South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, No. 2 (2024), pp. 439-452

DOI: 10.56827/SEAJMMS.2024.2002.31 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

SEPARATION AXIOMS AND COMPACTNESS IN FERMATEAN FUZZY SOFT TOPOLOGY

Archana Kumari Prasad, Anita Singh Banafar*, Jyoti Pandey Bajpai* and Samajh Singh Thakur*

Department of Mathematics,
Swami Vivekanand Government College,
Lakhnadon - 480886, Madhya Pradesh, INDIA

E-mail : akkumariprasad@gmail.com

*Department of Mathematics,
Jabalpur Engineering College,
Jabalpur - 482011, Madhya Pradesh, INDIA

E-mail : anita.banafar1@gmail.com, jyotipbajpai@gmail.com,
ssthakur@jecjabalpur.ac.in

(**Received:** *Apr. 01, 2024* **Accepted:** *Aug. 18, 2024* **Published:** *Aug. 30, 2024)*

Abstract: In this paper, we define and explore several properties of Fermatean fuzzy soft T_i, i = 0,1,2, Fermatean fuzzy soft regular, Fermatean fuzzy soft T_3, Fermatean fuzzy soft normal, and Fermatean fuzzy soft T_4 axioms using Fermatean fuzzy soft points. We also discuss some Fermatean fuzzy soft invariance properties namely Fermatean fuzzy soft topological property and Fermatean fuzzy soft hereditary property. Furthermore Fermatean fuzzy soft compactness is defined and its characterizations and preserving properties under Fermatean fuzzy soft continuous mappings are figured out.

Keywords and Phrases: Fermatean fuzzy soft sets, Fermatean fuzzy soft topology, Fermatean fuzzy soft separation axioms and Fermatean fuzzy soft compactness.

2020 Mathematics Subject Classification: 54A40, 54C05, 03E72.

1. Introduction

In 1965 Zadeh [29] created fuzzy sets as an extension of classical sets. After the occurrence of fuzzy sets [29], many generalizations of fuzzy sets such as Intuitionistic fuzzy sets [10], soft sets [18], Pythagorean fuzzy sets [27], SR-fuzzy sets [1], nth- power root fuzzy sets [6], (m, n)-fuzzy Sets [2], and Fermatean fuzzy sets [23] have been introduced and studied. Many hybridized classes with soft sets and above classes of sets such as Fuzzy soft sets [15], Intuitionistic fuzzy soft sets [16], Pythagorean fuzzy soft sets [22], (a, b)-fuzzy soft sets [3], and Fermatean fuzzy soft sets [25] have been invented and studied. In the recent past topological structures over these classes of fuzzy sets have been studied [4, 5, 7, 8, 9, 11, 12, 24, 26, 14, 19, 28, 13]. Recently Prasad et. al [20] created topological structure on Fermation fuzzy soft sets and studied basic topological concepts in Fermatean fuzzy soft topological spaces. In another paper Prasad et.al [21] defined the mappings on Fermatean fuzzy soft classes and studied their continuity in Fermatean fuzzy soft topological spaces. Compactness and separation axioms are major area of research in topology,and till today these concepts are not extended to Fermatean fuzzy soft sets. To fill this gap we introduced Fermatean fuzzy soft T_i (i=0,1,2,3,4) axioms and fermatean fuzzy soft compactness and studied some of their properties and characterizations in Fermatean fuzzy soft topological spaces. Fermatean fuzzy soft separation axioms and feramtean fuzzy soft compactness would be useful for the development of the theory of Fermatean fuzzy soft topology to solve the complicated problems containing uncertainties in economics, engineering, medical, environmental, and in general man-machine systems of various types.

Table 1: Abbreviations and their descriptions.

Abbreviation	**Description**
$\mathcal{FFS}(\mathbb{P})$	Family of all Fermatean fuzzy sets of $\mathbb{P}$
$\mathcal{FFS}$	Fermatean fuzzy soft
$\mathcal{FFSS}$	Fermatean fuzzy soft set
$\mathcal{FFSS}(\mathbb{P}, \Sigma)$	Family of all Fermatean fuzzy soft sets over $\mathbb{P}$ relative to Σ
$\mathcal{FFST}$	Fermatean fuzzy soft topology
$\mathcal{FFSTS}$	Fermatean fuzzy soft topological space
$\mathcal{FFSC}(\mathbb{P}, \Sigma)$	Family of all $\mathcal{FFS}$-closed sets of $(\mathbb{P}, \Sigma)$

