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APPROXIMATING LOCAL SOLUTION OF AN INITIAL VALUE PROBLEM OF NONLINEAR FIRST ORDER ORDINARY HYBRID DIFFERENTIAL EQUATIONS WITH MAXIMA

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Abstract: In this paper, we prove a couple of approximation results for local existence and uniqueness of the solution of an initial value problem of nonlinear first order ordinary hybrid differential equations with maxima under weaker partial compactness and partial Lipschitz type conditions using the Dhage monotone iteration method based on the recent hybrid fixed point theorems of Dhage. An approximation result for the Ulam-Hyers stability of the local solution of the considered hybrid differential equation with maxima is also established. Our main abstract results are also illustrated with a couple of numerical examples.

Keywords and Phrases: Initial value problem, Hybrid fixed point principle, Dhage monotone iteration method, Approximation result, Ulam-Hyers stability.

2020 Mathematics Subject Classification: 34A12, 34A34, 34A45, 47H10.

1. Introduction

The class of differential equations with maxima is the special case of the class of functional differential equations with delay and calls for the present value of the unknown function depends upon the previous maximum value of the function involved in the differential equations. Such differential equations occur in the automatic control theory, signal processing and allied areas of mathematics. A variety of results concerning the solution of such kind of differential equations such as existence, uniqueness, stability and controllability etc., are available in the literature. See Bainov and Hristova [2], Otrocol [22], Otrocol and Rus [23], Dhage and Otrocol [17], Dhage and Dhage [8, 9, 10, 11] and references therein. The usual existence and uniqueness result are generally proved under compactness and Lipschitz type conditions, whereas differential inequalities are proved under certain monotonicity conditions. Very recently, the approximations results for existence and uniqueness of solutions are proved under partial compactness and partial Lipschitz type conditions. But in that case one has to assume the existence of either a lower or an upper solution of the related differential equations. Here, we prove the approximation results for local solution of an IVP of nonlinear first order hybrid differential equation with maxima without the requirement of lower or upper solution as well as usual compactness and Lipschitz type conditions as done in Dhage and Dhage [14, 15, 16] which is the main motivation of the present paper.

The rest of the paper is organized as follows. Section 2 deals with the statement of the problem and Section 3 deals with the auxiliary results and main hybrid fixed point theorems involved in the Dhage iteration method. The hypotheses and main approximation results for the local existence and uniqueness of solution are given in Section 4. The approximation of the Ulam-Hyer stability is discussed in Section 5 and a couple of illustrative examples are presented in Section 6. Finally, some concluding remarks are mentioned in Section 7.

2. Statement of the Problem

Given a closed and bounded interval $J = [t_0, t_0 + a]$ in $\mathbb{R}$ for some $t_0, a \in \mathbb{R}$ with $a > 0$, we consider the initial value problem (in short IVP) of nonlinear first order ordinary hybrid differential equation (HDE) with maxima,

$$\left.\begin{aligned} x'(t) &= f\big(t, x(t), M_x(t)\big), \quad t \in J, \\ x(t_0) &= \alpha_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1}$$

where $M_x(t) = \max\limits_{\xi \in [t_0, t]} x(\xi)$ and the function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies some hybrid, that is, mixed hypotheses from algebra, analysis and topology to be specified later.

Definition 2.1. *A function $x \in C(J, \mathbb{R})$ is said to be a solution of the HDE (1) with maxima if it satisfies the equations in (1) on J, where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J. If the solution x lies in a neighborhood $\mathcal{N}(x_0)$ of some point $x_0 \in C(J, \mathbb{R})$, then we say it is a local solution or neighborhood solution (in short nbhd solution) of the HDE (1) with maxima on J.*

Remark 2.1. *It is well-known that an open ball $B(x, r)$ in $C(J, \mathbb{R})$ centered at a point x of radius $r > 0$ is a neighborhood of the point x, so if a solution x^* of the HDE (1) with maxima lies in a closed ball $\overline{B(x, r)}$ in $C(J, \mathbb{R})$, then it is a local solution in view of the fact that $\overline{B(x, r)} \subset B(x, r+\epsilon)$ for every $\epsilon > 0$. Note that the idea of local or nbhd-solution is different from the usual notion of a local solution as mentioned in Coddington* [3].

