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EXPONENTIAL STABILITY OF NON-UNIFORM EULER-BERNOULLI BEAM WITH A INDEFINITE DAMPING UNDER A FORCE CONTROL IN VELOCITY AND ANGULAR VELOCITY

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Abstract: In this paper we study the Riesz basis property and the exponential stability of a damped Euler-Bernoulli beam system with variables coefficients. The beam is clamped at one end and controlled at the free end by a force control in velocity and angular velocity. The exponential stability of the system is obtained using the Riesz basis approach.

Keywords and Phrases: Euler-Bernoulli beam, C_0-semigroups, exponential stability, Riesz basis.

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1. Introduction

We study the fundamental Riesz Basis Property and the exponential stability of a damped flexible Euler-Bernoulli beam. The beam is clamped at one end and controlled at the free end by a control force in velocity and angular velocity. The vibrations are described by the following system :

$$m(x)y_{tt} + (EI(x)y_{xx})_{xx} + \gamma(x)y_t = 0, \quad 0 < x < 1, t > 0, \tag{1}$$

$$y(0,t) = y_x(0,t) = 0, \quad t > 0, \tag{2}$$

$$EI(1)y_{xx}(1,t) = -\beta y_{xt}(1,t) \quad t > 0, \tag{3}$$

$$(EI(.)y_{xx}(.,t))_x(1) = \alpha y_t(1,t) \quad t > 0, \tag{4}$$

with initial conditions

$$y(x,0) = y_0(x), \qquad 0 \leqslant x \leqslant 1, \tag{5}$$

$$y_t(x,0) = y_1(0), \qquad 0 \leqslant x \leqslant 1, \tag{6}$$

where α, β are two positive constants. $y(x,t)$ stands for a transversal deviation of the beam at position x and time t; a subscript letter denotes the partial derivation with respect that variable. The length of the beam is chosen to be unity, $EI(x)$ is the stiffness of the beam, $m(x)$ is the mass density and $\gamma(.)$ is a continuous coefficient function of feedback damping. The Riesz basis property means that the eigenvectors of the system form a Riesz basis for the Hilbert space, the state space, which is one of the fundamental properties of linear vibrating systems.

Our work is based on the study in [1] where the authors studied the above system without damping and numerically obtained exponential stability. In this work, we will use the same approach as in [17], where the authors showed the exponential stability of such a system, but with variable coefficients subjected to force controls in position and velocity. In [17] and [4], the authors used the method of Wang to show the exponential stability of the systems studied.

Different recent methods have been used to study several types of Euler Bernoulli beam system see Kumarasamy [12] and Hasanov ([9, 10]). We adapt the asymptotic technique in our study. There are two main steps in the study of systems with variable coefficients. The first step is to transform the first equation of the system to be studied into a uniform equation by a successive transformation of space, and state. The second step is to determine an asymptotic expression of the values of the system using these uniform equations. This basic idea comes mainly from the work of Birkhoff [2] and [3]. This approach has been used to study Euler-Bernoulli beam equations with variable coefficients (See B.Z Guo [6], Guo and Wang [7], J. M. Wang [18] or J. M. Wang, G. Q. Xu and S. P. Yung [21]).

We use a result due to Wang and al. [19], which studies the problem at the eigenvalues of the beam in the form of an equation ordinary differential $L(f) = \lambda f$

with boundary conditions λ-polynomial . In our work we disrupted the system studied in [1] by adding velocity and angular velocity controls. The content of this work is as follows :

In Section 2, we convert the system (1)-(4) into an evolution problem in an appropriate Hilbert space, and then prove that the system is associated with a C_0-semigroup of linear operators with a compact solvent generator. The problem is thus well formulated. Asymptotic expressions for eigenvalues and eigenfunctions are also given. Sections 3 and 4 are devoted to proving the fundamental Riesz Basis Property and the exponential stability of the system (1)-(4), respectively.

2. Eigenvalue problem

We introduce the following usual Sobolev spaces:

$$H_E^2(0,1) = \{u \in H^2 : u(0) = u_x(0) = 0\}.$$

Consider Hilbert space

$$\mathcal{H} = H_E^2(0,1) \times L^2(0,1)$$

Spaces $L^2(0,1)$ and $H^k(0,1)$ are defined as follows:

$$L^2(0,1) = \left\{u : [0,1] \to \mathcal{C} \mid \int_0^1 \mid u \mid^2 dx < \infty\right\}$$

$$H^k(0,1) = \left\{u : [0,1] \to \mathcal{C} \mid u, u^{(1)}, ..., u^{(k)} \in L^2(0,1)\right\}$$

Either $w = (f_1, f_2)^T$ et $v = (g_1, g_2)^T$ elements of $\mathcal{H}$.

Hilbert's Space $\mathcal{H} = H_E^2(0,1) \times L^2(0,1)$ is therefore provided with the inner-product

$$\langle w, v\rangle = \int_0^1 m(x) f_2(x)\overline{g_2(x)}dx + \int_0^1 EI(x) f_1''(x)\overline{g_1''(x)}dx, \tag{7}$$

and $\|.\|_{\mathcal{H}}$ is its associated norm.

Either A_γ an unbounded linear operator and $D(A_\gamma)$ its domain of operator. We have:

$$D\left(A_\gamma\right) = \left\{ \begin{array}{l} \{(f,g)^T \in (H^4(0,1) \cap H_E^2(0,1)) \times H_E^2(0,1) : \\ -E(1)f''(1) = \beta g'(1), \ \ (EI\left(.\right) f''\left(.\right))'(1) = \alpha g(1) \end{array} \right\}. \tag{8}$$

Knowing that:

$$m(x)y_{tt} + (EI(x)y_{xx})_{xx} + \gamma(x)y_t = 0.$$

We draw

$$y_{tt} = -\frac{1}{m(x)}((EI(x)y_{xx})_{xx} + \gamma(x)y_t).$$

Then: Set $z = (y, y_t)$ which implies that $z_t = (y_t, y_{tt})$ then
$z_t = (y_t, -\frac{1}{m(x)}((EI(x)y_{xx})_{xx} + \gamma(x)y_t)$
set

$$A_\gamma(f,g)^T = \left(g(x), -\frac{1}{m(x)}((EI(x)f''(x))' + \gamma(x)g(x))\right)^T$$

We can now write the system $(1)-(4)$ as a first-order evolution problem. of first order

$$\begin{cases} \frac{d}{dt}z(t) = \mathcal{A}_\gamma z(t) \\ z(0) = z_0 \in \mathcal{H} \quad \text{initial condition,} \end{cases} \tag{9}$$

where $z(t) = (y, y_t)^T$, $z(0) = (y_0, y_1)^T$.
In [1] the authors have shown that the operator A_0 denotes the undamped case $\gamma(x) = 0$ is m-dissipative and therefore generates a semigroup of contractions. Set

$$\Gamma_\gamma(f,g)^T = A_\gamma(f,g)^T - A_0(f,g)^T = (0, \frac{-\gamma(x)g(x)}{m(x)})^T \ \forall (f,g) \in D(A_\gamma).$$

$$A_\gamma = \Gamma_\gamma + A_0$$

Theorem 2.1. *Let A_γ and A_0 be the operators defined above. A_0 is m-dissipative and generates a C_O-semigroup on H denoted by $\{T(t)\}_{t\geq 0}$ from which A_γ generates a C_0-semigroup $\{e^{A_\gamma t}\}_{t\geq 0}$ on H denoted by $\{S(t)\}_{t\geq 0}$.*
Proof. Γ_γ is a bounded linear operator on H.
For any $(f,g)^T \in D(A_\gamma)$ we have:

$$\langle \Gamma_\gamma(f,g), (f,g)\rangle = \langle (0, -\frac{\gamma(x)}{m(x)}g(x))^T, (f,g)^T\rangle.$$

$$\langle \Gamma_\gamma(f,g), (f,g)\rangle = -\int_0^1 \frac{\gamma(x)}{m(x)}|g(x)|^2 dx.$$

