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GENERALIZED MITTAG-LEFFLER TYPE POISSON DISTRIBUTION

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Abstract: The main object of this paper is to introduce generalized Mittag-Leffler type Poisson distribution. We obtain results regarding moments about the origin, Moment generating function, probability generating function and characteristic function for this distribution. It should be worthy to note that the above results reduce to corresponding results of Mittag-Leffler type Poisson distribution and Poisson distribution for specific values of parameters.

Keywords and Phrases: Generalized Mittag-Leffler function, Poisson distribution, Moments, Moment generating function.

2020 Mathematics Subject Classification: 60E05.

1. Introduction

The probability distribution play an important role in Mathematics and Statistics. In 2017, Porwal and Dixit [4] investigate Mittag-Leffler type Poisson distribution with the help of Mittag-Leffler function. They obtain moments about the origin, mean, variance and Moment generating function for this distribution. In fact this distribution is a generalization of Poisson distribution. After the appearance of this paper, further properties and applications of this distribution in univalent function theory and image processing are investigated by various researchers [1, 2]. Motivating with the above mentioned work Srivastava et al. [8] introduced the Miller-Ross-type Poisson distribution and Porwal et al. [5] investigated Wright distribution and its application in univalent functions. This opens a new direction of research to define discrete probability distribution with the help of special functions. It is worth noting that the Mittag-Leffler function holds significant importance in mathematics and modern sciences. This importance has led to its generalization by several mathematicians over time. This motivates us to define generalized Mittag-Leffler type Poisson distribution. Now, let's recall the definition of the Mittag-Leffler function and its various generalizations. In 1903, Swedish Mathematician Mittag-Leffler [3] introduced the function

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \tag{1.1}$$

where z is a complex variable and $\Gamma(\alpha)$ is a Gamma function, $\alpha \geq 0$. This function is known as Mittag-Leffler function and have a great importance in the study of Mathematics, Physics, Engineering, Science and Technology.

It is a direct generalization of the exponential function for $\alpha = 1$.

The generalization of $E_\alpha(z)$ was studied by Wiman [9] in 1905 and he defined the function as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \tag{1.2}$$

$$(\alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0).$$

which is known as Wiman's function or generalized Mittag-Leffler function. For $\beta = 1$ in the above function we obtain Mittag-Leffler function defined by (1.1) and for $\alpha = 1, \beta = 1$ it reduce to exponential function.

In 1971, Prabhakar [6] introduced the function

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n(\gamma)_n}{\Gamma(\alpha n + \beta)n!}, \tag{1.3}$$

$$(\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0)$$

where $(\gamma)_n$ is the Pochhammer symbol.

The function $E^{\gamma}_{\alpha,\beta}$ is a most natural generalization of the exponential function e^x, Mittag-Leffler function $E_\alpha(z)$ and Wiman's function $E_{\alpha,\beta}(z)$. In continuation of the study, in 2007 Shukla and Prajapati [7] investigated the function $E^{\gamma,q}_{\alpha,\beta}(z)$ and defined as

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\mathrm{qn}}}{\Gamma(\alpha n + \beta) n!} z^n \tag{1.4}$$

This function is defined for $\alpha, \beta, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\gamma) > 0,\ q \in (0,1) \cup \mathbb{N}$ and $(\gamma)^{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma\gamma}$ denotes the generalized Pochhammer symbol which is particular reduces to $q^{qn} \prod_{r=1}^{q} \left(\frac{\gamma+r-1}{q}\right)_n if q \in \mathbb{N}$.

The function $E^{\gamma,q}_{\alpha,\beta}(z)$ converges absolutely for all z if $q < \mathbb{R}\alpha + 1$. It is an entire function of order $(\mathbb{R}\alpha)^{-1}$.

2. Generalized Mittag-Leffler type Poisson distribution

In 2017, Porwal and Dixit [4] introduce a new probability distribution with the help of Mittag-Leffler function and its probability mass function by

$$P(m, \alpha, \beta, k) = \frac{m^k}{E_{\alpha,\beta}(m)\Gamma(\alpha k + \beta)}, k = 0, 1, 2 \dots$$

where $m > 0, \alpha > 0, \beta > 0$. This distribution is known as the Mittag-Leffler type Poisson distribution. Furthermore, several researchers have explored its applications, particularly in geometric function theory [2]. This inspires us to define a new probability distribution. Therefore, in this work, we define the probability mass function of generalized Mittag-Leffler type Poisson distribution in the following way

$$P(X) = \frac{m^k (\gamma)_{qk}}{E^{\gamma,q}_{\alpha,\beta}(m) k! \Gamma(\alpha k + \beta)}, k = 0, 1, 2 \dots \tag{2.1}$$

where $m > 0, \gamma > 0, \alpha > 0, \beta > 0, q \in (0,1) \cup \mathbb{N}$.

