frac{\epsilon'}{\sqrt{\mu^{(k-1)}/2\ \pi\ \ (M+1/2)}} \\ &\Rightarrow \|E_{\mu^{k-1},M}(f)\|_2<\epsilon. \end{aligned}$$

Therefore, for each $\epsilon>0\ \ \exists\,\delta>0$ such that

$$\forall\ x,y\ \in\Omega\Rightarrow\|E_{\mu^{k-1},M}(f)\|_2<\epsilon.$$

Thus the establishment of Theorem 1 is now complete.

Theorem 2. *Let f be a absolutely continuous signal where $\Omega=(0,1]$ & $X=\mathbb{R}$ and the pseudo-Chebyshev wavelet series of the function f an order one i.e. $k=1$ is given by*

$$\sum_{m=0}^{\infty}P_{1,1}^{m,\mu}\psi_{1,m}^{\mu}=P_{1,1}^{0,\mu}\psi_{1,0}^{\mu}+P_{1,1}^{1,\mu}\psi_{1,1}^{\mu}+P_{1,1}^{2,\mu}\psi_{1,2}^{\mu}+\cdots.$$

Then there exist $M\in\mathbb{N}$ such that $P_{1,1}^{m,\mu}(f)=0,$ for each non negative integers $m>M$.

Proof of Theorem 2

The proof for Theorem 2 can be constructed following the same reasoning used for proving Theorem 1 with consideration for the class of absolutely continuity signals taking $k=1,n=1$ in Theorem 1.

4. Illustrative Example

In this section, we calculate the approximation of a function

$$f(\tau)=\begin{cases} 2x^{1/2}-4x^{3/2}+8x^{5/2}+9x^{7/2}-15x^{9/2};\ x\in\Omega, \\ 0;\ x\notin\Omega. \end{cases}$$

by the pseudo-Chebyshev wavelet approximation method.

Put in Theorem 1, $k=1, n=1$ & $\mu=2$, we have

$$P^{\mu}(f)(x)=\sum_{m=0}^{\infty}\left\langle f,\psi^{\mu}_{1,m}\right\rangle_{\omega_{1,1}}\psi^{\mu}_{1,m}(x)=\sum_{m=0}^{\infty}f^{1,m}\psi^{\mu}_{1,m}(x)=f_0,$$

then we say that $f\approx f_0$ by the orthogonal projection operators $P^{\mu}_{n}(f)$ of an order $k=1$.

The calculated values of the projection operators and its errors $P^{\mu}_{1,1}$, $P^{\mu}_{1,2}$, $P^{\mu}_{1,3}$, $P^{\mu}_{1,4}$, $P^{\mu}_{1,5}$, $P^{\mu}_{1,6}$, $P^{\mu}_{1,7}$, $E^{\mu}_{1,1}$, $E^{\mu}_{1,2}$, $E^{\mu}_{1,3}$, $E^{\mu}_{1,4}$ $E^{\mu}_{1,5}$ $E^{\mu}_{1,6}$ $E^{\mu}_{1,7}$, i.e. $P^{\mu}_{1,M}$ & $E^{\mu}_{1,M}$ for $1\le M\le 7$ & $\mu\ge 2$, are given by Table 1 .

x	0.0000	0.1000	0.2000	0.3000	0.4000	0.5000	0.6000	0.7000	0.8000	0.9000	1.0000
f_x	0.0000	0.5336	0.7012	0.8991	1.1840	1.5468	1.9210	2.1799	2.1323	1.5174	0.0000
$P^{\mu}_{1,1}f_x$	0.0000	0.4867	0.6883	0.8430	0.9734	1.0883	1.1922	1.2877	1.3766	1.4601	1.5391
$E^{\mu}_{1,1}f_x$	0.0000	0.0469	0.0129	0.0561	0.2106	0.4585	0.7288	0.8922	0.7557	0.0573	1.5391
$P^{\mu}_{1,2}f_x$	0.0000	0.8721	1.1495	1.3051	1.3884	1.4197	1.4100	1.3661	1.2927	1.1933	1.0703
$E^{\mu}_{1,2}f_x$	0.0000	0.3385	0.4482	0.4060	0.2045	0.1271	0.5110	0.8138	0.8396	0.3242	1.0703
$P^{\mu}_{1,3}f_x$	0.0000	0.2475	0.6911	1.1545	1.5624	1.8617	2.0103	1.9727	1.7176	1.2170	0.4453
$E^{\mu}_{1,3}f_x$	0.0000	0.2861	0.0101	0.2554	0.3784	0.3149	0.0893	0.2072	0.4147	0.3004	0.4453
$P^{\mu}_{1,4}f_x$	0.0000	0.5479	0.6510	0.8478	1.1769	1.5882	1.9792	2.2082	2.1022	1.4606	0.0586
$E^{\mu}_{1,4}f_x$	0.0000	0.0143	0.0503	0.0513	0.0071	0.0414	0.0582	0.0283	0.0301	0.0568	0.0586
$P^{\mu}_{1,5}f_x$	0.0000	0.5336	0.7012	0.8991	1.1840	1.5468	1.9210	2.1799	2.1323	1.5174	0.0000
$E^{\mu}_{1,5}f_x$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$P^{\mu}_{1,6}f_x$	0.0000	0.5336	0.7012	0.8991	1.1840	1.5468	1.9210	2.1799	2.1323	1.5174	0.0000
$E^{\mu}_{1,6}f_x$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$P^{\mu}_{1,7}f_x$	0.0000	0.5336	0.7012	0.8991	1.1840	1.5468	1.9210	2.1799	2.1323	1.5174	0.0000
$E^{\mu}_{1,7}f_x$	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Table 1: Compare between truncated $f_0=\sum_{m=0}^{M-1}P^{m,\mu}_{f}\psi^{\mu}_{m}$ and exact function f