As $\gamma(.)$ is a continuous function on $[0,1]$, we have:

$$|\langle \Gamma_\gamma(f,g), (f,g)\rangle|\langle \leq M(\int_0^1 [|g(x)|^2 + |f''(x)|^2]dx)$$

$$M = \sup_{x\in[0,1]} |\frac{\gamma(x)}{m(x)}|$$

Where from

$$|\langle \Gamma_\gamma(f,g), (f,g)\rangle|\langle \leq M\|(f,g)^T\|^2_{\mathcal{H}}.$$

So Γ_γ is a bounded operator on H.
Since the operator A_0 generates a C_0-semigroup of contractions noted $\{T(t)\}_{t\geq 0} = \{e^{A_0 t}\}_{t\geq 0}$ on H (see [1]), such as

$$\| T(t) \| \leq Ce^{\omega t}. \quad for \quad M, \omega \in \mathcal{R} \quad and \quad t \in \mathcal{R}_+$$

and that Γ_γ is a bounded operator on H.
Then $A_\gamma = \Gamma_\gamma + A_0$ is a generator of a C_0-semigroup of contractions denoted $\{S(t)\}_{t\geq 0} = \{e^{A_\gamma t}\}_{t\geq 0}$ on H, such as

$$\| S(t) \| \leq Ce^{(\omega + C\|\Gamma_\gamma\|)t} \quad for \quad M, \omega \in \mathcal{R} \quad and \quad t \in \mathcal{R}_+$$

And this, thanks to the perturbation theorem of a generator of a semigroup by a bounded linear operator. (See Theorem 1 ([14])).

Next, let's show that the spectrum $\sigma(A_\gamma)$ consists entirely of isolated eigenvalues.

Theorem 2.2. *The operator A_γ has a compact resolvent and $0 \in \rho(A_\gamma)$. Therefore the spectrum $\sigma(A_\gamma)$ consists entirely of isolated eigenvalues.*
Proof. Clearly we only need to prove that $0 \in \rho(A_\gamma)$ and that A_γ^{-1} is compact on H. For $\psi = (g_1, g_2) \in H$ we look for a unique $\phi = (f_1, f_2) \in D(A_\gamma)$ such as : $A_\gamma \phi = \psi$
In other words, such that the following equations are verified:

$$f_2(x) = g_1(x) \quad g_1 \in H_E^2(0,1) \tag{10}$$

$$-\frac{1}{m(x)}((EI(x)f_1(x)''(x))' + \gamma(x)f_2(x)) = g_2(x) \tag{11}$$

$$f_1(0) = f_1^{'}(0) = 0 \tag{12}$$

$$EI(1)f_1^{''}(1) = -\beta f_2'(1) \tag{13}$$

$$EI(1)f_1^{'''}(1) = \alpha f_2(1) \tag{14}$$

Using the equation (11) :

$$-EI(x)f_1^{(4)}(x) - \gamma(x)f_2(x) = m(x)g_2(x)$$

we have :

$$f_1^{(4)}(x) = -\frac{\gamma(x)}{m(x)} f_2(x) - \frac{m(x)}{EI(x)} g_2(x)$$

By integration we obtain for all $0 \leq x \leq 1$

$$\int_x^1 f_1^{''''}(r)dr = -\int_x^1 [-\frac{\gamma(r)}{m(r)} f_2(r) - \frac{m(r)}{EI(r)} g_2(r)]dr$$

Which give:

$$f_1^{'''}(1) - f_1^{'''}(x) = -\int_x^1 [-\frac{\gamma(r)}{m(r)} f_2(r) - \frac{m(r)}{EI(r)} g_2(r)]dr$$

From equation (4) we obtain:

$$\frac{\alpha}{EI(1)} f_2(1) - f_1^{'''}(x) = -\int_x^1 [-\frac{\gamma(r)}{m(r)} f_2(r) - \frac{m(r)}{EI(r)} g_2(r)]dr$$

as

$$f_2(x) = g_1(x)$$

$$f_1^{'''}(x) = -\int_x^1 [-\frac{\gamma(r)}{m(r)} f_2(r) - \frac{m(r)}{EI(r)} g_2(r)]dr + \frac{\alpha}{EI(1)} g_1(1)$$

By a new integration we have:

$$\int_x^1 f_1^{'''}(\eta)d\eta = \int_x^1 \int_\eta^1 [-\frac{\gamma(r)}{m(r)} f_2(r) - \frac{m(r)}{EI(r)} g_2(r)]drd\eta + \int_x^1 \frac{\alpha}{EI(1)} g_1(1)d\eta$$

$$f_1^{''}(1) - f_1^{''}(x) = \int_x^1 \int_\eta^1 [-\frac{\gamma(r)}{m(r)} f_2(r) - \frac{m(r)}{EI(r)} g_2(r)]drd\eta + \frac{\alpha}{EI(1)} g_1(1)[\eta]_x^1$$

using equation (3) we have:

$$-\frac{\beta}{EI(1)} f_2'(1) - f_1''(x) = \int_x^1 \int_\eta^1 [-\frac{\gamma(r)}{m(r)} f_2(r) - \frac{m(r)}{EI(r)} g_2(r)]drd\eta + \frac{\alpha}{EI(1)} g_1(1)(1-x)$$

$$f_1''(x) = -\int_x^1 \int_\eta^1 \left[-\frac{\gamma(r)}{m(r)} f_2(r) - \frac{m(r)}{EI(r)} g_2(r)\right] drd\eta$$
$$- \frac{\alpha}{EI(1)} g_1(1)(1-x) - \frac{\beta}{EI(1)} g_1'(1)$$

$$\int_0^x f_1^{''}(\xi)d\xi = -\int_0^x \int_\xi^1 \int_\eta^1 \left[-\frac{\gamma(r)}{m(r)}f_2(r) - \frac{m(r)}{EI(r)}g_2(r)\right] drd\eta d\xi - \frac{\alpha}{EI(1)}g_1(1)\int_0^x (1-\xi)d\xi - \frac{\beta}{EI(1)}g_1'(1)\int_0^x d\xi$$

$$f_1'(x) - f_1'(0) = -\int_0^x \int_\xi^1 \int_\eta^1 \left[-\frac{\gamma(r)}{m(r)}f_2(r) - \frac{m(r)}{EI(r)}g_2(r)\right] drd\eta d\xi - \frac{\alpha}{EI(1)}g_1(1)\int_0^x (1-\xi)d\xi - \frac{\beta}{EI(1)}g_1'(1)\int_0^x d\xi$$

From equation (2) we obtain:

$$\int_0^s f_1^{'}(x)dx = -\int_0^s \int_0^x \int_\xi^1 \int_\eta^1 [-\frac{\gamma(r)}{m(r)}f_2(r) - \frac{m(r)}{EI(r)}g_2(r)]drd\eta d\xi dx - \frac{\alpha}{EI(1)}g_1(1)\int_0^s \int_0^x (1-\xi)d\xi dx - \frac{\beta}{EI(1)}g_1'(1)\int_0^s \int_0^x d\xi dx$$

$$f_1(x) = -\int_0^x \int_0^s \int_\xi^1 \int_\eta^1 \left[-\frac{\gamma(r)}{m(r)}f_2(r) - \frac{m(r)}{EI(r)}g_2(r)\right] drd\eta d\xi ds - \frac{\alpha}{EI(1)}g_1(1)\int_0^x \int_0^s (1-\xi)d\xi ds - \frac{\beta}{EI(1)}g_1'(1)\int_0^x \int_0^s d\xi ds$$

Then,

$$(f_1, f_2)^T \in D(A_\gamma)$$

Therefore

$$F = (f_1(x), f_2(x))^T = A_\gamma^{-1}G = (B(x), g_1(x))^T$$

with

$$B(x) = -\int_0^x \int_0^s \int_\xi^1 \int_\eta^1 \left[-\frac{\gamma(r)}{m(r)}f_2(r) - \frac{m(r)}{EI(r)}g_2(r)\right] drd\eta d\xi ds - \frac{\alpha}{EI(1)}g_1(1)(\frac{1}{2}x^2 - \frac{1}{6}x^3) - \frac{\beta}{EI(1)}g_1'(1)x^2$$

Hence

$$F = (f_1(x), f_2(x))^T = A_\gamma^{-1}G = (B(x), g_1(x))^T$$

Eventually A_γ^{-1} exists, hence $0 \in \rho(A_\gamma)$. Then Sobolev's injection theorem implies that A_γ^{-1} is a compact operator on H.
Therefore the spectrum $\sigma(A_\gamma)$ consists entirely of isolated eigenvalues.