It is easy to verify that the expression $P(X)$ given in (2.1) is a probability mass function because $P(X) \geq 0$ and $\sum_{k=0}^{\infty} P(x) = 1$. In this paper, we obtain first four moments about the origin, moment generating function, probability generating function and characteristic function for this distribution. It should be worthy to note that for $\gamma = 1$,$q = 1$ this distribution reduce to Mittag-Leffler type Poisson distribution studied in [4]. Further, for $\gamma = 1, q = 1, \alpha = 1, \beta = 1$ it reduce to Poisson distribution.

Definition 2.1. *The r^{th} moment of a discrete probability distribution about $X = 0$ is defined by*

$$\mu_r^{'} = \sum_{k=0}^{\infty} k^r P(X).$$

Here $\mu_1^{'}$ is known as mean of the distribution and variance of the distribution is given by $\mu_2^{'} - \left(\mu_1^{'}\right)^2$.

In our first result we give first four moments about the origin.

Theorem 2.1. *The first four moments about the origin of generalized Mittag-Leffler type Poisson distribution is given by the relation*

$$\mu_1^{'} = m(\gamma)_q \frac{E_{\alpha,\beta+\alpha}^{\gamma+q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}.$$

$$\mu_2^{'} = m^2(\gamma)_{2q} \frac{E_{\alpha,\beta+2\alpha}^{\gamma+2q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)} + m(\gamma)_q \frac{E_{\alpha,\beta+\alpha}^{\gamma+q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}.$$

$$\mu_3^{'} = m^3(\gamma)_{3q} \frac{E_{\alpha,\beta+3\alpha}^{\gamma+3q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)} + 3m^2(\gamma)_{2q} \frac{E_{\alpha,\beta+2\alpha}^{\gamma+2q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)} + m(\gamma)_q \frac{E_{\alpha,\beta+\alpha}^{\gamma+q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}.$$

$$\mu_4^{'} = m^4(\gamma)_{4q} \frac{E_{\alpha,\beta+4\alpha}^{\gamma+4q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)} + 6m^3(\gamma)_{3q} \frac{E_{\alpha,\beta+3\alpha}^{\gamma+3q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)} + 7m^2(\gamma)_{2q} \frac{E_{\alpha,\beta+2\alpha}^{\gamma+2q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}$$
$$+ m(\gamma)_q \frac{E_{\alpha,\beta+\alpha}^{\gamma+q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}.$$

Proof.

$$\begin{aligned}
\mu_1^{'} &= \sum_{k=0}^{\infty} k \frac{m^k (\gamma)_{\mathrm{qk}}}{E_{\alpha,\beta}^{\gamma,q}(m) k! \Gamma(\alpha k + \beta)} \\
&= \sum_{k=1}^{\infty} \frac{m^k (\gamma)_{\mathrm{qk}}}{E_{\alpha,\beta}^{\gamma,q}(m)(k-1)! \Gamma(\alpha k + \beta)} \\
&= \sum_{k=0}^{\infty} \frac{m^{k+1} (\gamma)_{q(k+1)}}{E_{\alpha,\beta}^{\gamma,q}(m)(k)! \Gamma\left(\alpha(k+1) + \beta\right)}
\end{aligned}$$

(Using the Pochhammer symbol)

$$= m \sum_{k=0}^{\infty} \frac{m^k (\gamma)_q (\gamma + q)_{qk}}{E_{\alpha,\beta}^{\gamma,q}(m)(k)! \Gamma\left(\alpha(k+1) + \beta\right)}$$

$$= m(\gamma)_q \sum_{k=0}^{\infty} \frac{m^k(\gamma+q)_{qk}}{E_{\alpha,\beta}^{\gamma,q}(m)(k)!\Gamma(\alpha k+\alpha+\beta)}.$$

$$\mu_1^{'} = m(\gamma)_q \frac{E_{\alpha,\beta+\alpha}^{\gamma+q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}. \tag{2.2}$$