The HDE (1) with maxima is well-known in the subject of nonlinear analysis and is very widely studied in the literature for a variety of different aspects of the solution by using different methods form analysis and topology, in particular from nonlinear functional analysis. When $f(t, x, y) = f(t, x)$, HDE (1) reduces to

$$\left.\begin{aligned} x'(t) &= f\big(t, x(t)\big), \quad t \in J, \\ x(t_0) &= \alpha_0 \in \mathbb{R}, \end{aligned}\right\} \tag{2}$$

and when $f(t, x, y) = f(t, y)$, the HDE (1) reduces to

$$\left.\begin{aligned} x'(t) &= f\big(t, M_x(t)\big), \quad t \in J, \\ x(t_0) &= \alpha_0 \in \mathbb{R}, \end{aligned}\right\} \tag{3}$$

The HDE (2) has been discussed in Dhage and Dhage [14] while HDE (3) is new to the literature as far as approximation of the local solution is concerned. Therefore the approximation results of this paper include the similar results for the HDEs (2) and (3) as special cases. The existence of local solution of the HDE (1) is proved by using the Schauder fixed point principle, see for example, Coddington [3], Lakshmikantham and Leela [21], Granas and Dugundji [18] and references therein. The approximation result for uniqueness of solution is proved by using the Banach fixed point theorem under a Lipschitz condition which is considered to be very strong in the area of nonlinear analysis. But to the knowledge the present authors, the approximation result for local existence and uniqueness theorems without using the Lipschitz condition is not discussed so far in the theory of nonlinear differential equations. In this paper, we discuss the approximation results for local existence and uniqueness of solution of the considered HDE (1) with maxima under weaker Lipschitz condition via construction of the algorithms based on the monotone iteration method and a hybrid fixed point theorem of Dhage [7]. Also see Dhage *et al.* [12] and references therein.

3. Auxiliary Results

We place the problem of HDE (1) in the function space $C(J, \mathbb{R})$ of continuous, real-valued functions defined on J. We introduce a supremum norm $\|\cdot\|$ in $C(J, \mathbb{R})$

defined by

$$\|x\| = \sup_{t \in J} |x(t)|, \tag{4}$$

and an order relation $\preceq$ in $C(J, \mathbb{R})$ by the cone K given by

$$K = \{x \in C(J, \mathbb{R}) \mid x(t) \geq 0 \ \forall\ t \in J\}. \tag{5}$$

Thus,

$$x \preceq y \iff y - x \in K, \tag{6}$$

or equivalently,

$$x \preceq y \iff x(t) \leq y(t) \ \forall\ t \in J.$$

It is known that the Banach space $C(J, \mathbb{R})$ together with the order relations $\preceq$ becomes an ordered Banach space which we denote for convenience, by $\big(C(J, \mathbb{R}), K\big)$. We denote the open and closed spheres centered at $x_0 \in C(J, \mathbb{R})$ of radius r, for some $r > 0$, by

$$B_r(x_0) = \{x \in C(J, \mathbb{R}) \mid \|x - x_0\| < r\} = B(x, r),$$

and

$$B_r[x_0] = \{x \in C(J, \mathbb{R}) \mid \|x - x_0\| \leq r\} = \overline{B(x, r)},$$

receptively. It is clear that $B_r[x_0] = \overline{B_r(x_0)}$. Let $M > 0$ be a real number. Denote

$$B_r^M[x_0] = \big\{x \in B_r[x_0] \,\big|\, |x(t_1) - x(t_2)| \leq M\,|t_1 - t_2| \ \text{ for } \ t_1, t_2 \in J\big\}. \tag{7}$$

Then, we have the following result.

Lemma 3.1. *The set $B_r^M[x_0]$ is compact in $C(J, \mathbb{R})$.*
Proof. By definition, $B_r[x_0]$ is a closed and bounded subset of the Banach space $C(J, \mathbb{R})$. Moreover, $B_r^M[x_0]$ is an equicontinuous subset of $C(J, \mathbb{R})$ in view of the condition (4). Now, by an application of Arzelá-Ascoli theorem, $B_r^M[x_0]$ is compact set in $C(J, \mathbb{R})$ and the proof of the lemma is complete.

It is well-known that the hybrid fixed point theoretic technique is very much useful in the subject of nonlinear analysis for dealing with the nonlinear equations qualitatively. See Granas and Dugundji [18] and the references therein. Here, we employ the Dhage monotone iteration method or simply *Dhage iteration method* based on the following two hybrid fixed point theorems of Dhage [7] and Dhage *et al.* [12].

Theorem 3.1. [Dhage [7]] *Let S be a non-empty partially compact subset of a regular partially ordered Banach space $\big(E, \|\cdot\|, \preceq,\big)$ with every chain C in S is*

Janhavi set and let $\mathcal{T} : S \to S$ be a monotone nondecreasing, partially continuous mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to ξ^*.