Our work shall make use of the following result from Wang [19], which deals with the eigenvalue problem of beams in the form of an ordinary differential equation $L(u) = \lambda u$ with λ polynomial boundary conditions see Shkalikov [15]; Tretter [16]. Since the operator A_γ is m-dissipative, so the operator $I - A_\gamma$ is an isomorphism to $D(A_\gamma)$ on $\mathcal{H}$ and the resolvent of A_γ is linear continuous operator on $\mathbb{H}$, then Sobolev's embedding theorem implies that A_γ has compact resolvent. Therefore, the spectrum $\sigma(A_\gamma)$ consists entirely of isolated eigenvalues. Now we are ready to study the eigenvalue problem of $\mathbb{A}_\gamma$. Let $\lambda \in \sigma(A_\gamma)$ and $\Phi = (\phi, \Psi)$ be an eigenfunction of A_γ corresponding to λ. Then we have $\Psi = \lambda\phi$ and ϕ satisfies the following equation :

$$\begin{cases} \lambda^2 m(x)\phi(x) + (EI(x)\phi''(x))'' + \gamma(x)\phi(x) = 0, \qquad 0 < x < 1, \\ \phi(0) = \phi'(0) = 0 \\ \phi''(1) = -\frac{\lambda\beta}{EI(1)}\phi'(1) \\ \phi'''(1) = \dfrac{\alpha\lambda}{EI(1)}\phi(1). \end{cases} \tag{15}$$

Expanding (74) yields

$$\begin{cases} \phi^{(4)}(x) + \dfrac{2EI'(x)}{EI(x)}\phi'''(x) + \dfrac{EI''(x)}{EI(x)}\phi''(x) + \lambda\dfrac{\gamma(x)}{EI(x)}\phi(x) + \\ \dfrac{\lambda^2 m(x)}{EI(x)}\phi(x) = 0, \qquad 0 < x < 1, \\ \phi''(1) = -\frac{\lambda\beta}{EI(1)}\phi'(1) \\ \phi'''(1) = \dfrac{\alpha\lambda}{EI(1)}\phi(1). \end{cases} \tag{16}$$

In order to simplify our computations, we introduce a spatial scale transformation in x :

$$f(z) = \phi(x), \qquad z = \frac{1}{h}\int_0^x \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta \tag{17}$$

where

$$h = \int_0^1 \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta \tag{18}$$

Then, (17) together with its boundary conditions can be transformed into

$$\begin{cases} f^{(4)}(z) + a(z) f'''(z) + b(z) f''(z) + c(z) f'(z) + \lambda h^4 \dfrac{\gamma(x)}{m(x)} f(x) + \\ \lambda^2 h^4 f(z) = 0, \qquad\qquad 0 < z < 1, \\ f(0) = f'(0) = 0 \\ z_x^2(1) f''(1) + z_{xx}(1) f'(1) + \frac{\lambda\beta}{EI(1)} z'(1) f'(1) = 0 \\ f'''(1) + \dfrac{3z_{xx}(1)}{z_x^2(1)} f''(1) + \dfrac{z_{xxx}(1)}{z_x^3(1)} f'(1) - \frac{\lambda\alpha}{EI(1)} f(1) = 0, \end{cases} \tag{19}$$

with

$$a(z) = \frac{6z_{xx}}{z_x^2} + \frac{EI'(x)}{z_x EI(x)} \tag{20}$$

$$b(z) = \frac{3z_{xx}^2}{z_x^4} + \frac{6z_{xx}EI'(x)}{z_x^3 EI(x)} + \frac{EI'(x)}{z_x^2 EI(x)} + \frac{4z_{xx}}{z_x^3} \tag{21}$$

$$c(x) = \frac{z_{xxxx}}{z_x^4} + \frac{2z_{xxx}EI'(x)}{z_x^4 EI(x)} + \frac{z_{xx}EI''(x)}{z_x^4 EI(x)} \tag{22}$$

$$z_x = \frac{1}{h}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}}, \qquad z_x^4 = \frac{1}{h^4}\frac{m(x)}{EI(x)} \tag{23}$$

and

$$z_{xx} = \frac{1}{4h}\left(\frac{m(x)}{EI(x)}\right)^{\frac{-3}{4}} \frac{d}{dx}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}}. \tag{24}$$

If we define

$$d(x) = \frac{\gamma(x)}{m(x)}$$

The equation in (20) is for any $0 < z < 1$

$$f^{(4)}(z) + a(z) f'''(z) + b(z) f''(z) + c(z) f'(z) + \lambda h^4 d(x) f(x) + \lambda^2 h^4 f(z) = 0.$$

This can be further simplified by applying another invertible transformation :

$$g(z) = \exp\left(\frac{1}{4}\int_0^z a(\zeta)\, d\zeta\right) f(z), \qquad 0 < z < 1, \tag{25}$$

and we arrive at the following eigenvalue problem that is equivalent to the origins one :

$$\begin{cases} g^{(4)}(z) + b_1(z)\, g''(z) + c_1(z)\, g'(z) + d_1 g(z) + \lambda h^4 d(x)\, g(x) + \\ \lambda^2 h^4 g(z) = 0, \qquad\qquad 0 < z < 1, \\ g(0) = g'(0) = 0 \\ g''(1) + b_{11} g'(1) + b_{12} g(1) = 0 \\ g'''(1) + b_{21} g''(1) + b_{22} g'(1) + b_{23} g(1) = 0, \end{cases} \tag{26}$$

where

$$b_1(1) = -\frac{3}{2}a'(z) - \frac{3}{8}a^2(z) + b(z) \tag{27}$$

$$c_1(z) = \frac{1}{8}a^3(z) - \frac{1}{2}a(z)b(z) - a'(z) + c(z) \tag{28}$$

$$\begin{aligned} d_1(z) &= \frac{3}{16}a'^2(z) - \frac{1}{4}a'''(z) + \frac{3}{32}a'(z)a^2(2) - \frac{3}{256}a^4(z) + \\ & b(z)\left(\frac{1}{16}a^2(z) - \frac{1}{4}a'(z)\right) - \frac{a(z)c(z)}{4} \end{aligned} \tag{29}$$

$$b_{11} = -\frac{1}{2}a(1) + \frac{z_{xx}(1)}{z_x^2(1)} + \frac{(\beta\lambda)}{EI(1)}z_x(1) \tag{30}$$

$$b_{12} = \frac{\frac{1}{16}z_x^2(1)a^2(1) - \frac{1}{4}z_x^2(1)a'(1) - \frac{1}{4}z_{xx}(1)a(1)}{z_x^2(1)} - \frac{(\beta\lambda z_x(1)a(1))}{4EI(1)} \tag{31}$$

$$b_{21} = -\frac{3}{4}a(1) + \frac{3z_{xx}(1)}{z_x^2(1)} + \frac{EI'(1)}{z_x(1)EI(1)} \tag{32}$$

$$b_{22} = -\frac{3}{4}a'(1) + \frac{3}{16}a^2(1) - \frac{3z_{xx}(1)a(1)}{2z_x^2(1)} - \frac{a(1)EI'(1)}{EI(1)z_x(1)} + \frac{z_{xxx}(1)}{z_x^3(1)} + \frac{z_{xx}(1)EI'(1)}{z_x^3(1)EI(1)} \tag{33}$$

$$\begin{aligned} b_{23} &= -\frac{1}{4}a''(1) + \frac{3}{16}a'(1)a(1) - \frac{1}{64}a^3(1) - \frac{3z_{xx}(1)a'(1)}{4z_x^2(1)} + \frac{3z_{xx}(1)a^2(1)}{16z_x^2(1)} - \\ & \frac{z_{xxx}(1)a(1)}{4z_x^3(1)} - \frac{(\alpha\lambda)}{z_x^3(1)EI(1)} - \frac{z_{xxx}(1)a(1)EI'(1)}{4z_x^3(1)EI(1)} + \frac{a'(1)EI(1)}{4z_x(1)EI(1)}. \end{aligned} \tag{34}$$

