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BACKWARD DIFFERENTIATION FORMULA BASED NUMERICAL METHOD TO SOLVE FISHER EQUATION

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Abstract: This paper presents a novel numerical approach, which is based on the Method of Lines. This method semi-discretizes the problem and produces a system of ordinary differential equations (ODEs) in time. To solve this system, a stiff solver, BDF2, is used, which yields very precise results. The linearization is handled by the Taylor series method. To validate the numerical method, various test examples are considered. These formulas find extensive applications across various scientific and engineering domains.

Keywords and Phrases: Fisher equation; Taylor series; Backward differentiation formula; Method of lines.

2020 Mathematics Subject Classification: 65M20, 92D25, 65L06, 65L07.

1. Introduction

In this paper, we focus on the one-dimensional Fisher equation,

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \alpha u(1-u), \quad 0 \le x \le 1, \quad t > 0, \tag{1}$$

with initial condition

$$u(x,0) = u_0(x), \qquad 0 \le x \le 1$$

and the boundary conditions

$$u(0,t) = f_1(t), \qquad 0 \le t \le T$$
$$u(1,t) = f_2(t), \qquad 0 \le t \le T$$

A smooth function of the variables is defined as $u_0(x)$, f_1, and f_2, where $u(x,t)$ population density, D is coefficient factor and α is the reactive factor. The nonlinear reaction-diffusion equation was originally introduced by Fisher in 1937 [10]. The Fisher equation is commonly known as the KPP equation, an abbreviation for Kolmogorov-Petrovsky-Piscounov. However, the Fisher equation is the more well-known name for it. Equation (1) characterizes a nonlinear model of a physical system featuring linear diffusion and nonlinear evolution, as described in [1]. Numerous disciplines, including science and industry, have made significant use of Fisher's equation [11], [2], [4], [5], [15]. The interplay between diffusion and reaction is thus described by equation (1) [6].

The mathematical features of Fisher's equation have been thoroughly discussed in the literature. The overviews of Fisher's equation provided by Brazhnik and Tyson [28], Izadi and Srivastava [16], Kawahara and Tanaka [18], and Larson [19] are highly informative and well-regarded. Subsequently, numerous researchers have conducted numerical solutions for Fisher's equation. To investigate numerical approaches for Fisher's equation, Parekh and Puri [23] and Twizell et al. [27] introduced both implicit and explicit finite difference algorithms. The modified form of a nonlinear Fisher's reaction-diffusion equation solved by radial basis functions (RBFs) based on differential quadrature methods (DQMs) [13]. The Fisher equation in bounded domains, By Faedo–Galerkin's method and with a homogeneous Dirichlet conditions [12], The Fisher equation is solved by extended homogeneous balance method and it is used to solve many non-linear equation liker Fisher's equation and Burgers-Fisher equation [9]. The Fisher's equation is solved by the Lie symmetries of the generalized Fisher equation in [25].
Many researchers have worked to create numerical methods for solving partial differential equations (PDEs). The domain is discretized into a limited number of areas, backward differentiation formulas are the numerical techniques used to compute the solutions of PDEs. Several numerical solutions of equation (1) were studied under initial and boundary conditions see [26], [8], [24], [7], [14], [17] [29]. In this study, we utilize the Method of Lines (MOL) to tackle the one-dimensional Fisher equation, and we compute spatial derivatives using finite difference methods.

Consequently, the suggested approach [22], converts non-linear partial differential equations into a system of nonlinear ordinary differential equations over time. A Taylor series expansion and the Backward Differentiation Formula of Order Two (BDF-2) will be used to solve this problem. The proposed scheme is more effective because linear algebraic equations are immediately solved after being expanded using the Taylor series method. The accuracy of the suggested technique in both space and time is of order two.
This paper is organized into seven sections. The first section introduces the topic. The second section details the numerical scheme. In the third section, we present the stability analysis. The fourth section covers the Backward Differentiation Formula of order two (BDF2). Section five discusses the results and examples. The sixth section provides the conclusion, and the seventh section lists the abbreviations.

2. Numerical Scheme

We employ a regular grid to divide the solution domain of equation (1). The space between $[0, 1]$ is divided into N equal subintervals, and the time interval $[0, T]$ is divided into M equal subintervals. In the spatial dimension, we set the mesh width as $\Delta x = 1/N$, and the points x_j are defined as $x_j = j\Delta x$ for $j = 0, 1, ..., N$. For the temporal dimension, we set t^l as $t^l = l\Delta t$ for $l = 0, 1, ..., M$, where $\Delta t = T/M$ represents the mesh width in time.