$$\begin{aligned}
\mu_2^{'} &= \sum_{k=0}^{\infty} k^2 \frac{m^k(\gamma)_{qk}}{E_{\alpha,\beta}^{\gamma,q}(m)k!\Gamma(\alpha k+\beta)} \\
&= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \left[\sum_{k=0}^{\infty} \frac{k(k-1)m^k(\gamma)_{qk}}{k!\Gamma(\alpha k+\beta)} + \sum_{k=0}^{\infty} \frac{km^k(\gamma)_{qk}}{k!\Gamma(\alpha k+\beta)} \right] \\
&= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \left[\sum_{k=2}^{\infty} \frac{m^k(\gamma)_{qk}}{(k-2)!\Gamma(\alpha k+\beta)} + \sum_{k=1}^{\infty} \frac{m^k(\gamma)_{qk}}{(k-1)!\Gamma(\alpha k+\beta)} \right] \\
&= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \left[\sum_{k=0}^{\infty} \frac{m^{k+2}(\gamma)_{q(k+2)}}{k!\Gamma\left(\alpha(k+2)+\beta\right)} + \sum_{k=0}^{\infty} \frac{m^{k+1}(\gamma)_{q(k+1)}}{k!\Gamma\left(\alpha(k+1)+\beta\right)} \right] \\
&= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \left[m^2 \sum_{k=0}^{\infty} \frac{m^k(\gamma)_{q(k+2)}}{k!\Gamma(\alpha k+2\alpha+\beta)} + m \sum_{k=0}^{\infty} \frac{m^k(\gamma)_{q(k+1)}}{k!\Gamma(\alpha k+\alpha+\beta)} \right] \\
&= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \left[m^2(\gamma)_{2q} \sum_{k=0}^{\infty} \frac{m^k(\gamma+2q)_{qk}}{k!\Gamma(\alpha k+2\alpha+\beta)} + m(\gamma)_q \sum_{k=0}^{\infty} \frac{m^k(\gamma+q)_{qk}}{k!\Gamma(\alpha k+\alpha+\beta)} \right].
\end{aligned}$$

$$\mu_2^{'} = m^2(\gamma)_{2q} \frac{E_{\alpha,\beta+2\alpha}^{\gamma+2q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)} + m(\gamma)_q \frac{E_{\alpha,\beta+\alpha}^{\gamma+q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}. \tag{2.3}$$

$$\begin{aligned}
\mu_3^{'} &= \sum_{k=0}^{\infty} k^3 \frac{m^k(\gamma)_{qk}}{E_{\alpha,\beta}^{\gamma,q}(m)k!\Gamma(\alpha k+\beta)} \\
&= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \left[\sum_{k=0}^{\infty} \frac{k(k-1)(k-2)m^k(\gamma)_{qk}}{k!\Gamma(\alpha k+\beta)} + 3\sum_{k=0}^{\infty} \frac{k(k-1)m^k(\gamma)_{qk}}{k!\Gamma(\alpha k+\beta)} + \sum_{k=0}^{\infty} \frac{km^k(\gamma)_{qk}}{k!\Gamma(\alpha k+\beta)} \right] \\
&= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \left[\sum_{k=3}^{\infty} \frac{m^k(\gamma)_{qk}}{(k-3)!\Gamma(\alpha k+\beta)} + 3\sum_{k=2}^{\infty} \frac{m^k(\gamma)_{qk}}{(k-2)!\Gamma(\alpha k+\beta)} + \sum_{k=1}^{\infty} \frac{m^k(\gamma)_{qk}}{(k-1)!\Gamma(\alpha k+\beta)} \right] \\
&= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \left[\sum_{k=0}^{\infty} \frac{m^{k+3}(\gamma)_{q(k+3)}}{(k)!\Gamma\left(\alpha(k+3)+\beta\right)} + 3\sum_{k=0}^{\infty} \frac{m^{k+2}(\gamma)_{q(k+2)}}{(k)!\Gamma\left(\alpha(k+2)+\beta\right)} + \sum_{k=0}^{\infty} \frac{m^{k+1}(\gamma)_{q(k+1)}}{(k)!\Gamma\left(\alpha(k+1)+\beta\right)} \right] \\
&= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \left[\begin{array}{r} m^3(\gamma)_{3q} \sum_{k=0}^{\infty} \frac{m^k(\gamma+3q)_{qk}}{k!\Gamma(\alpha k+3\alpha+\beta)} + 3m^2(\gamma)_{2q} \sum_{k=0}^{\infty} \frac{m^k(\gamma+2q)_{qk}}{k!\Gamma(\alpha k+2\alpha+\beta)} \\ +m(\gamma)_q \sum_{k=0}^{\infty} \frac{m^k(\gamma+q)_{qk}}{k!\Gamma(\alpha k+\alpha+\beta)} \end{array} \right].
\end{aligned}$$