To further solve the eigenvalue problem $(1)-(4)$, we follow the procedure in Birkhoff ([2], [3]) and Naimark (1967) [11] and divide the complex plane into eight distinct sectors,

$$S_k = \left\{ z \in \mathbb{C} : \frac{k\pi}{4} \le \arg z \le \frac{(k+1)\pi}{4} \right\}, \quad k = 0, 1, 2, ..., 7 \tag{35}$$

and let ω_1, ω_2, ω_3, ω_4 be the roots of equation $\theta^4+1=0$ that are arranged so that

$$\operatorname{Re}(\rho\omega_1) \leq \operatorname{Re}(\rho\omega_2) \leq \operatorname{Re}(\rho\omega_3) \leq \operatorname{Re}(\rho\omega_4), \quad \forall \rho \in S_k. \tag{36}$$

Obviously, in sector S_1, we can choose $\omega_1 = \exp(i\frac{3}{4}\pi) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $\omega_2 = \exp(i\frac{1}{4}\pi) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, $\omega_3 = \exp(i\frac{5}{4}\pi) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$, $\omega_4 = \exp(i\frac{7}{4}\pi) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$,

which satisfy the inequalities in (37) and choices can also be made for other sectors. In the rest of this section, we shall derive the asymptotic behavior of the eigenvalue of the sectors S_1 and S_2 because the same will hold for the other sectors with similar proofs. Setting $\lambda = \frac{\rho^2}{h^2}$, in each sector S_k, we have the following result about the asymptotic fundamental solutions of system $(1)-(4)$.

Lemma 2.1. *For $\rho \in S_k$ with ρ large enough, the equation : $g^{(4)}(z)+b_1(z)g''(z)+c_1(z)g'(z)+d_1 g(z)+\rho^2 h^2 d(z)g(z)+\rho^4 g(z)=0$, $0<z<1$, has four linearly independent asymptotic fundamental solutions,*

$$\Phi_s(z,\rho) = \exp \rho\omega_s z\left(1+\frac{\Phi_{s,1}(z)}{\rho}+O\left(\rho^{-2}\right)\right), \quad s=1,2,3,4 \tag{37}$$

and hence their derivatives for $s=1,2,3,4$ and $j=1,2,3$ are given by

$$\frac{d^j}{dz^j}\Phi_s(z,\rho) = (\rho\omega_s)^j \exp \rho\omega_s z\left(1+\frac{\Phi_{s,1}(z)}{\rho}+O\left(\rho^{-2}\right)\right) \tag{38}$$

where

$$\Phi_{s,1}(z) = -\frac{1}{4\omega_s^3}\int_0^z \omega_s^2 b(\zeta)\,d\zeta, \quad \Phi_{s,1}(0)=0 \quad for\ s=1,2,3,4 \tag{39}$$

and

$$\Phi_{s,1}(z) = -\frac{1}{4\omega_s}\int_0^1 b_1(\zeta)\,d\zeta - \frac{h^2}{4\omega_s^3}\int_0^1 d(\zeta)\,d\zeta = \frac{1}{\omega_s}\mu_1 + \frac{1}{\omega_s^3}\mu_2, \tag{40}$$

with $\mu_1 = -\frac{1}{4}\int_0^1 b_1(\zeta)\,d\zeta$ *et* $\mu_2 = -\frac{h^2}{4}\int_0^1 d(\zeta)\,d\zeta$.

Proof. The proof is a direct result in Birkhoff ([2], [3]) and Naimark [13] from which we deduce the required results (38) and (39).

Lemma 2.2. *For $\rho \in S_1$, if we set $\delta = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, then we have following inequalities :*

$$\operatorname{Re}(\rho\omega_1) \leq -|\rho|\,\delta, \quad \operatorname{Re}(\rho\omega_4) \geq |\rho|\,\delta, \ \omega_4 = -\omega_1 \text{ and } e^{\rho\omega_1} = O\left(\rho^{-2}\right) \text{ when } |\rho| \to +\infty.$$

Furthermore, substituting (38) and (39) into the boundary conditions (27), we obtain asymptotic expressions for the boundary conditions for large enough $|\rho|$:

$$U_4\left(\Phi_s,\rho\right)=\Phi_s\left(0,\rho\right)=1+O\left(\rho^{-2}\right)=[1]_2\,,\quad s=1,2,3,4, \tag{41}$$

$$U_3\left(\Phi_s,\rho\right)=\Phi_s'\left(0,\rho\right)=\rho\omega_S\left(1+O\left(\rho^{-2}\right)\right) \tag{42}$$

$$U_3\left(\Phi_s,\rho\right)=\rho\omega_S\,[1]_2\,,\quad s=1,2,3,4 \tag{43}$$

$$\begin{aligned}U_2\left(\Phi_s,\rho\right)=\left(\rho\omega_S\right)^2e^{\rho\omega_S}[1+((\zeta_{21}+\zeta_{22}\omega_s^{-2})\omega_s^{-2}+(\zeta_{23}\omega_s^{2}+\zeta_{24}\\+\zeta_{25}\omega_s^{-2})\rho^{-1}\omega_s^{-3}+\zeta_{26}\rho\omega_s^{-3}]_2\end{aligned} \tag{44}$$

$$U_1\left(\Phi_s,\rho\right)=\left(\rho\omega_S\right)^3e^{\rho\omega_S}\left[1+\zeta_{11}\omega_s^{-2}+(\mu_1+b_{21}(1))\rho^{-1}\omega_s^{-1}+(\zeta_{12}+\zeta_{13}\omega_s^{-2})\rho^{-1}\omega_s^{-3}\right]_2 \tag{45}$$

$$\zeta_{21}=\frac{\beta\mu_1}{z_x(1)h^2EI(1)}-\frac{\beta a(1)}{4h^2z_x(1)EI(1)} \tag{46}$$

$$\zeta_{23}=\mu_1-\frac{1}{2}a(1)+\frac{z_{xx}(1)}{z_x^2(1)}+\frac{\alpha}{z_x(1)EI(1)} \tag{47}$$

$$\zeta_{25}=-\frac{\beta a(1)\mu_2}{4z_x(1)h^2EI(1)} \tag{48}$$

$$\zeta_{24}=\mu_2-\frac{\beta a(1)\mu_1}{4z_x(1)h^2EI(1)} \tag{49}$$

$$\zeta_{12}=\mu_2-\frac{\alpha}{z_x^3(1)h^2EI(1)} \tag{50}$$

$$\zeta_{22}=\frac{\beta\mu_2}{z_x(1)h^2EI(1)} \tag{51}$$

$$\zeta_{26}=\frac{\beta}{z_x(1)h^2EI(1)} \tag{52}$$

Theorem 2.3. ([13], [11]) *For $n=2m,\rho\in S_0$ and $l\in 0,1,....$ An asymptotic expansion of the characteristic determinant $\Delta(\rho)$ is given by:*

$$\Delta=\rho^\gamma\exp(\rho\Omega)([\Theta_{-1}(\rho)]_l\exp(-\rho\omega_m)+[\Theta_0(\rho)]_l+[\Theta_1(\rho)]_l\exp(\rho\omega_m))$$

$$\gamma=\Sigma_{j=1}^n k_j\Omega=\Omega_{m+2}+...+\Omega_m$$

and

$$\Theta_i(\rho)=\Theta_{i0}+\frac{\Theta_{i1}}{\rho}+...+\frac{\Theta_{il}}{\rho^{l-1}},\quad i=-1,0,1$$

For constants Θ_{i0}, Θ_{i1}*, the same representations are also true in all other sectors* S_k*.*

From the asymptotic developments of the characteristic determinant, we can now define the notions of regularity, strong regularity, almost regularity and the normalized boundary conditions.