2. Preliminaries

Definition 2.1. [23] *Let $\mathbb{P}$ be an initial universal set. A structure $\nu = \{< p, m_\nu(p), n_\nu(p) >: p \in \mathbb{P}\}$ where $m_\nu : \mathbb{P} \to [0, 1]$ and $n_\nu : \mathbb{P} \to [0, 1]$ denotes the degree of membership and the degree of nonmembership of each $p \in \mathbb{P}$ to ν is* *called Fermatean fuzzy set in $\mathbb{P}$ if $0 \leq m_{\nu}^{3}(p) + n_{\nu}^{3}(p) \leq 1, \ \forall \ p \in \mathbb{P}$.*

Definition 2.2. [25] *Let $\mathbb{P}$ be a universe of discourse, Σ be the set of parameters and $\Upsilon \subseteq \Sigma$. A pair (ξ, Υ) is called Fermatean fuzzy soft set ($\mathcal{FFSS}$) over $\mathbb{P}$, where $\xi : \Upsilon \rightarrow \mathcal{FFS}(\mathbb{P})$ and $\mathcal{FFS}(\mathbb{P})$ is a family of all Fermatean fuzzy set of $\mathbb{P}$.*

The collection of all Fermatean fuzzy soft sets over $\mathbb{P}$ relative to Σ is denoted by $\mathcal{FFSS}(\mathbb{P}, \Sigma)$.

Definition 2.3. [20] *A subfamily Γ of $\mathcal{FFSS}(\mathbb{P}, \Sigma)$ is called a Fermatean fuzzy soft topology ($\mathcal{FFST}$) on $\mathbb{P}$ if:*

(a) $\tilde{\Phi}, \tilde{\mathbb{P}} \in \Gamma$.

(b) $(\nu_i, \Sigma) \in \Gamma, \ \forall \ i \in \Lambda \Rightarrow \cup_{i \in \Lambda} (\nu_i, \Sigma) \in \Gamma$.

(c) $(\nu_1, \Sigma), (\nu_2, \Sigma) \in \Gamma \Rightarrow (\nu_1, \Sigma) \cap (\nu_2, \Sigma) \in \Gamma$.

If Γ is a $\mathcal{FFST}$ on $\mathbb{P}$ then the structure $(\mathbb{P}, \Gamma, \Sigma)$ is called a Fermatean fuzzy soft topological space ($\mathcal{FFSTS}$) over $\mathbb{P}$ and the members of Γ are called Fermatean fuzzy soft open ($\mathcal{FFS}$-open) sets and their complements are called Fermatean fuzzy soft closed ($\mathcal{FFS}$-closed). The family of all $\mathcal{FFS}$-closed sets of $(\mathbb{P}, \Sigma)$ is denoted by $\mathcal{FFSC}(\mathbb{P}, \Sigma)$.

Definition 2.4. *Let $(\mathbb{P}, \Gamma, \Sigma)$ be a $\mathcal{FFSTS}$ and $(\xi, \Sigma) \in \mathcal{FFSS}(\mathbb{P}, \Sigma)$. Then the interior and closure of (ξ, Σ) denoted respectively by $Int(\xi, \Sigma)$ and $Cl(\xi, \Sigma)$ are defined as follows:*

$$Int(\xi, \Sigma) = \cup \{(\nu, \Sigma) \in \Gamma : (\nu, \Sigma) \subset (\xi, \Sigma)\}.$$
$$Cl(\xi, \Sigma) = \cap \{(\nu, \Sigma) \in \mathcal{FFSC}(\mathbb{P}, \Sigma) : (\xi, \Sigma) \subset (\nu, \Sigma)\}.$$

Definition 2.5. [21] *Let $\mathcal{FFSS}(\mathbb{P}, \Sigma)$ and $\mathcal{FFSS}(\mathbb{Q}, \Omega)$ be families of $\mathcal{FFSS}$s over $\mathbb{P}$ and $\mathbb{Q}$ respectively. Then $f_{\psi\varphi} : \mathcal{FFSS}(\mathbb{P}, \Sigma) \rightarrow \mathcal{FFSS}(\mathbb{Q}, \Omega)$ is called a Fermatean fuzzy soft mapping, where $\psi : \mathbb{P} \rightarrow \mathbb{Q}$ and $\varphi : \Sigma \rightarrow \Upsilon$.*

(a) Let $(\xi, \Sigma) \in \mathcal{FFSS}(\mathbb{P}, \Sigma)$. The image of (ξ, Σ) under $f_{\psi\varphi}$ is written as $f_{\psi\varphi}(\xi, \Sigma) = (\psi(\xi), \varphi(\Sigma))$ is a $\mathcal{FFSS}$ in $(\mathbb{Q}, \Upsilon)$ such that

$$m_{\psi(\xi)}(\iota)(q) = \begin{cases} \sup\limits_{\epsilon \in \varphi^{-1}(\iota) \cap \Upsilon, \ p \in \psi^{-1}(q)} m_{\xi(\epsilon)}(p), & \psi^{-1}(q) \neq \phi, \\ 0 & \textit{otherwise} \end{cases}$$

and

$$n_{\psi(\xi)}(\iota)(q) = \begin{cases} \inf\limits_{\epsilon\in\varphi^{-1}(\iota)\cap\Upsilon,\ p\in\psi^{-1}(q)} n_{\xi(\epsilon)}(p), & \psi^{-1}(q) \neq \phi, \\ 1 & otherwise \end{cases}$$

$\forall\ \epsilon \in \Sigma$, $p \in \mathbb{P}, \iota \in \Omega$ *and* $q \in \mathbb{Q}$.