2.1. Method of Lines(MOL)

The numerical techniques used in this paper involve linearization, BDF2 (Backward Differentiation Formula), and the Method of Lines (MOL). The method of lines (MOL), which is a semi-discretization technique, discretizes only in the spatial dimension. We approximate a spatial derivative using a central difference scheme, resulting in $N + 1$ equally spaced points and a spacing interval of $\Delta x = 1/N$, we get

$$\frac{\partial u}{\partial t}(x_j, t) = \frac{u_{j+1}(t) - u_{j-1}(t)}{2\Delta t}, \quad j = 1, 2, ..., N$$

$$\frac{\partial^2 u}{\partial x^2}(x_j, t) = \frac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{(\Delta x)^2}, \quad j = 1, 2, ..., N$$

Substituting into Fisher equation Eq. (1), we establish a set of initial conditions for a system of nonlinear ordinary differential equations. These initial conditions include the boundary conditions, $u_0(t) = 0$, $u_N(t) = 0$ and consider $D = 1$

$$\frac{du_j(t)}{dt} = \frac{u_{j-1}(t) - 2u_j(t) + u_{j+1}(t)}{(\Delta x)^2} + \alpha u_j(1 - u_j)$$

$$u_j(0) = u_0(x_j), \quad j = 1, 2, ..., N-1$$

When $u_j(t) = u(x_j, t)$, these differential equations can be formulated in the matrix form of size $(N-1) \times (N-1)$,

$$\frac{dU}{dt} = F(U, t), \tag{2}$$

$$U(0) = U_0$$

Where, $U(t) = [u_1(t), ..., u_N(t)]^T$.
With the following elements f_j, F is a function of U that exhibits non-linearity.

$$f_j(u_1, u_2, ..., u_{N-1,t}) = \lambda u_{j-1}(t) + u_j(\alpha - \alpha u_j - 2\lambda) + \lambda u_{j+1}(t) \tag{3}$$

Where, $\lambda = \frac{1}{(\Delta x)^2}, \quad j = 1, 2, ..., N-1$
The set of ordinary differential equations (2) forms a nonlinear system, which can be resolved by time integration. With a time step of $\Delta t = 1/M$, we divide the time period into $M+1$ equally spaced intervals. Following is the Backward Differentiation Formula for the time integration of order two.

Theorem 1. *Investigating the initial value problem (IVP) involves treating* $\mathbf{F}(\mathbf{U}, \mathbf{t})$ *as a continuous function and exploring it in the following manner:*

$$\frac{dU}{dt} = F(U, t), \quad U(t_0) = a,$$

Allows for the existence of a solution denoted as $U = f(t)$ *within the interval* $|(t - t_0)| \leq \delta$*, where* $\delta > 0$.
Proof. Considering the functions defined as follows:

$$f_j(u_1, u_2, ..., u_{K-1,t}) = \lambda u_{j-1}(t) + u_j(\alpha - \alpha u_j - 2\lambda) + \lambda u_{j+1}(t),$$

Given that j ranges from 1 to $l-1$, and recognizing that the functions are clearly continuous, we can confidently assert the existence of a solution for this Initial Value Problem (IVP).

Theorem 2. *Let* $\mathbf{C^1}$ *denote the set of functions that are differentiable and have continuous first derivatives.*
If $\mathbf{F}(\mathbf{U}, \mathbf{t}) \in \mathbf{C^1}$*, then a unique solution exists for the initial value problem (IVP).*

Proof. The partial derivatives of the functions described in equation (4) can be expressed as follows:

$$\frac{\partial f_j}{\partial u_j} = \alpha - 2\alpha u_j - 2\lambda, \quad j = 1, 2, ..., K-1, \tag{4}$$

$$\frac{\partial f_j}{\partial u_{j+1}} = \lambda, \quad j = 1, 2, ..., K-1, \tag{5}$$

$$\frac{\partial f_j}{\partial u_{j-1}} = \lambda, \quad j = 1, 2, ..., K-1, \tag{6}$$

$$\frac{\partial f_j}{\partial u_i} = 0, \quad j = 1, 2, ..., K-1, \quad i \neq j-1, j, j+1. \tag{7}$$

Every partial derivative of the function is in existence and remains continuous across the entire domain, thereby confirming that $\mathbf{F}(\mathbf{U}, \mathbf{t}) \in \mathbf{C^1}$. Consequently, the IVP possesses a distinct solution.
The forthcoming section will explore an examination of the stability of the nonlinear system.