Definition 2.1. *We say that the boundary conditions of the problem are strongly regular if the zeros of the characteristic determinant* $\Delta(\rho)$ *are asymptotically simple and different from each other by a positive number* $\delta > 0$ *that is to say:*

$$\Theta_{00}^2 - 4\Theta_{-10}\Theta_{10} \neq 0.$$

Note that $\lambda = \dfrac{\rho^2}{h^2} \neq 0$ is the eigenvalue $(1) - (4)$ if and only if ρ satisfies the characteristic determinant

$$\Delta(\rho) = \begin{vmatrix} U_4(\Phi_1,\rho) & U_4(\Phi_2,\rho) & U_4(\Phi_3,\rho) & U_4(\Phi_4,\rho) \\ U_3(\Phi_1,\rho) & U_3(\Phi_2,\rho) & U_3(\Phi_3,\rho) & U_3(\Phi_4,\rho) \\ U_2(\Phi_1,\rho) & U_2(\Phi_2,\rho) & U_2(\Phi_3,\rho) & U_2(\Phi_4,\rho) \\ U_1(\Phi_1,\rho) & U_1(\Phi_2,\rho) & U_1(\Phi_3,\rho) & U_1(\Phi_4,\rho) \end{vmatrix} = 0, \qquad (53)$$

Substituting the asymptotic expressions of the boundary conditions into (58) and using lemma 2.2 we obtain

$$\Delta(\rho) = \left| \begin{matrix} [1]_2 & [1]_2 \\ \rho\omega_1[1]_2 & \rho\omega_2[1]_2 \\ 0 & (\rho\omega_2)^2 e^{\rho\omega_2}\left[1 + (\zeta_{21} + \zeta_{22}\omega_2^{-2})\omega_2^{-2} + F + \zeta_{26}\rho\omega_2^{-3}\right]_2 \\ 0 & (\rho\omega_2)^3 e^{\rho\omega_2}\left[1 + \zeta_{11}\omega_2^{-2} + (\mu_1 + b_{21}(1))\rho^{-1}\omega_2^{-1} + (\zeta_{12} + \zeta_{13}\omega_2^{-2})\rho^{-1}\omega_2^{-3}\right]_2 \end{matrix} \right.$$

$$\begin{matrix} [1]_2 \\ \rho\omega_3[1]_2 \\ (\rho\omega_3)^2 e^{\rho\omega_3}\left[1 + (\zeta_{21} + \zeta_{22}\omega_3^{-2})\omega_3^{-2} + F + \zeta_{26}\rho\omega_3^{-3}\right]_2 \\ (\rho\omega_3)^3 e^{\rho\omega_3}\left[1 + \zeta_{11}\omega_3^{-2} + (\mu_1 + b_{21}(1))\rho^{-1}\omega_3^{-1} + (\zeta_{12} + \zeta_{13}\omega_3^{-2})\rho^{-1}\omega_3^{-3}\right]_2 \end{matrix}$$

$$\left. \begin{matrix} 0 \\ 0 \\ (\rho\omega_4)^2 e^{\rho\omega_4}\left[1 + (\zeta_{21} + \zeta_{22}\omega_4^{-2})\omega_4^{-2} + F + \zeta_{26}\rho\omega_4^{-3}\right]_2 \\ (\rho\omega_4)^3 e^{\rho\omega_4}\left[1 + \zeta_{11}\omega_4^{-2} + (\mu_1 + b_{21}(1))\rho^{-1}\omega_4^{-1} + (\zeta_{12} + \zeta_{13}\omega_4^{-2})\rho^{-1}\omega_4^{-3}\right]_2 \end{matrix} \right| .$$

In sector S_1, the choices are : $\omega_1^2 = -i,\ \omega_2^2 = i,\ \omega_3^2 = i,\ \omega_4^2 = -i,\ \omega_3^{-4} - \omega_4^{-4} = 0,$ $\omega_2^{-4} - \omega_4^{-4} = 0,\ \ \omega_2^{-1}\omega_4 = -i,\ \ \omega_3 = -\omega_2,\ \omega_4 - \omega_3 = \sqrt{2},\ \ \omega_1 - \omega_3 = \sqrt{2}i,$ $\omega_2 - \omega_1 = \sqrt{2},\ \ \omega_4 - \omega_2 = -i\sqrt{2},\ \omega_2^{-2} - \omega_4^{-2} = -2i,\ \ \omega_3^{-2} - \omega_4^{-2} = -2i,\ \ \omega_3^2\omega_4^2 = 1,$

$\omega_2^2\omega_4^2 = 1,$

a straightforward simplification will arrive at the following result, which also true on all other sectors S_k (see Naimark,[13] 1967, $pp.56-74$).

Theorem 2.4. *Let $\Delta(\rho)$ be the characteristic determinant of the eigenvalue problem (27). In sector S_1, an asymptotic expression of $\Delta(\rho)$ is given by :*

$$\Delta(\rho) = \left\{2\sqrt{2}\zeta_{26}\rho^7 e^{\rho\omega_4}\{e^{\rho\omega_2} + ie^{-\rho\omega_2} + \left[\mu_3 e^{\rho\omega_2} + \mu_4 e^{-\rho\omega_2}\right]\rho^{-1} + O\left(\rho^{-2}\right)\right\}, \quad (54)$$

where

$$\begin{cases} \mu_3 = \frac{\sqrt{2}}{2\zeta_{26}}\left(1+\zeta_{21}-\zeta_{22}-\zeta_{11}+\zeta_{11}\times\zeta_{21}+\zeta_{11}\times\zeta_{22}+\zeta_{26}(\mu_1+b_{21})-\zeta_{26}\times\zeta_{12}-\zeta_{11}\times\zeta_{13}\right), \\ \mu_4 = \frac{\sqrt{2}}{2\zeta_{26}}\left(1-\zeta_{21}-\zeta_{22}+\zeta_{11}+\zeta_{11}\times\zeta_{21}-\zeta_{11}\times\zeta_{22}-\zeta_{26}(\mu_1+b_{21})-\zeta_{26}\times\zeta_{12}+\zeta_{26}\times\zeta_{13}\right) \end{cases} \quad (55)$$

Thus, the boundary eigenvalue problem (27) is strongly regular.

Proof. Using the expressions of the fourth roots of -1 given we have :

$$\Delta(\rho) = \left\{2\sqrt{2}\zeta_{26}\rho^7 e^{\rho\omega_4}\{e^{\rho\omega_2} + ie^{-\rho\omega_2} + \left[\mu_3 e^{\rho\omega_2} + \mu_4 e^{-\rho\omega_2}\right]\rho^{-1} + O\left(\rho^{-2}\right)\right\}, \quad (56)$$

According to the theorem 2.3 and definition 2.1 above we have

$$\Theta_{00}^2 = 0, \Theta_{-10} = 2\sqrt{2}\zeta_{26}i, \Theta_{10} = 2\sqrt{2}\zeta_{26}$$

and

$$\Theta_{00}^2 - 4\Theta_{-10}\Theta_{10}\beta^2 i \neq 0$$

Therefore the boundary conditions of the eigenvalue problem are strongly regular. Which means that the eigenvalues are simple and different from each other according to the definition 1.