Theorem 3.2. [Dhage [7]] *Let $B_r[x]$ denote the partial closed ball centered at x of radius r, in a regular partially ordered Banach space $\big(E, \|\cdot\|, \preceq,\big)$ and let $\mathcal{T} : E \to E$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying*

$$\|x_0 - \mathcal{T}x_0\| \leq (1-q)r, \tag{8}$$

for some real number $r > 0$, then $\mathcal{T}$ has a unique comparable fixed point x^* in $B_r[x_0]$ and the sequence $\{x_n\}_{n=0}^\infty$ of successive iterations converges monotonically to x^*. Furthermore, if every pair of elements in X has a lower or upper bound, then x^* is unique.

Remark 3.1. *We note that every pair of elements in a partially ordered set* (**poset**) *$(E, \preceq)$ has a lower or upper bound if $(E, \preceq)$ is a lattice, that is, $\preceq$ is a lattice order in E. In this case the poset $(E, \|\cdot\|, \preceq)$ is called a* **partially lattice ordered Banach space**. *There do exist several lattice partially ordered Banach spaces which are useful for applications in nonlinear analysis. For example, every Banach lattice is a partially lattice ordered Banach space. The details of the lattice structure of the Banach spaces appear in Birkhoff* [1].

As a consequence of Remark 3.1, we obtain

Theorem 3.3. *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in a regular partially lattice ordered Banach space $\big(E, \|\cdot\|, \preceq,\big)$ and let $\mathcal{T} : E \to E$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (8), then $\mathcal{T}$ has a unique fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to ξ^*.*

If a Banach X is partially ordered by an order cone K in X, then in this case we simply say X is an **ordered Banach space** which we denote it by (X, K). Similarly, an ordered Banach space (X, K), where partial order $\preceq$ defined by the con K is a lattice order, then (X, K) is called the **lattice ordered Banach space**. Clearly, an ordered Banach space $\big(C(J, \mathbb{R}), K\big)$ of continuous real-valued functions defined on the closed and bounded interval J is lattice ordered Banach space, where the cone K is given by $K = \{x \in CJ, \mathbb{R}) \mid x \succeq 0\}$. The details of the cones and their properties appear in Guo and Lakshmikantham [19]. Then, we have the following useful results concerning the ordered Banach spaces proved in Dhage [5,6].

Lemma 3.2. [Dhage [5,6]] *Every ordered Banach space (X, K) is regular.*

Lemma 3.3. [Dhage [5,6]] *Every partially compact subset S of an ordered Banach space (X, K) is a Janhavi set in X.*

As a consequence of Lemmas 3.2 and 3.3, we obtain the following hybrid fixed point theorem which we need in what follows.

Theorem 3.4. [Dhage [7] and Dhage *et al.* [12]] *Let S be a non-empty partially compact subset of an ordered Banach space (X, K) and let $\mathcal{T} : S \to S$ be a partially continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq Tx_0$ or $x_0 \succeq Tx_0$, then $\mathcal{T}$ has a fixed point $x^* \in S$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations converges monotonically to x^*.*

Theorem 3.5. [Dhage [7]] *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in an ordered Banach space $\big(X, K\big)$ and let $\mathcal{T} : (X, K) \to (X, K)$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (8), then $\mathcal{T}$ has a unique comparable fixed point x^* in $B_r[x_0]$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive iterations converges monotonically to x^*. Furthermore, if every pair of elements in X has a lower or upper bound, then x^* is unique.*

Theorem 3.6. *Let $B_r[x]$ denote the partial closed ball centered at x of radius r for some real number $r > 0$, in a lattice ordered Banach space $\big(X, K\big)$ and let $\mathcal{T} : (X, K) \to (X, K)$ be a monotone nondecreasing and partial contraction operator with contraction constant q. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying (8), then $\mathcal{T}$ has a unique fixed point ξ^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_0^{\infty}$ of successive iterations converges monotonically to ξ^*.*

The details of the notions of partial order, Janhavi set, regularity, monotonicity, partial continuity, partial closure, partial compactness and partial contraction etc. and related applications appear in Dhage [4, 5, 6], Dhage and Dhage [8], Dhage *et al.* [12, 13] and references therein.

4. Local Approximation Results

We consider the following set of hypotheses in what follows.

(H_1) The function f is continuous and bounded on $J \times \mathbb{R} \times \mathbb{R}$ with bound M_f.

(H_2) There exist constants $\ell_1 > 0$ and $\ell_2 > 0$ such that

$$0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq \ell_1(x_1 - y_1) + \ell_2(x_2 - y_2)$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \geq y_1$, $x_2 \geq y_2$, where $(\ell_1 + \ell_2)\, a < 1$.

(H_3) $f(t,x,y)$ is nondecreasing in x and y for each $t \in J$.

(H_4) $f(t,\alpha_0,\alpha_0) \geq 0$ for all $t \in J$.

Then we have the following useful lemma.