3. Stability Analysis

In the context of nonlinear stability analysis, Lyapunov's stability theory stands out as a fundamental mathematical instrument. To assess stability, a significant approach involves determining the eigenvalues of the Jacobian matrix at the equilibrium point of a nonlinear autonomous system.
When we look at the nonlinear system described in Eq. (3), because it operates on its own without external influences, we observe the following:

$$\frac{dU}{dt} = F(U),$$
$$U(0) = U_0$$

In the context where F represents a nonlinear function of U, the elements f_j can be expressed as follows:

$$f_j(u_1, u_2, ..., u_{K-1,t}) = \lambda u_{j-1}(t) + u_j(\alpha - \alpha u_j - 2\lambda) + \lambda u_{j+1}(t),$$

For $j = 1, 2, \ldots, K-1$, we can expand F as a Taylor series centered around the equilibrium point $U^* = 0$.

$$\begin{aligned} F(U) &\approx F(U^*) + F(U^*)(U - U^*) \\ &\approx f'(U^*)U \end{aligned}$$

We will investigate the system's stability as described in Equation (5) by employing Lyapunov's Indirect Method.

3.1. Lyapunov's Indirect Method

Consider the equilibrium point at $x = 0$ for the equation $\dot{x} = f(x)$. Here, $f : D \longrightarrow \mathbb{R}^k$ is a continuously differentiable function, and D represents a neighborhood around the origin. let

$$A = \frac{\partial f}{\partial x}|_{x=0}$$

then
1. The origin exhibits asymptotic stability when the real part of each eigenvalue λ_i of matrix A satisfies $Re(\lambda_i) \leq 0$.
2. The origin is deemed unstable if there exists at least one eigenvalue A_i of matrix A such that $Re(\lambda_i) > 0$.

For the nonlinear system described in Equation (5), we can provide the Jacobian matrix as follows:

$$F(U^{(l+1)}) = F(U^{(l)}) + J_F^{(l)}(U^{(l+1)} - U^l) + O(\Delta t^2) \tag{8}$$

Where,

$$J_F^{(l)} = \begin{pmatrix} (\frac{\partial f_1}{\partial u_1})^{(l)} & (\frac{\partial f_1}{\partial u_2})^{(l)} & \cdots & (\frac{\partial f_1}{\partial u_{l-1}})^{(l)} \\ \vdots & & & \\ (\frac{\partial f_{l-1}}{\partial u_1})^{(l)} & (\frac{\partial f_{l-1}}{\partial u_2})^{(l)} & \cdots & (\frac{\partial f_{l-1}}{\partial u_{l-1}})^{(l)} \end{pmatrix}$$

The Jacobian matrix, denoted as matrix 'A' and evaluated at the equilibrium point, can be represented as a tridiagonal matrix, and its specific form is:

$$A = \begin{pmatrix} P_1 & \lambda & & & \\ \lambda & P_2 & \lambda & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda & P_{l-2} & \lambda \\ & & & \lambda & P_{l-1} \end{pmatrix}$$

Given that $P_j = \alpha - 2\alpha u_j - 2\lambda$, where $j = 1, 2, ..., l-1$.

4. Backward Differentiation Formula of order two (BDF2)

$$U^{(l+1)} = \frac{4}{3}U^{(l)} - \frac{1}{3}U^{l-1} + \frac{2}{3}(\Delta t)F(U^{l+1}, t^{l+1}), \quad l = 2, ..., M \tag{9}$$

Referring to BDF1 [21], we obtain the solution at the initial time level denoted as U_1, which is the first-order backward differentiation formula.

$$U^{(l+1)} = U^l + (\Delta t)F(U^{l+1}, t^{l+1}), \quad l = 0, 1, ..., M-1$$

U_0 represents the initial condition, and $U(l)$ is a vector denoted as $[u_1^{(l)}, u_2^{(l)}, \ldots, u_{l-1}^{(l)}]$. Because the system (3) exhibits non-linearity, it necessitates the solution of a non-linear algebraic equation at each time step. One way to handle this issue is by utilizing the linearization technique. linearization through the Taylor series,

$$F(U^{(l+1)}) = F(U^{(l)}) + J_F^{(l)}(U^{(l+1)} - U^l) + O(\Delta t^2) \tag{10}$$