(b) *Let* $(\delta, \Omega) \in \mathcal{FFSS}(\mathbb{Q}, \Omega)$. *The inverse image of* (δ, Ω) *under* $f_{\psi\varphi}$, *denoted by* $f^{-1}_{\psi\varphi}((\delta, \Omega))$ *is a* $\mathcal{FFSS}$ *in* $(\mathbb{P}, \Sigma)$ *given by:*

$$m_{\psi^{-1}(\delta)}(\epsilon)(p) = m_{\delta(\varphi(\epsilon))}(\psi(p))$$

and

$$n_{\psi^{-1}(\delta)}(\epsilon)(p) = n_{\delta(\varphi(\epsilon))}(\psi(p))$$

$\forall\ \epsilon \in \Sigma$ *and* $p \in \mathbb{P}$

Definition 2.6. [21] *Let* $(\mathbb{P}, \Gamma_1, \Sigma)$ *and* $(\mathbb{Q}, \Gamma_2, \Omega)$ *be two* $\mathcal{FFSTS}$*s. Then the Fermatean fuzzy soft mapping* $f_{\psi\varphi} : (\mathbb{P}, \Gamma_1, \Sigma) \to (\mathbb{Q}, \Gamma_1, \Omega)$ *is said to be:*

(a) $\mathcal{FFS}$*-continuous if* $f^{-1}_{\psi\varphi}((\mu, \Omega)) \in \Gamma_1$, $\forall\ (\mu, \Omega) \in \Gamma_2$.

(b) $\mathcal{FFS}$*-open if* $f_{\psi\varphi}(\xi, \Sigma) \in \Gamma_2$, $\forall\ (\xi, \Sigma) \in \Gamma_1$.

(c) $\mathcal{FFS}$*-closed if* $f_{\psi\varphi}(\xi, \Sigma) \in \mathcal{FFSC}(\mathbb{Q}, \Omega)$, $\forall\ (\xi, \Sigma) \in \mathcal{FFSC}(\mathbb{P}, \Sigma)$.

Lemma 2.7. [21] *Let* $(\mathbb{P}, \Gamma_1, \Sigma)$ *and* $(\mathbb{Q}, \Gamma_2, \Omega)$ *be two* $\mathcal{FFSTS}$*s and* $f_{\psi\varphi} : (\mathbb{P}, \Gamma_1, \Sigma) \to (\mathbb{Q}, \Gamma_2, \Omega)$ *is a* $\mathcal{FFS}$*-continuous and* $\mathcal{FFS}$*-open bijective mapping. Then* $f_{\psi\varphi}(Cl(\lambda, \Sigma)) = Cl(f_{\psi\varphi}(\lambda, \Sigma))$, $\forall(\lambda, \Sigma) \in \mathcal{FFS}(\mathbb{P}, \Sigma)$.

3. Fermatean Fuzzy Soft Separation Axioms

In this section, we define Fermatean fuzzy soft separation axioms namely $\mathcal{FFS}-T_i$ axioms, for (i = 0, 1, 2,3,4) using $\mathcal{FFS}$- points and discuss several properties and their relationship with the help of examples.

Definition 3.1. *A* $\mathcal{FFSS}$ (ϖ, Σ) *is said to be a* $\mathcal{FFS}$*-point denoted by* ϵ_ϖ *if for every* $\epsilon \in \Sigma$, $\varpi(\epsilon) \neq \{(p, 0, 1)\ :\ p \in \mathbb{P}\}$ *and* $\varpi(\tilde{\epsilon}) = \{(p, 0, 1)\ :\ p \in \mathbb{P}\}$, $\forall\ \tilde{\epsilon} \in \Sigma - \epsilon$. *Note that, any* $\mathcal{FFS}$*-point* ϵ_ϖ *(say) is also considered as singleton* $\mathcal{FFS}$ *subset of the* $\mathcal{FFSS}$ (ϖ, Σ).

Definition 3.2. *A* $\mathcal{FFS}$*-point* ϵ_ϖ *is said to be in the* $\mathcal{FFSS}(\nu, \Sigma)$, *that is* $\epsilon_\varpi \in (\nu, \Sigma)$, *if* $\varpi(\epsilon) \subset \nu(\epsilon)$, *for every* $\epsilon \in \Sigma$.