Lemma 4.1. *If $h \in L^1(J,\mathbb{R})$, then the IVP of ordinary first order linear differential equation*

$$x'(t) = h(t), \ \ t \in J, \quad x(t_0) = \alpha_0, \tag{9}$$

is equivalent to the integral equation

$$x(t) = \alpha_0 + \int_{t_0}^{t} h(s)\,ds, \ \ ,t \in J. \tag{10}$$

Theorem 4.1. *Suppose that the hypotheses (H_1), (H_3) and (H_4) hold. Furthermore, if $M_f a \leq r$ and $M_f \leq M$, then the HDE (1) with maxima has a solution x^* in $B_r^M[x_0]$, where, $x_0 \equiv \alpha_0$, and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} x_0(t) &= \alpha_0, \ \ t \in J, \\ x_{n+1}(t) &= \alpha_0 + \int_{t_0}^{t} f\big(s, x_n(s), M_{x_n}(s)\big)\,ds, \ \ t \in J, \end{aligned}\right\} \tag{11}$$

where $n = 0, 1, \ldots$; is monotone nondecreasing and converges to x^*.

Proof. Set $X = C(J,\mathbb{R})$. Clearly, X is an ordered Banach space. Let x_0 be a constant function on J such that $x_0(t) = \alpha_0$ for all $t \in J$ and define a closed ball $B_r^M[x_0]$ in X defined by (6). By Lemma 3.1, $B_r^M[x_0]$ is a compact subset of X. By Lemma 4.1, the HDE with maxima (1) is equivalent to the nonlinear hybrid integral equation (HIE)

$$x(t) = \alpha_0 + \int_{t_0}^{t} f\big(s, x(s), M_x(s)\big)\,ds, \ \ t \in J. \tag{12}$$

Now, define an operator $\mathcal{T}$ on $B_r^M[x_0]$ into X by

$$\mathcal{T}x(t) = \alpha_0 + \int_{t_0}^{t} f\big(s, x_n(s), M_{x_n}(s)\big)\,ds, \ \ t \in J. \tag{13}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 3.4 on $B_r^M[x_0]$ in the following series of steps.

Step I: *The operator $\mathcal{T}$ maps $B_r^M[x_0]$ into itself.*

Firstly, we show that $\mathcal{T}$ maps $B_r^M[x_0]$ into itself. Let $x \in B_r^M[x_0]$ be arbitrary element. Then,

$$\begin{aligned} |\mathcal{T}x(t) - x_0(t)| &= \left| \int_{t_0}^{t} f\big(s, x_n(s), M_{x_n}(s)\big)\, ds \right| \\ &\le \int_{t_0}^{t} \left| f\big(s, x_n(s), M_{x_n}(s)\big) \right| ds \\ &< M_f \int_{t_0}^{t_0+a} ds \\ &= M_f a \le r. \end{aligned}$$

Taking the supremum over t in the above inequality yields

$$\|\mathcal{T}x - x_0\| \le M_f a \le r$$

which implies that $\mathcal{T}x \in B_r[x_0]$ for all $x \in B_r^M[x_0]$. Next, let $t_1, t_2 \in J$ be arbitrary. Then, we have

$$\begin{aligned} |\mathcal{T}x(t_1) - \mathcal{T}x(t_2)| &\le \left| \int_{t_1}^{t_2} \left| f\big(s, x_n(s), M_{x_n}(s)\big) \right| ds \right| \\ &\le M_f |t_1 - t_2| \\ &\le M\, |t_1 - t_2|. \end{aligned}$$

Therefore, $\mathcal{T}x \in B_r^M[x_0]$ for all $x \in B_r^M[x_0]$ As a result, we have $\mathcal{T}\left(B_r^M[x_0]\right) \subset B_r^M[x_0]$.

Step II: *$\mathcal{T}$ is a monotone nondecreasing operator.*

Let $x, y \in B_r^M[x_0]$ be any two elements such that $x \succeq y$. Then, from continuity of the function y we have an element $\xi^* \in [t_0, t]$ such that $y(\xi^*) = \max_{\xi \in [t_0,t]} y(\xi)$. But $x(\xi^*) \ge y(\xi^*)$. Consequently, $M_x(t) \ge M_y(t)$ for each $t \in J$. Hence,

$$\begin{aligned} \mathcal{T}x(t) &= \alpha_0 + \int_{t_0}^{t} f\big(s, x(s), M_x(s)\big)\, ds \\ &\ge \alpha_0 + + \int_{t_0}^{t} f\big(s, y(s), M_y(s)\big)\, ds \\ &= \mathcal{T}y(t), \end{aligned}$$

for all $t \in J$. So, $\mathcal{T}x \succeq \mathcal{T}y$, that is, $\mathcal{T}$ is monotone nondecreasing on $B_r^M[x_0]$.