Where,

$$J_F^{(l)} = \begin{pmatrix} (\frac{\partial f_1}{\partial u_1})^{(l)} & (\frac{\partial f_1}{\partial u_2})^{(l)} & \cdots & (\frac{\partial f_1}{\partial u_{N-1}})^{(l)} \\ \vdots & & & \\ (\frac{\partial f_{N-1}}{\partial u_1})^{(l)} & (\frac{\partial f_{N-1}}{\partial u_2})^{(l)} & \cdots & (\frac{\partial f_{N-1}}{\partial u_{N-1}})^{(l)} \end{pmatrix}$$

at the l^{th} time level, is the Jacobian matrix. Equation (9) is substituted in Equation (10) to provide,

$$U^{(l+1)} = \frac{4}{3}U^{(l)} - \frac{1}{3}U^{l-1} + \frac{2\Delta t[F(U^{(l)}) + J_F^{(l)}(U^{(l+1)}) - U^{(l)}]}{3}, \quad l = 2, ..., M \tag{11}$$

$$\begin{aligned} (I - \frac{2\Delta t}{3}J_F^{(l)})U^{(l+1)} &= (\frac{4}{3}I - \frac{2\Delta t}{3}J_F^{(l)})U^{(l)} + \frac{2\Delta t}{3}F(U^{(l)}) - \frac{1}{3}(U^{(l)}) \\ U^{(l+1)} =&(I - \frac{2\Delta t}{3}J_F^{(l)})^{-1}(\frac{4}{3}I - \frac{2\Delta t}{3}J_F^{(l)})U^{(l)} \\ &+ (I - \frac{2\Delta t}{3}J_F^{(l)})^{-1}\frac{2\Delta t}{3}F(U^{(l)}) - (I - \frac{2\Delta t}{3}J_F^{(l)})^{-1}\frac{1}{3}(U^{(l)}) \end{aligned} \tag{12}$$

Here, $J_F^{(l)}$ represents the Jacobian matrix at the l^{th} time step. The mentioned approach becomes linearized as a consequence. Avoid Newton's method, we only need to solve the computationally faster Equation (12), which consists of linear algebraic equations at each time step.

5. Results and Discussion

To illustrate the efficiency and suitability of the proposed numerical approach, multiple test experiments were carried out. For different α values and varying final time points, we have compared the calculated solution with the exact solution to evaluate their consistency.

Example 1. Consider the Fisher Equation (1) for $\alpha = 1$ and $\alpha = 6$

$$u_t = u_{xx} + \alpha u(1-u),$$

subject to the initial condition

$$u(x,0) = \frac{1}{(1+e^{\sqrt{\frac{\alpha}{6}}x})^2}$$

where the exact solution is presented in [27] given by

$$u(x,t) = \frac{1}{(1+e^{\sqrt{\frac{\alpha}{6}}x-\frac{5}{6}\alpha t})^2}.$$

Table 1: Numerical and exact results (BDF-2) for Example-1 are compared at different points of space at $\Delta t = 0.000005$, final time $T = 1$ and $\alpha = 1$

	Determined Result				**Exact solution**
x	$N = 20$	$N = 40$	$N = 80$	$N = 100$	
0	0.271254811	0.271254811	0.271254811	0.271254811	0.271254811
0.1	0.260738368	0.260738402	0.260738411	0.260738412	0.260738428
0.2	0.250420991	0.250421051	0.250421066	0.250421068	0.250421096
0.3	0.240311552	0.240311630	0.240311650	0.240311652	0.240311688
0.4	0.230418230	0.230418319	0.230418341	0.230418344	0.230418385
0.5	0.220748486	0.220748579	0.220748602	0.220748605	0.220748648
0.6	0.211309043	0.211309133	0.211309156	0.211309158	0.211309201
0.7	0.202105868	0.202105949	0.202105969	0.202105971	0.202106010
0.8	0.193144165	0.193144228	0.193144244	0.193144246	0.193144276
0.9	0.184428366	0.184428403	0.184428412	0.184428413	0.184428430
1	0.175962132	0.175962132	0.175962132	0.175962132	0.175962132

Table 2: Numerical and exact results (BDF-2) for Example-1 are compared at different points of space at $\Delta t = 0.000005$, final time $T = 1$ and $\alpha = 6$

	Determined Result				**Exact solution**
x	$N = 20$	$N = 40$	$N = 80$	$N = 100$	
0	0.387455619	0.387455619	0.387455619	0.387455619	0.387455619
0.1	0.358425003	0.358426076	0.358426345	0.358426377	0.358426914
0.2	0.329980634	0.329982645	0.329983147	0.329983208	0.329984205
0.3	0.302312544	0.302315293	0.302315981	0.302316063	0.302317425
0.4	0.275597376	0.275600623	0.275601435	0.275601532	0.275603147
0.5	0.249993802	0.249997280	0.249998151	0.249998255	0.250000000
0.6	0.225638652	0.225642075	0.225642931	0.225643034	0.225644772
0.7	0.202643927	0.202646993	0.202647760	0.202647852	0.202649430
0.8	0.181094865	0.181097255	0.181097853	0.181097925	0.181099172
0.9	0.161049109	0.161050483	0.161050827	0.161050868	0.161051594
1	0.142536957	0.142536957	0.142536957	0.142536957	0.142536957