$$\mu_3^{'} = m^3(\gamma)_{3q}\frac{E_{\alpha,\beta+3\alpha}^{\gamma+3q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}+3m^2(\gamma)_{2q}\frac{E_{\alpha,\beta+2\alpha}^{\gamma+2q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}+m(\gamma)_q\frac{E_{\alpha,\beta+\alpha}^{\gamma+q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}. \tag{2.4}$$

$$\mu_4^{'} = \sum_{k=0}^{\infty} k^4 \frac{m^k(\gamma)_{qk}}{E_{\alpha,\beta}^{\gamma,q}(m)k!\Gamma(\alpha k+\beta)}$$

$$= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)}\left[\sum_{k=0}^{\infty}\frac{k(k-1)(k-2)(k-3)m^k(\gamma)_{qk}}{k!\Gamma(\alpha k+\beta)}+6\sum_{k=0}^{\infty}\frac{k(k-1)(k-2)m^k(\gamma)_{qk}}{k!\Gamma(\alpha k+\beta)}+7\sum_{k=0}^{\infty}\frac{k(k-1)m^k(\gamma)_{qk}}{k!\Gamma(\alpha k+\beta)} + \sum_{k=0}^{\infty}\frac{km^k(\gamma)_{qk}}{k!\Gamma(\alpha k+\beta)}\right]$$

$$= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)}\left[\sum_{k=4}^{\infty}\frac{m^k(\gamma)_{qk}}{(k-4)!\Gamma(\alpha k+\beta)}+6\sum_{k=3}^{\infty}\frac{m^k(\gamma)_{qk}}{(k-3)!\Gamma(\alpha k+\beta)}+7\sum_{k=2}^{\infty}\frac{m^k(\gamma)_{qk}}{(k-2)!\Gamma(\alpha k+\beta)} + \sum_{k=1}^{\infty}\frac{m^k(\gamma)_{qk}}{(k-1)!\Gamma(\alpha k+\beta)}\right]$$

$$= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)}\left[\sum_{k=0}^{\infty}\frac{m^{k+4}(\gamma)_{q(k+4)}}{(k)!\Gamma(\alpha(k+4)+\beta)}+6\sum_{k=0}^{\infty}\frac{m^{k+3}(\gamma)_{q(k+3)}}{(k)!\Gamma(\alpha(k+3)+\beta)} +7\sum_{k=0}^{\infty}\frac{m^{k+2}(\gamma)_{q(k+2)}}{(k)!\Gamma(\alpha(k+2)+\beta)}+\sum_{k=0}^{\infty}\frac{m^{k+1}(\gamma)_{q(k+1)}}{(k)!\Gamma(\alpha(k+1)+\beta)}\right]$$

$$= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)}\left[m^4(\gamma)_{4q}\sum_{k=0}^{\infty}\frac{m^k(\gamma+4q)_{qk}}{k!\Gamma(\alpha k+4\alpha+\beta)}+6m^3(\gamma)_{3q}\sum_{k=0}^{\infty}\frac{m^k(\gamma+3q)_{qk}}{k!\Gamma(\alpha k+3\alpha+\beta)} +7m^2(\gamma)_{2q}\sum_{k=0}^{\infty}\frac{m^k(\gamma+2q)_{qk}}{k!\Gamma(\alpha k+2\alpha+\beta)}+m(\gamma)_q\sum_{k=0}^{\infty}\frac{m^k(\gamma+q)_{qk}}{k!\Gamma(\alpha k+\alpha+\beta)}\right].$$

$$\mu_4^{'} = m^4(\gamma)_{4q}\frac{E_{\alpha,\beta+4\alpha}^{\gamma+4q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}+6m^3(\gamma)_{3q}\frac{E_{\alpha,\beta+3\alpha}^{\gamma+3q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}+7m^2(\gamma)_{2q}\frac{E_{\alpha,\beta+2\alpha}^{\gamma+2q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}+ +m(\gamma)_q\frac{E_{\alpha,\beta+\alpha}^{\gamma+q,q}(m)}{E_{\alpha,\beta}^{\gamma,q}(m)}. \tag{2.5}$$

Definition 2.2. *The moment generating function (m.g.f.) of a random variable X is denoted by $M_X(t)$ and defined by*

$$M_X(t) = \sum_{k=0}^{\infty} e^{tk}P(X). \tag{2.6}$$

Definition 2.3. *The probability generating function (p.g.f.) of a random variable*

X is denoted by $\mathcal{P}_X(s)$ and defined by

$$\mathcal{P}_X(s) = \sum_{k=0}^{\infty} s^k P(X). \tag{2.7}$$

Definition 2.4. *The characteristic function of a random variable X is denoted by $\varphi_X(t)$ and defined by*

$$\varphi_X(t) = \sum_{k=0}^{\infty} e^{itk} P(X). \tag{2.8}$$

Theorem 2.2. *The moment generating function of generalized Mittag-Leffler type Poisson distribution is given by*

$$M_X(t) = \frac{E_{\alpha,\beta}^{\gamma,q}(e^t m)}{E_{\alpha,\beta}^{\gamma,q}(m)}. \tag{2.9}$$