Definition 3.3. *A $\mathcal{FFSTS}$ $(\mathbb{P},\Gamma,\Sigma)$ is said to be $\mathcal{FFS}$-T_0, if for every pair of distinct $\mathcal{FFS}$-points ϵ_α and ϵ_β over $\mathbb{P}$, $\exists$ $(\varpi,\Sigma),(\mu,\Sigma)\in\Gamma$ such that $\epsilon_\alpha\in(\varpi,\Sigma)$ but $\epsilon_\beta\notin(\varpi,\Sigma)$ or $\epsilon_\beta\in(\mu,\Sigma)$ but $\epsilon_\alpha\notin(\mu,\Sigma)$.*

Example 3.4. All discrete $\mathcal{FFSTS}$s are $\mathcal{FFS}$-T_0, because for any two distinct $\mathcal{FFS}$-points ϵ_α and ϵ_β over $\mathbb{P}$, $\exists$ a $\mathcal{FFS}$-open sets such that $\{\epsilon_\alpha\}$ such that $\epsilon_\alpha\in\{\epsilon_\alpha\}$ and $\epsilon_\beta\notin\{\epsilon_\alpha\}$.

Theorem 3.5. *Every $\mathcal{FFS}$-subspace of a $\mathcal{FFS}$-T_0 space is $\mathcal{FFS}$-T_0.*
Proof. Let $(\mathbb{Y},\Gamma_{\mathbb{Y}},\Sigma)$ be a $\mathcal{FFS}$-subspace of a $\mathcal{FFS}$-T_0 space $(\mathbb{P},\Gamma,\Sigma)$. Let ϵ_α and ϵ_β be two distinct $\mathcal{FFS}$-points over $\mathbb{Y}$. Then ϵ_α and ϵ_β are distinct $\mathcal{FFS}$-points over $\mathbb{P}$. Since $(\mathbb{P},\Gamma,\Sigma)$ is $\mathcal{FFS}$-T_0,$\exists$ a $\mathcal{FFS}$-open set containing one of the $\mathcal{FFS}$-point but not other. Without loss of generality, let $(\varpi,\Sigma)\in\Gamma$ such that $\epsilon_\alpha\in(\varpi,\Sigma)$ but $\epsilon_\beta\notin(\varpi,\Sigma)$. Put $(\varpi,\Sigma)_{\mathbb{Y}}=(\varpi,\Sigma)\cap\mathbb{Y}$. Then $(\varpi,\Sigma)_{\mathbb{Y}}\in\Gamma_{\mathbb{Y}}$ such that $\epsilon_\alpha\in(\varpi,\Sigma)_{\mathbb{Y}}$ but $\epsilon_\beta\notin(\varpi,\Sigma)_{\mathbb{Y}}$. Hence $(\mathbb{Y},\Gamma_{\mathbb{Y}},\Sigma)$ is $\mathcal{FFS}$-T_0.

Definition 3.6. *A $\mathcal{FFSTS}$ $(\mathbb{P},\Gamma,\Sigma)$ is said to be $\mathcal{FFS}$-T_1, if for every pair of distinct$\mathcal{FFS}$-points ϵ_α and ϵ_β over $\mathbb{P}$, $\exists$ $(\varpi,\Sigma),(\mu,\Sigma)\in\Gamma$ such that $\epsilon_\alpha\in(\varpi,\Sigma)$ but $\epsilon_\beta\notin(\varpi,\Sigma)$ and $\epsilon_\beta\in(\mu,\Sigma)$ but $\epsilon_\alpha\notin(\mu,\Sigma)$.*

Example 3.7. Every discrete $\mathcal{FFSTS}$ is a $\mathcal{FFS}$-T_1, because, for any two distinct $\mathcal{FFS}$-point ϵ_α, ϵ_β over $\mathbb{P}$, $\exists$ $\mathcal{FFS}$-open sets $\{\epsilon_\alpha\}$ and $\{\epsilon_\beta\}$ such that $\epsilon_\alpha\in\{\epsilon_\alpha\}$ but $\epsilon_\beta\notin\{\epsilon_\beta\}$ and $\epsilon_\alpha\notin\{\epsilon_\beta\}$ but $\epsilon_\beta\in\{\epsilon_\beta\}$.

Remark 3.8. *Every $\mathcal{FFS}-T_1$ space is $\mathcal{FFS}-T_0$. But the converse may not be true. For,*

Example 3.9. Let $\mathbb{P}=\{p_1,p_2\}$, $\Sigma=\{\epsilon_1,\epsilon_2\}$ and $\mathcal{FFSS}$ (μ,Σ) is defined as follows:

$$(\mu,\Sigma)=\begin{array}{c} \\ p_1 \\ p_2 \end{array}\begin{array}{c} \begin{array}{cc} \epsilon_1 & \epsilon_2 \end{array} \\ \begin{pmatrix} (1.0,0.0) & (0.0,1.0) \\ (0.0,1.0) & (1.0,0.0) \end{pmatrix} \end{array}.$$

Let $\Gamma=\{\Phi,\tilde{\mathbb{P}},(\mu,\Sigma)\}$ be a $\mathcal{FFST}$ over $\mathbb{P}$. Then the $\mathcal{FFSTS}$ $(\mathbb{P},\Gamma,\Sigma)$ is $\mathcal{FFS}-T_0$ but not $\mathcal{FFS}-T_1$.