Table 3: The error (BDF-2) at different points in the space of Example-1 is compared with the exact solution at $\Delta t = 0.000005$, final time $T = 1$ for $\alpha = 1$

	Absolute errors			
x	$N = 20$	$N = 40$	$N = 80$	$N = 100$
0.1	6.030E-08	2.56E-08	1.70E-08	1.59E-08
0.2	1.052E-07	4.48E-08	2.97E-08	2.79E-08
0.3	1.364E-07	5.82E-08	3.86E-08	3.63E-08
0.4	1.552E-07	6.63E-08	4.41E-08	4.14E-08
0.5	1.624E-07	6.95E-08	4.63E-08	4.35E-08
0.6	1.580E-07	6.78E-08	4.52E-08	4.25E-08
0.7	1.413E-07	6.07E-08	4.05E-08	3.81E-08
0.8	1.110E-07	4.78E-08	3.20E-08	3.01E-08
0.9	6.470E-08	2.79E-08	1.87E-08	1.76E-08

Table 4: The error (BDF-2) at different points in the space of Example-1 is compared with the exact solution at $\Delta t = 0.000005$, final time $T = 1$ for $\alpha = 6$

	Absolute errors			
x	$N = 20$	$N = 40$	$N = 80$	$N = 100$
0.1	1.9114E-06	8.3800E-07	5.6970E-07	5.3750E-07
0.2	3.5711E-06	1.5606E-06	1.0579E-06	9.9760E-07
0.3	4.8803E-06	2.1315E-06	1.4440E-06	1.3615E-06
0.4	5.7717E-06	2.5246E-06	1.7124E-06	1.6149E-06
0.5	6.1977E-06	2.7196E-06	1.8495E-06	1.7451E-06
0.6	6.1205E-06	2.6974E-06	1.8410E-06	1.7382E-06
0.7	5.5034E-06	2.4375E-06	1.6703E-06	1.5783E-06
0.8	4.3062E-06	1.9166E-06	1.3187E-06	1.2470E-06
0.9	2.4853E-06	1.1110E-06	7.6710E-07	7.2590E-07