Step III: *$\mathcal{T}$ is partially continuous operator.*

Let C be a chain in $B_r^M[x_0]$ and let $\{x_n\}$ be a sequence in C converging to a point $x \in C$. Then, $M_{x_n} \to M_x$ in view of the inequality

$$|M_{x_n}(t) - M_x(t)| \leq \|x_n - x\|$$

for all $t \in J$. Then, by dominated cnonvergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{T}x_n &= \lim_{n\to\infty} \left[\alpha_0 + \int_{t_0}^{t} f\big(s, x_n(s), M_{x_n}(s)\big)\, ds\right] \\
&= \alpha_0 + \lim_{n\to\infty} \int_{t_0}^{t} f(s, x_n(s), M_{x_n}(s))\, ds \\
&= \alpha_0 + \int_{t_0}^{t} \left[\lim_{n\to\infty} f(s, x_n(s), M_{x_n}(s))\right] ds \\
&= \alpha_0 + \int_{t_0}^{t} f\big(s, x(s), M(s)\big)\, ds \\
&= \mathcal{T}x(t)
\end{aligned}$$

for all $t \in J$. Therefore, $\mathcal{T}x_n \to \mathcal{T}x$ pointwise on J. As $\{\mathcal{T}x_n\} \subset B_r^M[x_0]$, $\mathcal{T}x_n$ is an equicontinuous sequence of points in X. As a reult, we have that $\mathcal{T}x_n \to \mathcal{T}x$ uniformly on J. Hence $\mathcal{T}$ is partially continuous operator on $B_r^M[x_0]$.

Step IV: *The element $x_0 \in B_r^M[x_0]$ satisfies the order relation $x_0 \preceq \mathcal{T}x_0$.*

Since (H_4) holds, one has

$$\begin{aligned}
x_0(t) &= \alpha_0 + \int_{t_0}^{t} f(s, x_0(s), x_0(s))\, ds \\
&\leq x_0(t) + \int_{t_0}^{t} f(s, \alpha_0, \alpha_0)\, ds \\
&= \alpha_0 + \int_{t_0}^{t} f(s, x_0(s), x_0(s))\, ds \\
&= \mathcal{T}x_0(t)
\end{aligned}$$

for all $t \in J$. As a result, we have $x_0 \preceq \mathcal{T}x_0$. This shows that the constant function x_0 in $B_r^M[x_0]$ serves as to satisfy the operator inequality $x_0 \preceq \mathcal{T}x_0$.

Thus, the operator $\mathcal{T}$ satisfies all the conditions of Theorem 3.4, and so $\mathcal{T}$ has a fixed point x^* in $B_r^M[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the HIE

(12) and consequently the HDE with maxima (1) has a local solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (11) converges monotone nondecreasingly to x^*. This completes the proof.

Next, we prove an approximation result for existence and uniqueness of the solution simultaneously under weaker form of Lipschitz condition.

Theorem 4.2. *Suppose that the hypotheses* (H_1), (H_2) *and* (H_4) *hold. Furthermore, if*

$$M_f a \leq \big[1 - (\ell_1 + \ell_2)a\big]r, \quad (\ell_1 + \ell_2)\, a < 1, \tag{14}$$

for some real number $r > 0$, *then the HDE with maxima (1) has a unique solution* x^* *in* $B_r[x_0]$ *defined on* J, *where* $x_0 \equiv \alpha_0$, *and the sequence* $\{x_n\}_{n=0}^{\infty}$ *of successive approximations defined by (11) converges monotone nondecreasingly to* x^*.

Proof. Set $(X, K) = \big(C(J, \mathbb{R}), \preceq\big)$ which is a lattice w.r.t. the lattice operations *meet* and *join* defined by $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$ respectively, and so every pair of elements of X has a lower and an upper bound. Let x_0 be a constant function on J such that $x_0(t) = \alpha_0$ for all $t \in J$ and consider closed sphere $B_r[x_0]$ centred at $x_0 \in C(J, \mathbb{R})$ of radius r, for some fixed $r > 0$, in the partially ordered Banach space (X, K).

Define an operator $\mathcal{T}$ on X into X by (13). Clearly, $\mathcal{T}$ is monotone nondecreasing on X. To see this, let $x, y \in X$ be two elements such that $x \succeq y$. Then, by hypothesis (H_2),

$$\mathcal{T}x(t) - \mathcal{T}y(t) = \int_{t_0}^{t} \big[f\big(s, x(s), M_x(s)\big) - f(s, y(s)), M_y(s)\big]\, ds \geq 0,$$

for all $t \in J$. Therefore, $\mathcal{T}x \succeq \mathcal{T}y$ and consequently $\mathcal{T}$ is monotone nondecresing on X.