Theorem 3.10. *Every $\mathcal{FFS}$-subspace of a $\mathcal{FFS}$-T_1 space is $\mathcal{FFS}$-T_1.*

Definition 3.11. *A $\mathcal{FFSTS}$ $(\mathbb{P},\Gamma,\Sigma)$ is said to be$\mathcal{FFS}$-T_2, if for every pair of distinct$\mathcal{FFS}$-points ϵ_α and ϵ_β over $\mathbb{P}$, $\exists$ $(\varpi,\Sigma),(\mu,\Sigma)\in\Gamma$ such that $\epsilon_\alpha\in(\varpi,\Sigma)$, $\epsilon_\beta\in(\mu,\Sigma)$ and $(\varpi,\Sigma)\cap(\mu,\Sigma)=\Phi$.*

Remark 3.12. *Every $\mathcal{FFS}-T_2$ space is $\mathcal{FFS}-T_1$. But the converse may not be true. For,*

Example 3.13. Let $\mathbb{P} = \{p_1, p_2\}$, $\Sigma = \{\epsilon_1, \epsilon_2\}$ and $\mathcal{FFSS}$s (μ, Σ) , (ν, Σ), and (ζ, Σ) are defined as follows:

$$(\mu, \Sigma) = \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} \epsilon_1 & \epsilon_2 \end{array} \\ \begin{pmatrix} (1.0, 0.0) & (0.0, 1.0) \\ (0.0, 1.0) & (0.0, 1.0) \end{pmatrix} \end{array}.$$

$$(\nu, \Sigma) = \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} \epsilon_1 & \epsilon_2 \end{array} \\ \begin{pmatrix} (0.0, 1.0) & (0.0, 1.0) \\ (0.0, 1.0) & (1.0, 0.0) \end{pmatrix} \end{array}.$$

$$(\zeta, \Sigma) = \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} \epsilon_1 & \epsilon_2 \end{array} \\ \begin{pmatrix} (1.0, 0.0) & (0.0, 1.0) \\ (0.0, 1.0) & (1.0, 0.0) \end{pmatrix} \end{array}.$$

Let $\Gamma = \{\Phi, \tilde{\mathbb{P}}, (\mu, \Sigma), (\nu, \Sigma), (\zeta, \Sigma)\}$ be a $\mathcal{FFST}$ over $\mathbb{P}$. Then the $\mathcal{FFSTS}$ $(\mathbb{P}, \Gamma, \Sigma)$ is $\mathcal{FFS} - T_1$ but not $\mathcal{FFS} - T_2$.

Theorem 3.14. *A $\mathcal{FFSTS}$ $(\mathbb{P}, \Gamma, \Sigma)$ is $\mathcal{FFS}$-T_2 if and only if for any two distinct $\mathcal{FFS}$-points ϵ_α and ϵ_β, $\exists$ $(\varpi_1, \Sigma), (\varpi_2, \Sigma) \in \mathcal{FFSC}(\mathbb{P}, \Sigma)$ such that $\epsilon_\alpha \in (\varpi_1, \Sigma)$ but $\epsilon_\beta \notin (\varpi_1, \Sigma)$, $\epsilon_\alpha \notin (\varpi_2, \Sigma)$ but $\epsilon_\beta \in (\varpi_2, \Sigma)$ and $(\varpi_1, \Sigma) \cup (\varpi_2, \Sigma) = \mathbb{P}$.*

Proof. Necessity: Suppose that $(\mathbb{P}, \Gamma, \Sigma)$ is $\mathcal{FFS}$-T_2 and ϵ_α, ϵ_β are any two distinct $\mathcal{FFS}$-points over $\mathbb{P}$. Then by hypothesis $\exists$ (μ_1, Σ) , $(\mu_2, \Sigma) \in \Gamma$ such that $\epsilon_\alpha \in (\mu_1, \Sigma)$, $\epsilon_\beta \in (\mu_2, \Sigma)$ such that $(\mu_1, \Sigma) \cap (\mu_2, \Sigma) = \Phi$. Clearly $(\mu_1, \Sigma) \subset (\mu_2, \Sigma)^c$ and $(\mu_2, \Sigma) \subset (\mu_1, \Sigma)^c$. Hence $\epsilon_\alpha \in (\mu_2, \Sigma)^c$. Put $(\varpi_1, \Sigma) = (\mu_2, \Sigma)^c$. This gives $\epsilon_\alpha \in (\varpi_1, \Sigma)$ but $\epsilon_\beta \notin (\varpi_1, \Sigma)$. Also $\epsilon_\beta \in (\mu_1, \Sigma)^c$. Put $(\varpi_2, \Sigma) = (\mu_1, \Sigma)^c$. Therefore $\epsilon_\beta \in (\varpi_2, \Sigma)$ but $\epsilon_\alpha \notin (\varpi_2, \Sigma)$. Moreover, $(\varpi_1, \Sigma) \cup (\varpi_2, \Sigma) = (\mu_1, \Sigma)^c \cup (\mu_2, \Sigma)^c = ((\mu_1, \Sigma) \cap (\mu_2, \Sigma))^c = \Phi^c = \mathbb{P}$