Using the final determinant $\Delta(\rho)$, we can derive an asymptotic expression of the problem values. $\Delta(\rho) = 0$ Implies that :

$$2\sqrt{2}\zeta_{26}\rho^7 e^{\rho\omega_4}\{e^{\rho\omega_2} + ie^{-\rho\omega_2} + \left[\mu_3 e^{\rho\omega_2} + \mu_4 e^{-\rho\omega_2}\right]\rho^{-1} + O\left(\rho^{-2}\right)\} = 0$$

which equivalent to

$$e^{\rho\omega_2} + ie^{-\rho\omega_2} + \left[\mu_3 e^{\rho\omega_2} + \mu_4 e^{-\rho\omega_2}\right]\rho^{-1} + O\left(\rho^{-2}\right) = 0 \quad (57)$$

and can be rewritten as

$$e^{\rho\omega_2} + ie^{-\rho\omega_2} + O\left(\rho^{-1}\right) = 0. \quad (58)$$

Note that the following equation :

$$e^{\rho\omega_2} + ie^{-\rho\omega_2} = 0$$

has solutions

$$\rho_n = \left(n + \frac{3}{4}\right)\frac{\pi i}{\omega_2},\ n = 1, 2, ... \tag{59}$$

Let $\widetilde{\rho_n}$ be the solutions of (59). Applying Rouché's theorem see Naimark [13], 1967, p.70 to (59), we get the following expression

$$\widetilde{\rho_n} = \rho_n + \alpha_n = \left(n + \frac{3}{4}\right)\frac{\pi i}{\omega_2} + \alpha_n, \quad \alpha_n = O\left(n^{-1}\right), \quad n = N, N+1, ... \tag{60}$$

where N is a large positive integer. Substituting $\widetilde{\rho_n}$ into (58), and using the fact that $e^{\rho\omega_2} = -ie^{-\rho\omega_2}$, we obtain

$$e^{\alpha_n\omega_2} - e^{-\alpha_n\omega_2} + \mu_3\widetilde{\rho_n}^{-1}e^{\alpha_n\omega_2} - i\mu_4\widetilde{\rho_n}^{-1}e^{-\alpha_n\omega_2} + O\left(\widetilde{\rho_n}^{-2}\right) = 0.$$

Expanding the exponential function according to its Taylor series, we get

$$\alpha_n = -\frac{\mu_3}{2\omega_2\rho_n} - \frac{\mu_4}{2\omega_2\rho_n}i + O\left(n^{-2}\right), \quad n = N, N+1, ...$$

Therefore, we have

$$\widetilde{\rho_n} = \left(n + \frac{3}{4}\right)\frac{\pi i}{\omega_2} + \frac{\mu_3}{2\left(n + \frac{3}{4}\right)\pi}i - \frac{\mu_4}{2\left(n + \frac{3}{4}\right)\pi} + O\left(n^{-2}\right), \quad n = N, N+1, ...$$

Note that $\lambda_n = \dfrac{\rho_n^2}{h^2} \neq 0$, $\omega_2 = e^{i\frac{\pi}{4}}$ and $\omega_2^2 = i$. So we have

$$\lambda_n = -\frac{\sqrt{2}}{2h^2}\left(\mu_4 + \mu_3\right) + \frac{1}{h^2}\left[\frac{\sqrt{2}}{2}\left(\mu_3 - \mu_4\right) + \left(n + \frac{3}{4}\right)^2\pi^2\right]i + O\left(n^{-1}\right), \tag{61}$$

where $n = N, N+1, ...$ with N large enough. The same proof can be applied to sector S_2 because the eigenvalues of the problem (27) can obtained by a similar calculation with the choices

$$\omega_1 = \exp(i\frac{1}{4}\pi) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad \omega_2 = \exp(i\frac{3}{4}\pi) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,$$

$$\omega_3 = \exp(i\frac{7}{4}\pi) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i, \quad \omega_4 = \exp(i\frac{5}{4}\pi) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,$$

so that inequality (37) holds :

$$\operatorname{Re}(\rho\omega_1) \le \operatorname{Re}(\rho\omega_2) \le \operatorname{Re}(\rho\omega_3) \le \operatorname{Re}(\rho\omega_4), \quad \forall\rho \in S_2.$$

Hence, in sector S_2, we have the following asymptotic expression of $\Delta(\rho)$:

$$\Delta(\rho) = -\left\{2\sqrt{2}\zeta_{26}\rho^7 e^{\rho\omega_4}\{e^{\rho\omega_2} - ie^{-\rho\omega_2} - \left[\mu_3 e^{\rho\omega_2} + \mu_4 e^{-\rho\omega_2}\right]\rho^{-1} + O\left(\rho^{-2}\right)\right\}, \tag{62}$$

By a calculation similar to the one done in sector S1, we have the following:

$$\widetilde{\rho_n} = \left(n + \frac{3}{4}\right)\frac{\pi i}{\omega_2} - \frac{\mu_3}{2\left(n + \frac{3}{4}\right)\pi}i - \frac{\mu_4}{2\left(n + \frac{3}{4}\right)\pi} + O\left(n^{-2}\right), \quad n = N, N+1, ... \tag{63}$$

with N large enough. In the sector S_2,using $\lambda_n^2 \neq 0$, $\omega_2 = e^{i\frac{3}{4}\Pi}$ and $\omega_2^2 = -i$, we have:

$$\lambda_n = -\frac{\sqrt{2}}{2h^2}(\mu_4 + \mu_3) - \frac{1}{h^2}\left[\frac{\sqrt{2}}{2}(\mu_3 - \mu_4) + \left(n + \frac{3}{4}\right)^2\pi^2\right]i + O\left(n^{-1}\right), \tag{64}$$

where $n = N, N+1, ...$ with N large enough.

Here we notice that the eigenvalues generated in the other sectors S_k coincide with those of the sectors S_1 and S_2. By combining the first expression of λ_n with the second expression of λ_n we obtain the result on the following eigenvalues.

Theorem 2.5. *Let A_λ be defined above. Then an asymptotic expression of the eigenvalues of the problem* (2.10) *is given by:*

$$\lambda_n = -\frac{\sqrt{2}}{2h^2}(\mu_4 - \mu_3) \pm \frac{1}{h^2}\left[\frac{\sqrt{2}}{2}(\mu_4 + \mu_3) + \left(n + \frac{3}{4}\right)^2\pi^2\right]i + O\left(n^{-1}\right), \tag{65}$$

where $n = N, N+1, ...$ with N large enough.

In addition, the eigenvalues $\lambda_n (n = N, N+1, ...)$ with sufficiently large modulus are simple and distinct except for a finite number of them, and satisfy

$$\mu_4 - \mu_3 = -2\sqrt{2}\mu_2 + \frac{\sqrt{2}h}{\beta EI(1)}\left(\frac{m(1)}{EI(1)}\right)^{\frac{-3}{4}} \tag{66}$$

We have

$$\mu_2 = -\frac{h^2}{4}\int_0^1 d(\zeta)\,d\zeta, \quad d(x) = \frac{\gamma(x)}{m(x)}, \quad \frac{dz}{dx} = \frac{1}{h}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}}, \tag{67}$$

so,

$$\mu_2 = -\frac{h^2}{4}\int_0^1 \frac{\gamma(x)}{m(x)}\frac{1}{h}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx = -\frac{h}{4}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx. \tag{68}$$

Moreover, λ_n $(n = N, N+1, ...)$ with sufficiently large modulus are simple and distinct except for finitely many of them, and satisfy

$$\lim_{n\to+\infty} \mathrm{Re}\lambda_n = -\frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx - \frac{\alpha\beta + m(1)EI(1)}{h\beta EI(1)}\left(\frac{m(1)}{EI(1)}\right)^{\frac{-3}{4}}. \tag{69}$$

Lemma 2.3. (Wang [21]) *A sequence $\{\Phi(n) : n \geq 1\}$ in a Hilbert space $\mathcal{H}$, is a Riesz basis if and only if there exists a bounded invertible linear operator and with bounded inverse $\mathcal{T}$ on $\mathcal{H}$ such that:*

$$\mathcal{T}\Phi_n = e_n, \quad n \geq 1$$

where $\{e_n : n \geq 1\}$ is an orthonormal basis of $\mathcal{H}$.