Next, we show that $\mathcal{T}$ is a partial contraction on X. Let $x, y \in X$ be such that $x \succeq y$. Then, by hypothesis (H_2), we obtain

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}y(t)| &= \left| \int_{t_0}^{t} \big[f\big(s, x(s), M_x(s)\big) - f\big(s, y(s), M_y(s)\big)\big]\, ds \right| \\
&\leq \left| \int_{t_0}^{t} \big[\ell_1(x(s) - y(s)) + \ell_2(M_x(s) - M_y(s))\big]\, ds \right| \\
&= \int_{t_0}^{t} \big[\ell_1|x(s) - y(s)| + \ell_2|M_x(s) - M_y(s)|\big]\, ds \\
&< \int_{t_0}^{t_0+a} (\ell_1 + \ell_2)\|x - y\|\, ds \\
&= a\,(\ell_1 + \ell_2)\|x - y\|
\end{aligned}$$

for all $t \in J$, where $(\ell_1 + \ell_2)\,a < 1$. Taking the supremum over t in the above inequality yields

$$\|\mathcal{T}x - \mathcal{T}y\| \leq (\ell_1 + \ell_2)\,a\,\|x - y\|$$

for all comparable elements $x, y \in X$. This shows that $\mathcal{T}$ is a partial contraction on X with contraction constant $k\,a$. Furthermore, it can be shown as in the proof of Theorem 4.1 that the element $x_0 \in B_r^M[x_0]$ satisfies the relation $x_0 \preceq \mathcal{T}x_0$ in view of hypothesis (H$_4$). Finally, by hypothesis (H$_1$) and condition (14), one has

$$\begin{aligned}
\|x_0 - \mathcal{T}x_0\| &= \sup_{t\in J}\left|\int_{t_0}^{t} f\big(s, x_0, x_0\big)\,ds\right| \\
&\leq \sup_{t\in J}\int_{t_0}^{t} |f(s, \alpha_0, \alpha_0)|\,ds \\
&\leq M_f a \\
&\leq \big[1 - (\ell_1 + \ell_2)a\big]r
\end{aligned}$$

which shows that the condition (8) of Theorem 3.5 is satisfied. Hence $\mathcal{T}$ has a unique fixed point x^* in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the HIE (12) and consequently the HDE with maxima (1) has a unique local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (11) converges monotone nondecreasingly to x^*. This completes the proof.

Remark 4.1. *The conclusion of Theorems 4.1 and 4.2 also remains true if we replace the hypothesis (H$_4$) with the following one.*

(H$_4$) The function f satisfies $f(t, \alpha_0, \alpha_0) \leq 0$ for all $t \in J$.

In this case, the HDE (1) with maxima has a local solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (11) is monotone nonincreasing and converges to the solution x^*.

Remark 4.2. *If the initial condition in the equation (1) is such that $\alpha_0 > 0$, then under the conditions of Theorem 4.1, the HDE (1) with maxima has a local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (11) converges monotone nondecreasingly to the positive solution x^*. Similarly, under the conditions of Theorem 4.2, the HDE (1) with maxima has a unique local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (11) converges monotone nondecreasingly to the unique positive solution x^*.*

5. Approximation of Local Ulam-Hyers Stability

The Ulam-Hyers stability for various dynamic systems has already been discussed by several authors under the conditions of classical Schauder fixed point theorem (see Huang *et al.* [20], Tripathy [24] and references therein). Here, in the present paper, we discuss the approximation of the Ulam-Hyers stability of local solution of the HDE (1) with maxima under the conditions of hybrid fixed point principle stated in Theorem 3.5. We need the following definition in what follows.

Definition 5.1. *The HDE (1) with maxima is said to be locally Ulam-Hyers stable if for* $\epsilon > 0$ *and for each local solution* $y \in B_r[x_0]$ *of the inequality*

$$\left.\begin{aligned} &\big|y'(t) - f\big(t, y(t), M_y(t)\big)\big| \leq \epsilon, \quad t \in J,\\ &y(t_0) = \alpha_0 \in \mathbb{R}, \end{aligned}\right\} \tag{$*$}$$

there exists a constant $K_f > 0$ *such that*

$$\big|y(t) - \xi(t)\big| \leq K_f \epsilon \tag{$**$}$$

for all $t \in J$*, where* $\xi \in B_r[x_0]$ *is a local solution of the HDE with maxima (1) defined on* J*, where* $x_0 \equiv \alpha_0$*. The solution* ξ *of the HDE with maxima (1) is called Ulam-Hyers stable local solution on* J*.*