Sufficiency: Let ϵ_α and ϵ_β, be two distinct $\mathcal{FFS}$-points of $\mathbb{P}$. Then by hypothesis $\exists$ $(\varpi_1, \Sigma), (\varpi_2, \Sigma) \in FFSC(\mathbb{P}, \Sigma)$ such that $\epsilon_\alpha \in (\varpi_1, \Sigma)$, $\epsilon_\beta \notin (\varpi_1, \Sigma)$, $\epsilon_\alpha \notin (\varpi_2, \Sigma)$, $\epsilon_\beta \in (\varpi_2, \Sigma)$ and $(\varpi_1, \Sigma) \cup (\varpi_2, \Sigma) = \mathbb{P}$. Put $(\mu_1, \Sigma) = (\varpi_2, \Sigma)^c$ and $(\mu_2, \Sigma) = (\varpi_1, \Sigma)^c$. Then $(\mu_1, \Sigma), (\mu_2, \Sigma) \in \Gamma$ such that $\epsilon_\alpha \in (\mu_1, \Sigma)$ and $\epsilon_\beta \in (\mu_2, \Sigma)$. Moreover,$(\mu_1, \Sigma) \cap (\mu_2, \Sigma) = (\varpi_2, \Sigma)^c \cap (\varpi_1, \Sigma)^c = ((\varpi_1, \Sigma) \cup (\varpi_2, \Sigma))^c = \mathbb{P}^c = \Phi$. Hence, $(\mathbb{P}, \Gamma, \Sigma)$ is $\mathcal{FFS}$-T_2.

Theorem 3.15. *Every $\mathcal{FFS}$-subspace of a $\mathcal{FFS}$-T_2 space is $\mathcal{FFS}$-T_2.*

Proof. Let $(\mathbb{Y}, \Gamma_{\mathbb{Y}}, \Sigma)$ be a $\mathcal{FFS}$-subspace of a $\mathcal{FFS}$-T_2 space $(\mathbb{P}, \Gamma, \Sigma)$. Let ϵ_α and ϵ_β be two distinct $\mathcal{FFS}$-points over $\mathbb{Y}$. Then ϵ_α and ϵ_β are distinct $\mathcal{FFS}$-points over $\mathbb{P}$. Since $(\mathbb{P}, \Gamma, \Sigma)$ is $\mathcal{FFS}$-T_2, $\exists$ $(\varpi, \Sigma), (\mu, \Sigma) \in \Gamma$ such that $\epsilon_\alpha \in (\varpi, \Sigma)$, $\epsilon_\beta \in (\mu, \Sigma)$ and $(\varpi, \Sigma) \cap (\mu, \Sigma) = \Phi$. Put $(\varpi, \Sigma)_{\mathbb{Y}} = (\varpi, \Sigma) \cap \mathbb{Y}$ and $(\mu, \Sigma)_{\mathbb{Y}} = (\mu, \Sigma) \cap \mathbb{Y}$. Then $(\varpi, \Sigma)_{\mathbb{Y}}, (\mu, \Sigma)_{\mathbb{Y}} \in \Gamma_{\mathbb{Y}}$ such that $\epsilon_\alpha \in (\varpi, \Sigma)_{\mathbb{Y}}$ and $\epsilon_\beta \in (\mu, \Sigma)_{\mathbb{Y}}$. Moreover, $(\varpi, \Sigma)_{\mathbb{Y}} \cap (\mu, \Sigma)_{\mathbb{Y}} = ((\varpi, \Sigma) \cap \mathbb{Y}) \cap ((\mu, \Sigma) \cap \mathbb{Y}) = ((\varpi, \Sigma) \cap (\mu, \Sigma)) \cap \mathbb{Y} = \Phi \cap \mathbb{Y} = \Phi$. Hence $(\mathbb{Y}, \Gamma_{\mathbb{Y}}, \Sigma)$ is $\mathcal{FFS}$-T_2.