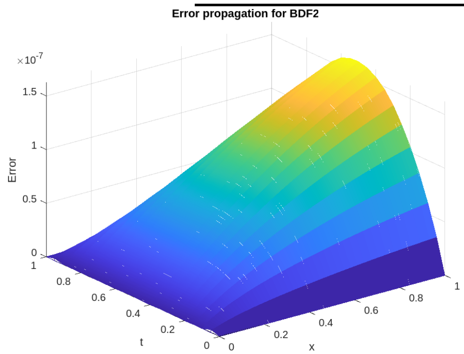

Figure 1: Solution at $\Delta t = 0.000005$, $N = 20$ for $\alpha = 1$

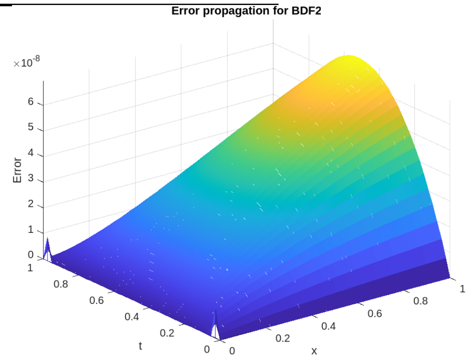

Figure 2: Solution at $\Delta t = 0.000005$, $N = 40$ for $\alpha = 1$

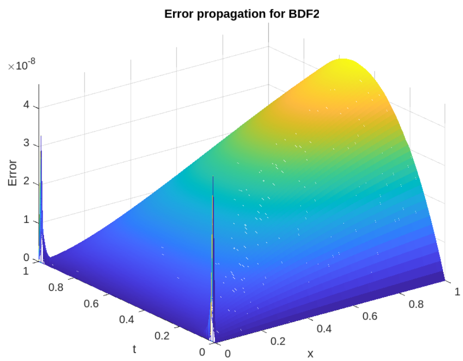

Figure 3: Solution at $\Delta t = 0.000005$, $N = 80$ for $\alpha = 1$

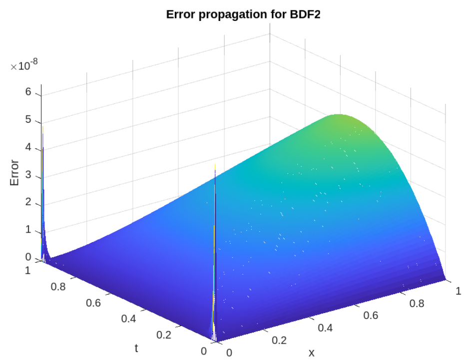

Figure 4: Solution at $\Delta t = 0.000005$, $N = 100$ for $\alpha = 1$

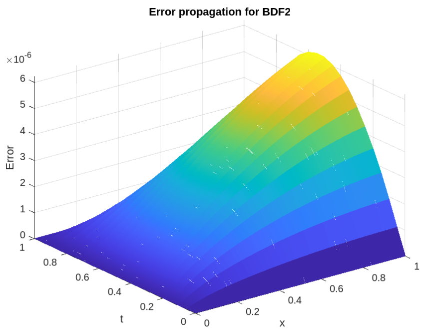

Figure 5: Solution at $\Delta t = 0.000005$, $N = 20$ for $\alpha = 6$

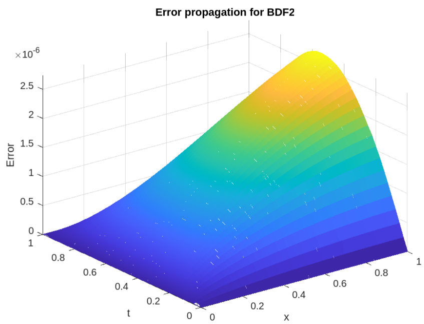

Figure 6: Solution at $\Delta t = 0.000005$, $N = 40$ for $\alpha = 6$

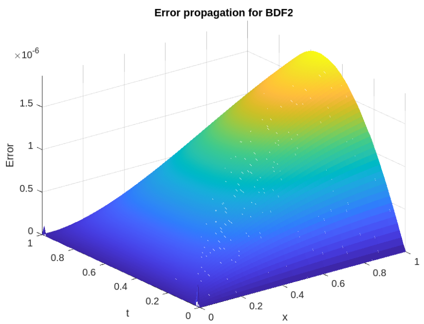

Figure 7: Solution at $\Delta t = 0.000005$, $N = 80$ for $\alpha = 6$

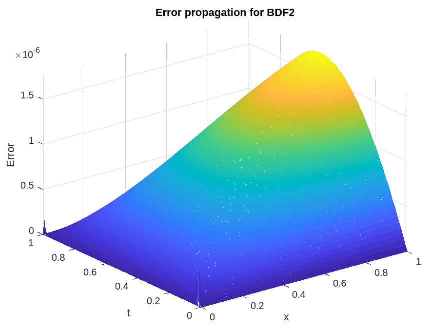

Figure 8: Solution at $\Delta t = 0.000005$, $N = 100$ for $\alpha = 6$

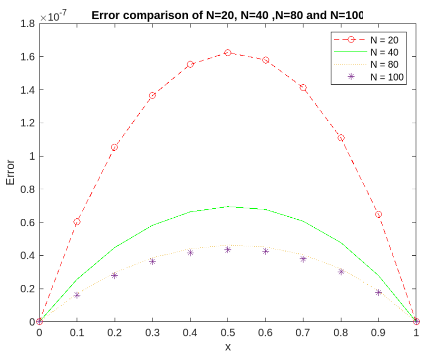

Figure 9: Absolute errors at $\Delta t = 0.000005$, for $\alpha = 1$

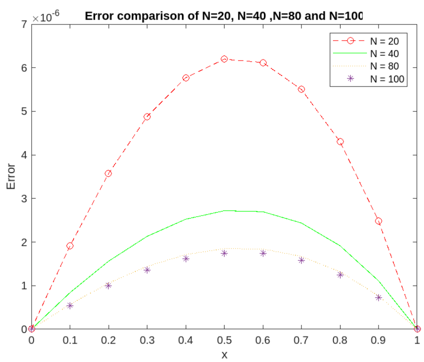

Figure 10: Absolute errors at $\Delta t = 0.000005$, for $\alpha = 6$

Table 5: Numerical and exact results for Example-1 are compared at $\Delta t = 0.00005$, $N = 20$ for $\alpha = 6$

T	x	**DQM [3], [20]**	**Present Method**	**Exact Solution**
0.5	0.25	0.81847	0.818403	0.818393
	0.75	0.72592	0.725835	0.725824
1.0	0.25	0.98293	0.982920	0.982919
	0.75	0.97208	0.972073	0.972071

Table 6: Numerical and exact results for Example-1 are compared at $\Delta t = 0.001$, $T = 0.25$ for $\alpha = 6$

value of x	**SIS [6]**	**Present Method**	**Exact Solution**
0.00	0.2627	0.262654	0.262654
0.25	0.2027	0.202584	0.202649
0.50	0.1516	0.151554	0.151602
0.75	0.1101	0.110053	0.110099
1.00	0.0778	0.077777	0.077777