Theorem 5.1. *Assume that all the hypotheses of Theorem 4.2 hold. Then the HDE (1) with maxima has a unique Ulam-Hyers stable local solution* $x^* \in B_r[x_0]$*, where* $x_0 \equiv \alpha_0$*, and the sequence* $\{x_n\}_{n=0}^{\infty}$ *of successive approximations given by (11) converges monotone nondecreasingly to* x^**.*

Proof. Let $\epsilon > 0$ be given and let $y \in B_r[x_0]$ be a solution of the functional inequality (*) on J, that is, we have

$$\left.\begin{aligned} &\big|y'(t) - f\big(t, y(t), M_y(t)\big)\big| \leq \epsilon, \quad t \in J,\\ &y(0) = \alpha_0 \in \mathbb{R}_+. \end{aligned}\right\} \tag{15}$$

By Theorem 4.2, the HDE (1) with maxima has a unique local solution $\xi \in B_r[x_0]$. Then by Lemma 3.1, one has

$$\xi(t) = \alpha_0 + \int_{t_0}^{t} f\big(s, \xi(s), M_\xi(s)\big)\, ds, \quad t \in J. \tag{16}$$

Now, by integration of (15) yields the estimate:

$$\left|y(t) - \alpha_0 - \int_{t_0}^{t} f\big(s, y(s), M_y(s)\big)\, ds\right| \leq a\, \epsilon, \tag{17}$$

for all $t \in J$.

Next, from (16) and (17) we obtain

$$\begin{aligned} \big|y(t) - \xi(t)\big| &= \left|y(t) - \alpha_0 - \int_{t_0}^{t} f\big(s, \xi(s), M_\xi(s)\big)\, ds\right| \\ &\leq \left|y(t) - \alpha_0 - \int_{t_0}^{t} f\big(s, y(s), M_y(s)\big)\, ds\right| \\ &\quad + \left|\int_{t_0}^{t} f\big(s, y(s), M_y(s)\big)\big)\, ds - \int_{t_0}^{t} f\big(s, \xi(s), M_\xi(s)\big)\, ds\right| \\ &\leq a\,\epsilon + \int_{t_0}^{t} \big|f\big(s, y(s), M_y(s)\big) - f\big(s, \xi(s), M_\xi(s)\big)\big|\, ds \\ &\leq a\,\epsilon + (\ell_1 + \ell_2)\, a(\|y - \xi\|). \end{aligned}$$

Now, taking the supremum over t, we obtain

$$\|y - \xi\| \leq a\,\epsilon + (\ell_1 + \ell_2)a\|y - \xi\|$$

or

$$\|y - \xi\| \leq \left[\frac{a\,\epsilon}{1 - (\ell_1 + \ell_2)a}\right]$$

where, $(\ell_1 + \ell_2)a < 1$. Letting $K_f = \left[\dfrac{a}{1 - (\ell_1 + \ell_2)a}\right] > 0$, we obtain

$$\big|y(t) - \xi(t)\big| \leq K_f \epsilon$$

for all $t \in J$. As a result, ξ is a Ulam-Hyers stable local solution of the HDE with maxima (1) on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (11) converges monotone nondecreasingly to ξ. Consequently the HDE with maxima (1) is a locally Ulam-Hyers stable on J. This completes the proof.

Remark 5.1. *If the given initial condition in the equation (1) is such that $\alpha_0 > 0$, then under the conditions of Theorem 5.1, the HDE with maxima (1) has a unique Ulam-Hyers stable local positive solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (11) converges monotone nondecreasingly to x^*.*

6. The Examples

In this section we give a couple of example to illustrate the absttract ideas invlved in our results, Theorems 4.1 and 4.2.

Example 6.1. Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the IVP of nonlinear first order HDE with maxima,

$$x'(t) = \tanh x(t) + \tanh M_x(t), \;\; t \in [0,1]; \quad x(0) = \frac{1}{4}. \tag{18}$$

Here $\alpha_0 = \frac{1}{4}$ and $f(t, x, y) = \tanh x + \tanh y$ for $(t, x, y) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$. We show that f satisfies all the conditions of Theorem 4.1. Clearly, f is bounded on $[0, 1] \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = 2$ and so the hypothesis (H$_1$) is satisfied. Also the function $f(t, x, y)$ is nondecreasing in x and y for each $t \in [0, 1]$. Therefore, hypothesis (H$_3$) is satisfied. Moreover, $f(t, \alpha_0, \alpha_0) = f(t, \frac{1}{4}, \frac{1}{4}) = 2 \tanh(\frac{1}{4}) \geq 0$ for each $t \in [0, 1]$, and so the hypothesis (H$_4$) holds. If we take $r = 2$ and $M = 1$, all the conditions of Theorem 4.1 are satisfied. Hence, the HDE with maxima (18) has a local solution x^* in the closed ball $B_1^1[\frac{1}{4}]$ of $C(J, \mathbb{R})$ which is positive in view of Remark 4.2. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$\begin{aligned} x_0(t) &= \frac{1}{4}, \;\; t \in [0,1], \\ x_{n+1}(t) &= \frac{1}{4} + \int_0^t [\tanh x_n(s) + \tanh M_{x_n}(s)]\, ds, \;\; t \in [0,1], \end{aligned}$$

converges monotone nondecreasingly to x^*.