Definition 3.16. *A $\mathcal{FFSTS}$ $(\mathbb{P},\Gamma,\Sigma)$ is said to be $\mathcal{FFS}$-regular if for every $(\varpi,\Sigma)\in\mathcal{FFSC}(\mathbb{P},\Sigma)$ and every $\mathcal{FFS}$-point ϵ_α over $\mathbb{P}$ such that $\epsilon_\alpha\notin(\varpi,\Sigma)$, $\exists$ $(\mu,\Sigma),(\nu,\Sigma)\in\Gamma$ such that $\epsilon_\alpha\in(\mu,\Sigma)$, $(\varpi,\Sigma)\subset(\nu,\Sigma)$ and $(\mu,\Sigma)\cap(\nu,\Sigma)=\Phi$.*

Example 3.17. Let $\mathbb{P}=\{p_1,p_2\}$, $\Sigma=\{\epsilon_1,\epsilon_2\}$ and $\mathcal{FFSS}$s (μ,Σ), (ν,Σ) are defined as follows:

$$(\mu,\Sigma)=\begin{array}{c} \\ p_1 \\ p_2\end{array}\begin{array}{c}\begin{array}{cc}\epsilon_1 & \epsilon_2\end{array}\\ \begin{pmatrix}(0.0,1.0) & (1.0,0.0)\\ (0.0,1.0) & (1.0,0.0)\end{pmatrix}\end{array},$$

$$(\nu,\Sigma)=\begin{array}{c} \\ p_1 \\ p_2\end{array}\begin{array}{c}\begin{array}{cc}\epsilon_1 & \epsilon_2\end{array}\\ \begin{pmatrix}(1.0,0.0) & (0.0,1.0)\\ (1.0,0.0) & (0.0,1.0)\end{pmatrix}\end{array}.$$

Let $\Gamma=\{\Phi,\tilde{\mathbb{P}},(\mu,\Sigma),(\nu,\Sigma)\}$ be a $\mathcal{FFST}$ over $\mathbb{P}$. Then the $\mathcal{FFSTS}$ $(\mathbb{P},\Gamma,\Sigma)$ is $\mathcal{FFS}$-regular.

Theorem 3.18. *Every $\mathcal{FFS}$-subspace of a $\mathcal{FFS}$-regular space is $\mathcal{FFS}$-regular.*

Proof. Suppose $(\mathbb{Y},\Gamma_{\mathbb{Y}},\Sigma)$ be a $\mathcal{FFS}$-subspace of a$\mathcal{FFS}$-regular space $(\mathbb{P},\Gamma,\Sigma)$. Let $(\varpi,\Sigma)_{\mathbb{Y}}\in\mathcal{FFSC}(\mathbb{Y},\Sigma)$ and let ϵ_α is a $\mathcal{FFS}$-point over $\mathbb{Y}$ such that $\epsilon_\alpha\notin(\varpi,\Sigma)_{\mathbb{Y}}$. Then by Theorem 3.15 [20], $\exists$ $(\varpi,\Sigma)\in\mathcal{FFSC}(\mathbb{P},\Sigma)$ such that $(\varpi,\Sigma)_{\mathbb{Y}}=(\varpi,\Sigma)\cap\mathbb{Y}$. It is clear that $\epsilon_\alpha\notin(\varpi,\Sigma)$, so by$\mathcal{FFS}$-regularity of $(\mathbb{P},\Gamma,\Sigma)$, $\exists$ $(\mu,\Sigma),(\nu,\Sigma)\in\Gamma$ such that $\epsilon_\alpha\in(\mu,\Sigma)$, $(\varpi,\Sigma)\subset(\nu,\Sigma)$ and $(\mu,\Sigma)\cap(\nu,\Sigma)=\Phi$. Put $(\mu,\Sigma)_{\mathbb{Y}}=(\mu,\Sigma)\cap\mathbb{Y}$ and $(\nu,\Sigma)_{\mathbb{Y}}=(\nu,\Sigma)\cap\mathbb{Y}$. Then $(\mu,\Sigma)_{\mathbb{Y}},(\nu,\Sigma)_{\mathbb{Y}}\in\Gamma_{\mathbb{Y}}$ such that $\epsilon_\alpha\in(\mu,\Sigma)_{\mathbb{Y}}$, $(\varpi,\Sigma)_{\mathbb{Y}}\subset(\nu,\Sigma)_{\mathbb{Y}}$. Moreover, $(\mu,\Sigma)_{\mathbb{Y}}\cap(\nu,\Sigma)_{\mathbb{Y}}=((\mu,\Sigma)\cap\mathbb{Y})\cap((\nu,\Sigma)\cap\mathbb{Y})=((\mu,\Sigma)\cap(\nu,\Sigma))\cap\mathbb{Y}=\Phi\cap\mathbb{Y}=\Phi$. Hence $(\mathbb{Y},\Gamma_{\mathbb{Y}},\Sigma)$ is $\mathcal{FFS}$-regular.