$$\begin{aligned}P^{\mu}(f) &= \sum_{m=0}^{\infty}P^{m,\mu}_{1,1}\psi^{\mu}_{1,m}\\ &\approx 1.4747\ \psi^{\mu}_{1,0}-0.4708\ \psi^{\mu}_{1,1}-0.6093\ \psi^{\mu}_{1,2}-0.3427\ \psi^{\mu}_{1,3}\\ &\quad -0.0519\ \psi^{\mu}_{1,4}+0+\cdots+0,\\ &\approx f_0^{(1,5)}=f_0=\sum_{m=0}^{5-1}P^{m,\mu}_{f}\psi^{\mu}_{m},\end{aligned}$$

and

$$E^{\mu}_{M}(f) = \inf_{P^{\mu}_{1,M}(f)}\|P^{\mu}_{1,M}(f)-f\|_2=\inf_{M}\sum_{m=M}^{\infty}\left\langle f,\psi^{\mu}_{1,m}\right\rangle_{\omega^{\mu}_{1,1}}\psi^{\mu}_{1,m}\approx 0,\ \text{for } M\ge 5.$$

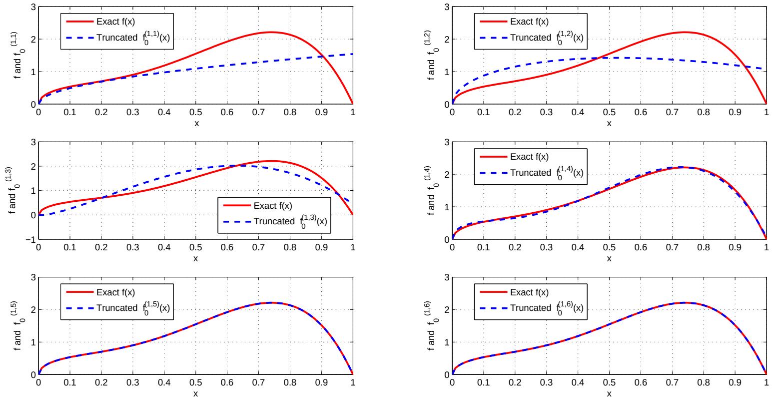

Figure 1: Graph of f and $f_0 = \sum_{m=o}^{M-1} f^{1,m}\psi^{\mu}_{1,m}$.

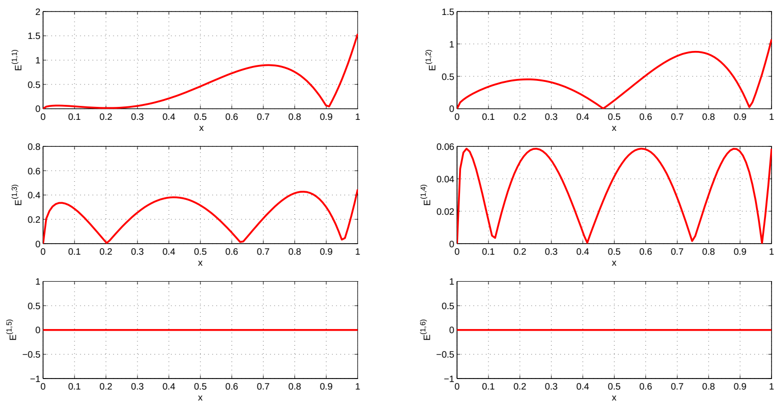

Figure 2: Graph of error functions $E^{\mu}_{1,M}f$.

5. Result Discussion and Conclusions

Since, by Theorems 1 an error functions $E_{\mu^{k-1},M}(f)$ and $E_M(f)$ of order k, & $k = 1$ respectively, by the Extended Pseudo-Chebyshev wavelet method using orthogonal projection operators, P_n, $n = 1, 2, 3, \cdots, \mu^{k-1}$, $\mu \geq 2$, are

$$0 \leq \| E^{\mu}_{\mu^{k-1},M}(f) \|_2 < \frac{\epsilon'}{\sqrt{\mu^{(k-1)}/2\ \pi\ \ (M + 1/2)}} \to 0, \text{ as } k \to \infty \text{ or } M \to \infty, \text{and}$$

$$0 \leq \| E^{\mu}_{M}(f) \|_2 < \frac{\epsilon'}{\sqrt{(\pi/2)\ \ (M + 1/2)}} \to 0 \text{ as } M \to \infty, \text{and}$$

$$\lim_{M\to\infty} E^{\mu}_{\mu^{k-1},M} = 0, \quad \lim_{M\to\infty} E^{\mu}_{M} = 0 \text{ and } P^{m,\mu}_{f} = 0, \ m \geq 5.$$

Therefore the finding in Theorems 1 and 2, by Extended Pseudo-Chebyshev wavelet approximations obtained from these discoveries signify the pinnacle achievements in wavelet analysis [45]. Moreover, the numerical findings illustrated in Table 1 and Figure 1, coupled with the absolute error depicted in Table 1 and Figure 2, further confirm the efficacy of this approach in effectively resolving the issue.

Acknowledgement

Authors are grateful to anonymous learned referees and all the editorial board members, for their exemplary guidance, valuable feedback and constant encouragement which improve the quality and presentation of this paper. I, Susheel Kumar, one of the author, am thankful to my supervisor, Prof. Shyam Lal, for his incomparable and inexpressible guidance.

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