Proof.

$$\begin{aligned} M_X(t) &= \sum_{k=0}^{\infty} e^{\mathrm{tk}} P(X) \\ &= \sum_{k=0}^{\infty} e^{\mathrm{tk}} \frac{m^k (\gamma)_{\mathrm{qk}}}{E_{\alpha,\beta}^{\gamma,q}(m) k! \Gamma(\alpha k + \beta)} \\ &= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \sum_{k=0}^{\infty} e^{\mathrm{tk}} \frac{m^k (\gamma)_{\mathrm{qk}}}{k! \Gamma(\alpha k + \beta)} \\ &= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \sum_{k=0}^{\infty} \frac{(e^t m)^k (\gamma)_{\mathrm{qk}}}{k! \Gamma(\alpha k + \beta)} \\ &= \frac{E_{\alpha,\beta}^{\gamma,q}(e^t m)}{E_{\alpha,\beta}^{\gamma,q}(m)}. \end{aligned}$$

Theorem 2.3. *The probability generating function for the generalized Mittag-Leffler type Poisson distribution is given by*

$$\mathcal{P}_X(s) = \frac{E_{\alpha,\beta}^{\gamma,q}(sm)}{E_{\alpha,\beta}^{\gamma,q}(m)}. \tag{2.10}$$

Proof.

$$\mathcal{P}_X(s) = \sum_{k=0}^{\infty} s^k P(X)$$

$$= \sum_{k=0}^{\infty} s^k \frac{m^k (\gamma)_{qk}}{k! E_{\alpha,\beta}^{\gamma,q}(m) \Gamma(\alpha k + \beta)}$$

$$= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \sum_{k=0}^{\infty} \frac{(sm)^k (\gamma)_{\mathrm{qk}}}{k! \Gamma(\alpha k + \beta)}$$

$$\mathcal{P}_X(s) = \frac{E_{\alpha,\beta}^{\gamma,q}(sm)}{E_{\alpha,\beta}^{\gamma,q}(m)}.$$

Theorem 2.4. *The characteristic function for the generalized Mittag-Leffler type Poisson distribution is given by*

$$\varphi_X(t) = \frac{E_{\alpha,\beta}^{\gamma,q}(e^{it} m)}{E_{\alpha,\beta}^{\gamma,q}(m)}. \tag{2.11}$$

Proof.

$$\varphi_X(t) = \sum_{k=0}^{\infty} e^{itk} P(X)$$

$$= \sum_{k=0}^{\infty} e^{tik} \frac{m^k (\gamma)_{qk}}{E_{\alpha,\beta}^{\gamma,q}(m) k! \Gamma(\alpha k + \beta) (\gamma)_{qk}}$$

$$= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \sum_{k=0}^{\infty} e^{tik} \frac{m^k (\gamma)_{qk}}{k! \Gamma(\alpha k + \beta)}$$

$$= \frac{1}{E_{\alpha,\beta}^{\gamma,q}(m)} \sum_{k=0}^{\infty} \frac{(e^{it} m)^k (\gamma)_{qk}}{k! \Gamma(\alpha k + \beta)}$$

$$\varphi_X(t) = \frac{E_{\alpha,\beta}^{\gamma,q}(e^{it} m)}{E_{\alpha,\beta}^{\gamma,q}(m)}.$$

Remark 2.1. *If we put $\gamma = 1, q = 1$ in the results of Theorems 2.1-2.4 then we obtain the corresponding results for Mittag=Leffler type Poisson distribution studied by Porwal and Dixit* [4].

Remark 2.2. *If we put $\gamma = 1, q = 1, \alpha = \beta = 1$ in the results of Theorems 2.1-2.4 then we obtain the corresponding results for Poisson distribution.*

4. Conclusion

In this paper, we introduce a generalized Mittag-Leffler type Poisson distribution. We present findings related to the moments, moment generating functions, probability generating functions, and characteristic functions of this distribution.

Given the widespread use of Mittag-Leffler and generalized Mittag-Leffler functions across various fields of mathematics and sciences, we anticipate that distributions defined by these functions will also play a significant role in mathematics and statistics. Additionally, the applications of various distribution series in geometric function theory are explored. Therefore, we believe that this paper will contribute valuable insights into univalent functions and probability distributions.

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