Example 2. Consider the generalized form in the range [0, 1] as follows:

$$u_t = u_{xx} + u(1 - u^\eta) \tag{13}$$

with an initial condition

$$u(x, 0) = \{\frac{1}{2}\tanh(-\frac{\eta}{2\sqrt{2\eta + 4}}x) + \frac{1}{2}\}^{\frac{2}{\eta}} \tag{14}$$

The exact solution is discussed in [30], [3] for by

$$u(x, t) = \{\frac{1}{2}\tanh(-\frac{\eta}{2\sqrt{2\eta + 4}}(x - \frac{\eta + 4}{\sqrt{2\eta + 4}}t)) + \frac{1}{2}\}^{\frac{2}{\eta}}. \tag{15}$$

Table 7: Numerical and exact solutions (BDF-2) for Example-2 are compared at various "spatial points" and at $\Delta t = 0.0005$, final time $T = 1$ for $\eta = 1$

	Determined Result				**Exact solution**
x	$N = 20$	$N = 40$	$N = 80$	$N = 100$	
0	0.271254811	0.271254811	0.271254811	0.271254811	0.271254811
0.1	0.260736853	0.260736886	0.260736895	0.260736895	0.260738428
0.2	0.250418342	0.250418400	0.250418415	0.250418417	0.250421096
0.3	0.240308101	0.240308176	0.240308195	0.240308198	0.240311688
0.4	0.230414285	0.230414371	0.230414392	0.230414395	0.230418385
0.5	0.220744344	0.220744433	0.220744455	0.220744458	0.220748648
0.6	0.211305002	0.211305089	0.211305110	0.211305113	0.211309201
0.7	0.202102247	0.202102325	0.202102344	0.202102346	0.202106010
0.8	0.193141319	0.193141380	0.193141395	0.193141397	0.193144276
0.9	0.184426699	0.184426734	0.184426743	0.184426744	0.184428430
1	0.175962132	0.175962132	0.175962132	0.175962132	0.175962132

Table 8: Absolute error (BDF-2) at different points in the space of Example-2 is compared with the exact solution at $\Delta t = 0.0005$, final time $T = 1$ for $\eta = 1$

	Absolute errors			
x	$N = 20$	$N = 40$	$N = 80$	$N = 100$
0.1	8.4660E-07	1.5415E-06	1.5328E-06	1.5329E-06
0.2	2.2096E-06	2.6955E-06	2.6809E-06	2.6791E-06
0.3	3.2119E-06	3.5117E-06	3.4928E-06	3.4906E-06
0.4	3.8822E-06	4.0141E-06	3.9927E-06	3.9902E-06
0.5	4.2402E-06	4.2152E-06	4.1929E-06	4.1902E-06
0.6	4.2913E-06	4.1122E-06	4.0906E-06	4.0879E-06
0.7	4.0240E-06	3.6848E-06	3.6654E-06	3.6631E-06
0.8	3.4090E-06	2.8963E-06	2.8810E-06	2.8792E-06
0.9	2.4008E-06	1.6962E-06	1.6871E-06	1.6869E-06

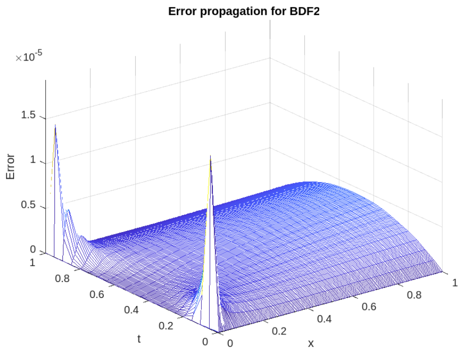

Figure 11: Solution at $\Delta t = 0.0005$, $N = 20$, and $\eta = 1$ for Example-2

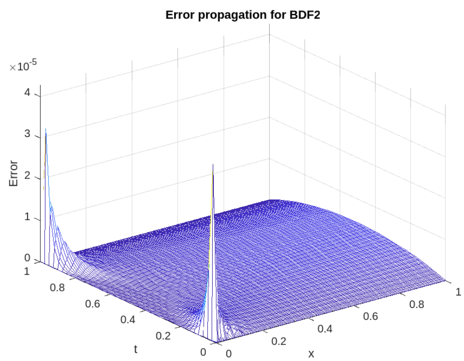

Figure 12: Solution at $\Delta t = 0.0005$, $N = 40$, and $\eta = 1$ for Example-2