3. Riesz Basis Property

In this subsection, we discuss the Riesz basis property of the eigenfunctions of operator A_γ of the system (9). We begin with showing that the generalized eigenfunctions of A_γ form an unconditional basis in Hilbert state space $\mathbb{H}$. For this aim, we introduce a transformation $\mathcal{L}$ via

$$\mathcal{L}(f, g) = (\phi, \psi)$$

where

$$\phi(x) = f(x), \quad \psi(x) = g(x), \quad z = \frac{1}{h}\int_0^x \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta, \tag{70}$$

with

$$h = \int_0^1 \left(\frac{m(\zeta)}{EI(\zeta)}\right)^{\frac{1}{4}} d\zeta. \tag{71}$$

It is easily seen that $\mathcal{L}$ is a bounded invertible operator on $\mathbb{H}$. Now we define the following ordinary differential operator :

$$\begin{cases} L\left(f\right)=f^{(4)}\left(z\right)+a\left(z\right)f'''\left(z\right)+b\left(z\right)f''\left(z\right)+c\left(z\right)f'\left(z\right), \\ \mu\left(z\right)=h^2d\left(z\right)=h^2\dfrac{\gamma\left(x\right)}{m\left(x\right)} \\ B_1\left(f\right)=f\left(0\right), \quad B_2\left(f\right)=f'\left(0\right), \\ B_3\left(f\right)=z_x^2\left(1\right)f''\left(1\right)+z_{xx}\left(1\right)f'\left(1\right)=0 \\ B_4\left(f\right)=f'''\left(1\right)+\dfrac{3z_{xx}\left(1\right)}{z_x^2\left(1\right)}f''\left(1\right)+\dfrac{z_{xxx}\left(1\right)}{z_x^3\left(1\right)}f'\left(1\right)-\dfrac{\left(\alpha+\lambda\beta\right)}{z_x^3\left(1\right)EI\left(1\right)}f\left(1\right)=0, \end{cases} \tag{72}$$

where the coefficients are given by (21)-(25). Let $\mathbb{A}$, $\eta\in\sigma\left(\mathbb{A}\right)$ be an eigenvalue of $\mathbb{A}$ and (f,g) be an eigenfunction, then we have $g=\eta f$ and f will satisfy the following equation :

$$f^{(4)}\left(z\right)+a\left(z\right)f'''\left(z\right)+b\left(z\right)f''\left(z\right)+c\left(z\right)f'\left(z\right)+\eta u\left(z\right)f\left(z\right)+\eta^2 f\left(z\right)=0,$$

with boundary conditions $B_j\left(f\right)=0,\ \ j=1,2,3,4$. Now by taking $\lambda=\dfrac{\eta}{h^2}$ and

$$\mathcal{L}\left(f,g\right)=\left(\phi\left(x\right),\psi\left(x\right)\right)$$

we see that $\psi=\lambda\phi$ and ϕ satisfies the equation

$$\begin{cases} \lambda^2 m(x)\phi\left(x\right)+\left(EI\left(x\right)\phi''\left(x\right)\right)''+\gamma\left(x\right)\phi\left(x\right)=0, \qquad 0<x<1, \\ \phi\left(0\right)=\phi'\left(0\right)=0 \\ \phi''\left(1\right)=-\frac{\lambda\beta}{EI(1)}\phi'\left(1\right) \\ \phi'''\left(1\right)=\dfrac{\alpha\lambda}{EI\left(1\right)}\phi\left(1\right). \end{cases} \tag{73}$$

From where we have the following result $\eta\in\sigma(\mathcal{A})\Leftrightarrow\lambda\in\sigma(\mathcal{A}_\gamma)$.

Theorem 3.1. *Consider the operator A_γ of the system* (9)*. Then the eigenvalues of the operator A_γ are all simple, except for a finite number of them, and the generalized eigenfunctions of the operator A_γ form a Riesz basis for Hilbert space H.*

Proof. In the previous section, we know that the problem with eigenvalues (20) has its strongly regular boundary conditions according to Theorem 0.4. and Definition 0.1., which implies that the eigenvalues are separate and simple except for a finite number of them. So the first assertion is true. Then, the strong regularity of the boundary conditions ensures that the sequence of generalized eigenfunctions

$F_n = (f_n, \eta_n f_n)^T$ of the operator $\mathcal{A}$ forms a Riesz basis for $\mathcal{H} = H$. Since $\mathcal{T}$ is bounded and invertible on $\mathcal{H} = H$, it follows that $\Psi_n = (\phi_n, \lambda_n \phi_n)^T = \mathcal{T} F_n$ also forms a Riesz basis on H (see [21])

Theorem 3.2. *Let $T_{\mathcal{A}}(t)_{t\geq 0}$ be a C_0-semigroup in a Hilbert space H and $\mathcal{A}$ its infinitesimal generator. Suppose that : (i) $\mathcal{B}$ is bounded on $\mathcal{H}$,*
(ii) the eigenvectors of $\mathcal{A}$ form a Riesz basis in $\mathcal{H}$
(iii) the spectrum $\sigma(\mathcal{A}) = \lambda_n$ of $\mathcal{A}$ is discrete and formed of simple eigenvalues for n sufficiently large (we recall that $\mathcal{A}+\mathcal{B}$ is an infinitesimal generator of C_0 -semigroup $T_{\mathcal{A}+\mathcal{B}_{t\geq 0}}$. Then the optimal energy decay rate is determined by the spectral abscissa of the operator associated with the C_0-semi-group $T_{\mathcal{A}+\mathcal{B}_{t\geq 0}}$.

Proof. In the previous section we showed that the boundary conditions of the eigenvalue problem are strongly regular.
So the eigenvalues are separate and simple, except for a finite number of them.
So the first assertion is true. Consequently the strong regularity of the boundary conditions ensures that the sequence of generalized eigenfunctions
$\mathbb{F} = (f_n, \eta_n f_n)^T$ of the operator $\mathcal{A}$ forms a Riesz basis for $\mathcal{H} = H$. Since $\mathcal{T}$ is bounded and invertible on $\mathcal{H} = H$, it follows that $\psi_n = (\phi_n, \lambda_n \phi_n)^T = \mathcal{T} F_n$ also forms a Riesz basis on H.

We are now able to study the exponential stability of the system (9). Since the Riesz basis property implies that the optimal rate of energy decay is determined by the spectral abscissa of the system operator (see Theorem 0.8) and that (40) describes the asymptote of $\sigma(A_\gamma)$, for any sufficiently small $\varepsilon > 0$, there is only a finite number of eigenvalues of A_γ in the following half-plane:

$$\Sigma : Re(\lambda) > -\frac{1}{2h}\int_0^1 \frac{\gamma(x)}{m(x)}\left(\frac{m(x)}{EI(x)}\right)^{\frac{1}{4}} dx - \frac{\alpha\beta + m(1)EI(1)}{h\beta EI(1)}\left(\frac{m(1)}{EI(1)}\right)^{\frac{-3}{4}} + \varepsilon. \tag{74}$$

We have two stability results which describe the influence of the γ coefficient function of friction on the exponential stability of the system. See (see Curtain and Zwart [5]).

4. Exponential stability

The following theorem gives conditions for obtaining the exponential stability of the system $(1)-(4)$.

Theorem 4.1. *If γ is continuous and positive or zero on the interval $[0,1]$, then the system $(1)-(4)$ is exponentially stable for all $\beta \geq 0$ and $\alpha > 0$. In this case there are the constants $M > 0$ and $\omega > 0$ such that the energy*

$$E(t) = \frac{1}{2} \left\| (y, y_t)^T \right\|_H^2$$

of the system $(1) - (4)$ *satisfied*

$$E(t) \leq ME(0) \exp(-\omega t), \quad \forall t \geq 0$$

For any initial condition $(y(x,0), y_t(x,0))^T \in H$.