Example 6.2. Given a closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the IVP of nonlinear first order HDE with maxima,

$$x'(t) = \frac{1}{4}\left[\tan^{-1} x(t) + \tan^{-1} M_x(t)\right], \;\; t \in [0,1]; \quad x(0) = \frac{1}{4}. \tag{19}$$

Here $\alpha_0 = \dfrac{1}{4}$ and $f(t, x, y) = \dfrac{1}{4}\left[\tan^{-1} x + \tan^{-1} y\right]$ for $(t, x, y) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$. We show that f satisfies all the conditions of Theorem 4.2. Clearly, f is bounded on $[0, 1] \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = \frac{22}{28}$ and so, the hypothesis (H$_1$) is satisfied. Next, let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \geq y_1$, $x_2 \geq y_2$, Then there exist a constants ξ_1 and ξ_2 with $x_1 < \xi < y_1$ satisfying

$$0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq \frac{1}{4}\left[\frac{1}{1+\xi^2}(x_1 - y_1) + \frac{1}{1+\xi^2}(x_2 - y_2)\right]$$

for all $t \in [0, 1]$. So the hypothesis (H$_2$) holds with $\ell_1 = \frac{1}{4} \cdot \frac{1}{1+\xi_1^2}(x_1 - y_1)$ and $\ell_2 = \frac{1}{4} \cdot \frac{1}{1+\xi_2^2}(x_2 - y_2)$. Moreover, $f(t, \alpha_0, \alpha_0) = f\left(t, \frac{1}{4}, \frac{1}{4}\right) = \frac{1}{4} \tan^{-1}\left(\frac{1}{4}\right) + \frac{1}{4} \tan^{-1}\left(\frac{1}{4}\right) \geq 0$

for each $t \in [0,1]$, and so the hypothesis (H_4) holds. If we take $r = 2$, then we have

$$M_f a = \frac{11}{14} \leq \Big(1 - \frac{1}{2}\Big) \cdot 2 = \big[1 - (\ell_1 + \ell_2)a\big] r$$

and so, the condition (14) is satisfied. Thus, all the conditions of Theorem 4.2 are satisfied. Hence, the HDE with maxima (19) has a unique local solution x^* in the closed ball $B_2[\frac{1}{4}]$ of $C(J,\mathbb{R})$. This further in view of Remark 4.2 implies that the HDE with maxima (19) has a unique local positive solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_0(t) = \frac{1}{4}, \quad t \in [0,1],$$

$$x_{n+1}(t) = \frac{1}{4} + \int_0^t \tan^{-1} x_n(s)\, ds + + \int_0^t \tan^{-1} M_{x_n}(s)\, ds, \quad t \in [0,1],$$

converges monotone nondecreasingly to x^*. Moreover, the unique local positive solution x^* is Ulam-Hyers stable on $[0,1]$ in view of Definition 5.1. Consequently the HDE with maxima (19) is a locally Ulam-Hyers stable on the interval $[0,1]$.

7. The Conclusion

Finally, while concluding this paper, we remark that unlike the Schauder fixed point theorem we do not require any convexity argument in the proof of main existence theorem, Theorem 4.1. Similarly, we do not require the usual Lipschitz condition in the proof of uniqueness theorem, Theorem 4.2, but a weaker one sided or partial Lipschitz condition is enough to serve the purpose. However, in both the cases we are able to achieve the existence of local solution by monotone convergence of the successive approximations which otherwise is not possible usual compactness and Lipschitz type condition. This indicates the advantage of our new Dhage monotone iteration method over the earlier methods using Schauder and Banach fixed point principles. Similarly, the Dhage mnotone iteration method is also useful in obtaining the local approximate Ulan-Hyer stable solution via monotonic convergence of the sequence of successive approximations under weaker partial Lipschitz type condition. The quoted numerical examples in Section 6 indicate the validity of our hypotheses and abstract results of this paper. Moreover, the differential equation (1) with maxima considered in this paper is of very simple form, however other complex nonlinear IVPs of HDE with maximas may be considered and the present study can also be extended to such sophisticated nonlinear differential equations with maxima with appropriate modifications. These and other such problems form the further research scope in the subject of nonlinear differential and integral equations with applications. Some of the results in this direction will be reported elsewhere.

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