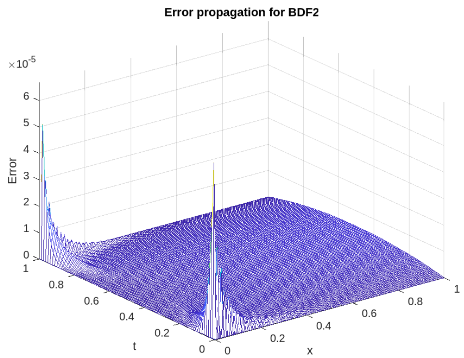

Figure 13: Solution at $\Delta t = 0.0005$, $N = 80$, and $\eta = 1$ for Example-2

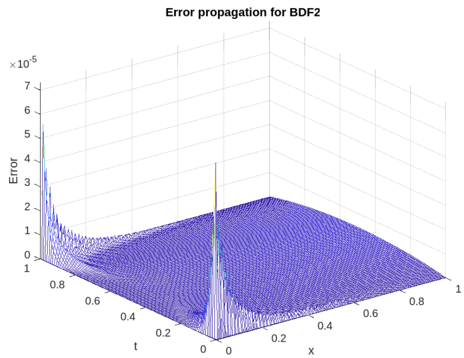

Figure 14: Solution at $\Delta t = 0.0005$, $N = 100$, and $\eta = 1$ for Example-2

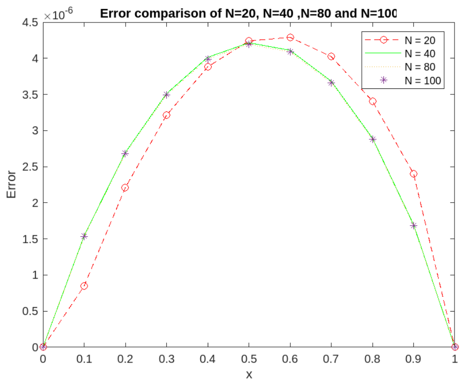

Figure 15: Absolute error at $\Delta t = 0.0005$ and $\eta = 1$. for Example-2

Table 9: Numerical and exact results for Example-2 are compared at different points of space at $\Delta t = 0.00005$, $N = 20$ and $\eta = 1$

T	x	**DQM [3], [20]**	**CFD6 [3], [20]**	**Present Method**	**Exact Solution**
0.5	0.25	0.33412	0.334094	0.334094	0.334094
	0.75	0.27838	0.278353	0.278353	0.278353
1.0	0.25	0.45576	0.455739	0.455739	0.455739
	0.75	0.39544	0.395411	0.395411	0.395411

In this paper, the Fisher equation is solved both logically and numerically using the proposed scheme BDF2. Comparison between numerical results and the exact solution with different numbers of partitions on the X-axis is presented in Tables 1 and 2 for different values of $\alpha = 1$ and 6 for Example-1, and in Table 5 for $\alpha = 1$ for Example-2. Our proposed scheme signifies consistency as the numerical results get closer to the exact solution when the number of partitions is increased. Tables 3 and 4 represent absolute errors for two values of $\alpha = 1$ and 6 for Example-1, while Table 6 also represents absolute errors for $\alpha = 1$ for Example-2. The table of absolute errors also indicates that as the number of partitions increases, the numerical results get closer to the exact solution. Figures 1 to 8 and 11 to 14 represent numerical results at different nodes for Examples 1 and 2. Additionally, Figures 9, 10 and 15 illustrate graphs of absolute errors at different nodes for both Example-1 and Example-2. In Tables 5 and 9, we conducted comparisons with existing numerical methods as detailed in references [3], [20]. The present method was compared at different time levels ($T = 0.5$ and 1) for $\alpha = 6$ for both Example-1 and Example-2. Similarly, in Table 6, we conducted a comparison with existing numerical methods as detailed in reference [6], comparing at the time level $T = 0.25$ and for $\alpha = 6$ for Example-1. It is clear that our present method provides more accuracy result compared to the existing methods in references [3], [20] and [6].

6. Conclusions

An attempt has been made to solve Fisher's equation using a second-order backward differentiation formula. The numerical method compares the Method of Lines and the second-order backward differentiation formula. The Method of Lines decomposes the Fisher's equation into a system of nonlinear ODEs, which is subsequently solved by BDF2. The Taylor series based linearization technique is used to handle the non-linearity. Several test examples are solved by the proposed numerical method for various values of α. The computational results are quite precise and exhibit good consistency with the exact solution. The results are also compared with a few existing methods and they are found to be more precise and accurate. The linearization technique used in the paper reduces both the cost and computation time, making the current numerical strategy more effective than previous schemes described in the literature.

7. Abbreviations

ODE - Ordinary Differential Equation
BDF2 - Backward Differentiation Formula of Order Two
RBF - Radial Basis Functions
DQM - Differential Quadrature Method
PDE - Partial Differential Equation
MOL - Method of Lines
SIS - Semi-Implicit Scheme

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