Proof. A_γ is a dissipative operator and $\{\exp(A_\gamma t)\}_{t \geq 0}$ is a contraction semigroup on H. To obtain exponential stability, it remains to show that there is no eigenvalue on the imaginary axis.

Let $\lambda = ir$ where $r \in \mathbb{R}^*$, be an eigenvalue of the operator A_γ on the imaginary axis.

Let $\Psi = (\phi, \psi)^T$ be the corresponding eigenfunction. So $\psi = \lambda \phi$.

We have $\gamma(x) \geq 0$ and for any $u = (f, g)^T \in D(\mathbb{A}_\gamma)$,

$\langle \mathbb{A}_\gamma u, u \rangle_{\mathbb{H}} = \left\langle \left(g(x), -\frac{1}{m(x)} \left(EI(x) f''(x) \right)'' + \gamma(x) g(x) \right), (f, g) \right\rangle$

$= \int_0^1 - (EI(x) f''(x))'' \overline{g(x)} + EI(x) \overline{f''(x)} g''(x) - \gamma(x) |g(x)|^2 dx + \alpha f(1) \overline{g(1)}$,

$= \int_0^1 EI(x) \left[\overline{f''(x)} g(x)'' - f''(x) \overline{g''(x)} \right] - \gamma(x) |g(x)|^2 dx + \alpha f(1) \overline{g(1)}$,

$$\operatorname{Re} \langle \mathbb{A}_\gamma u, u \rangle_{\mathbb{H}} = - \int_0^1 \gamma(x) |g(x)|^2 dx - \beta |g(1)|^2 \leq 0. \tag{75}$$

Thus A_γ is dissipative and $\{\exp(A_\gamma t)\}_{t \geq 0}$ is a contraction semi-group on H. Furthermore, the spectrum of A_γ admits the following asymptote:

$$Re\lambda \sim -\frac{1}{2h} \int_0^1 \frac{\gamma(x)}{m(x)} \left(\frac{m(x)}{EI(x)} \right)^{\frac{1}{4}} dx - \frac{\alpha\beta + m(1) EI(1)}{h\beta EI(1)} \left(\frac{m(1)}{EI(1)} \right)^{\frac{-3}{4}}.$$

If we can show that there are no eigenvalues on the imaginary axis then we will obtain exponential stability of the system studied because we will have $Re\lambda < 0$. Let $\lambda = ir$ with $r \in \mathbf{R}^*$ be an eigenvalue of the operator A_γ on the imaginary axis and $\Psi = (\phi, \psi)^T$ a corresponding eigenfunction, then $\psi = \lambda \phi$.

We have $Re\langle A_\gamma \Psi, \Psi \rangle_H = -(\alpha \mid \psi(1) \mid^2 + \beta \mid \psi_x(1) \mid^2 + \int_0^1 \gamma(x) \mid \psi(x) \mid^2 dx)$

$0 = \parallel \Psi \parallel_H^2 Re(\lambda) = Re\langle A_\gamma \Psi, \Psi \rangle_H = -\alpha \mid \psi(1) \mid^2 - \beta \mid \psi_x(x) \mid^2 - \int_0^1 \gamma(x) \mid \psi(x) \mid^2 dx$ Since $\gamma(x) \geq 0$ and $\psi(x)$ are continuous with $\alpha > 0$ and $\beta > 0$ we obtain

$$\psi(1) = 0, \quad \psi'(1) = 0 \quad \textit{and} \quad \gamma(x) \mid \psi(x) \mid^2 = 0 \quad \forall x \in [0, 1]. \tag{76}$$

Then $\phi(1) = 0$ because $\psi = \lambda\phi$.
The ϕ function satisfies the following differential equation:

$$\begin{cases} \lambda^2 m(x)\phi\left(x\right) + \left(EI\left(x\right)\phi''\left(x\right)\right)'' + \lambda\gamma\left(x\right)\phi\left(x\right) = 0, \quad & 0 < x < 1, \\ \phi\left(0\right) = \phi'\left(0\right) = \phi''\left(1\right) = \phi'''\left(1\right) = 0. \\ \phi\left(x\right) = 0, \quad & \forall x \in I \end{cases} \tag{77}$$

Note that ϕ satisfies the linear differential equation (77) and has an infinity of zeros in the open interval $]0.1[$.
Therefore, $\phi = 0$ by uniqueness of the solution of linear ordinary differential equations(See [17]). There are therefore no eigenvalues on the imaginary axis and we obtain $Re(\lambda) < 0$.

From Theorem 0.7 and as the optimal rate of decay of the energy of the system is determined by the spectral abscissa of the system operator, we deduce that the system $(1) - (4)$ is exponentially stable.
Next, still using an idea from Wang [20], we study the situation where the continuous function $\gamma(.)$ changes sign in $[0, 1]$.
We have the following theorem:

Theorem 4.2. *Let γ continue on [0,1]* $\quad \gamma = \gamma^{+} - \gamma^{-}$

$$\gamma_{+}(x) = max\left\{\gamma(x), 0\right\}, \gamma_{-}(x) = max\left\{-\gamma(x), 0\right\},$$

and let $A_{\gamma+}(f,g) = (g(x), -\gamma_{+}(x)g(x) - f_{xxxx}(x))^{\tau} \quad \forall \quad (f,g)^{T} \in D(A_{\gamma+}) = D(A_{\gamma})$ *and*

$$\Gamma_{-}(f,g) = (0, \gamma_{-}(x)g(x))^{\tau}, \quad \forall (f,g)^{\tau} \in H.$$

So A_{γ} can be written as: $A_{\gamma} = A_{\gamma+} + \Gamma_{-}$
Let $S(A_{\gamma}) = \sup\left\{Re\lambda/\lambda \in \sigma(A_{\gamma+})\right\}$.
If we have the following condition:

$$\max_{x\in[0,1]}\left\{\gamma_{-}(x)\right\} <\mid S(A_{\gamma+}) \mid,$$

then we obtain the exponential stability of the system (9)*.*
Proof. Γ_{-} is a self-adjoint operator. Set

$$\parallel \Gamma_{-} \parallel = \max_{x\in[0,1]}\left\{\gamma_{-}(x)\right\}$$

According to the Theorem and according to the definition of the operator $A_{\gamma+}$,

$\{e^{A_{\gamma+}t}\}_{t\geqslant 0}$ is a semi-group of contractions and $S(A_{\gamma+}) < 0$.
Moreover, thanks to the theory of semigroup perturbation for linear operators.
We obtain $\lambda \in \rho(A_\gamma)$ as long as $Re\lambda > S(A_{\gamma+}) + \parallel \Gamma_- \parallel$. We also have the following important result:

$$\omega(A_\gamma) = S(A_\gamma) \leqslant S(A_{\gamma+}) + \parallel \Gamma_- \parallel .$$

So for us to have exponential stability we must have

$$S(A_{\gamma+}) + \parallel \Gamma_- \parallel < 0.$$

Which implies that

$$\parallel \Gamma_- \parallel < \mid S(A_{\gamma+}) \mid .$$

Therefore, we can conclude the exponential stability system (9) if

$$\max_{x\in[0,1]} \{\gamma_-(x)\} < \mid S(A_{\gamma+}) \mid$$

5. Conclusion

In this paper, we studied the exponential stability of an Euler-Bernouilli beam with variable coefficients, damped and subjected to force control in velocity and angular velocity. We used the method developed by wang and al [17], because this method is adapted to Euler-Bernoulli beams with variable coefficients. We have therefore obtained interesting results on the of the Riesz basis property and the exponential stability of the system studied. In our study we found that this method is not suitable in the case where $EI(x) = m(x) = 1$. These results obtained in this article can be verified by a numerical method by the finite element or finite difference method.

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