Definition 3.19. *A $\mathcal{FFSTS}$ $(\mathbb{P},\Gamma,\Sigma)$ is called $\mathcal{FFS}$-T_3, if it is $\mathcal{FFS}$-regular and $\mathcal{FFS}$-T_1.*

Remark 3.20. *Every $\mathcal{FFS}-T_3$ space is $\mathcal{FFS}-T_2$. But the converse may not be true.*

Definition 3.21. *A $\mathcal{FFSTS}$ $(\mathbb{P},\Gamma,\Sigma)$ is said to be$\mathcal{FFS}$-normal if for every pair $(\varpi_1,\Sigma),(\varpi_2,\Sigma)\in TSFSC(\mathbb{P},\Sigma)$ such that $(\varpi_1,\Sigma)\cap(\varpi_2,\Sigma)=\Phi$, $\exists$ $(\mu_1,\Sigma),(\mu_2,\Sigma)\in\Gamma$ such that $(\varpi_1,\Sigma)\subset(\mu_1,\Sigma_1)$, $(\varpi_2,\Sigma_2)\subset(\mu_2,\Sigma)$ and $(\mu_1,\Sigma_1)\cap(\mu_2,\Sigma_2)=\Phi$.*

Example 3.22. Let $\mathbb{P}=\{p_1,p_2\}$,$\Sigma=\{\epsilon_1,\epsilon_2\}$ and TSFSSs (ϖ_1,Σ), (ϖ_2,Σ), (ϖ_3,Σ),(ϖ_4,Σ), (ϖ_5,Σ), (ϖ_6,Σ) are defined as follows:

$$(\varpi_1,\Sigma_1)=\begin{array}{c} \\ p_1 \\ p_2\end{array}\begin{array}{c}\begin{array}{cc}\epsilon_1 & \epsilon_2\end{array}\\ \begin{pmatrix}(0.0,1.0) & (0.6,0.4)\\ (0.0,1.0) & (0.5,0.5)\end{pmatrix}\end{array}$$

$$(\varpi_2,\Sigma_2)= \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} \epsilon_1 & \epsilon_2 \end{array} \\ \begin{pmatrix} (0.5,0.4) & (0.6,0.4) \\ (0.3,0.5) & (0.5,0.5) \end{pmatrix} \end{array}$$

$$(\varpi_3,\Sigma_3)= \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} \epsilon_1 & \epsilon_2 \end{array} \\ \begin{pmatrix} (1.0,0.0) & (0.8,0.3) \\ (1.0,0.0) & (0.6,0.4) \end{pmatrix} \end{array}$$

$$(\varpi_4,\Sigma_4)= \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} \epsilon_1 & \epsilon_2 \end{array} \\ \begin{pmatrix} (0.7,0.2) & (1.0,0.0) \\ (0.9,0.1) & (1.0,0.0) \end{pmatrix} \end{array}$$

$$(\varpi_5,\Sigma_5)= \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} \epsilon_1 & \epsilon_2 \end{array} \\ \begin{pmatrix} (0.7,0.2) & (0.8,0.3) \\ (0.9,0.1) & (0.6,0.4) \end{pmatrix} \end{array}$$

$$(\varpi_6,\Sigma_6)= \begin{array}{c} \\ p_1 \\ p_2 \end{array} \begin{array}{c} \begin{array}{cc} \epsilon_1 & \epsilon_2 \end{array} \\ \begin{pmatrix} (0.5,0.4) & (0.0,1.0) \\ (0.3,0.5) & (0.0,1.0) \end{pmatrix} \end{array}$$

Let $\Gamma = \{\Phi,\ \tilde{\mathbb{P}},\ (\varpi_1,\Sigma),\ (\varpi_2,\Sigma),\ (\varpi_3,\Sigma),\ (\varpi_4,\Sigma),\ (\varpi_5,\Sigma),\ (\varpi_6,\Sigma)\}$ be a $\mathcal{FFST}$ over $\mathbb{P}$. Then the $\mathcal{FFSTS}$ $(\mathbb{P},\Gamma_1,\Sigma)$ is $\mathcal{FFS}$-normal.

Theorem 3.23. *Every $\mathcal{FFS}$-closed subspace of a $\mathcal{FFS}$-normal space is $\mathcal{FFS}$-normal.*

Proof. Suppose $(\mathbb{Y},\Gamma_{\mathbb{Y}},\Sigma)$ be a $\mathcal{FFS}$-closed subspace of a $\mathcal{FFS}$-normal space $(\mathbb{P},\Gamma,\Sigma)$. Let $(\varpi_1,\Sigma),(\varpi_2,\Sigma) \in \mathcal{FFSC}(\mathbb{Y},\Sigma)$ such that $(\varpi_1,\Sigma)\cap(\varpi_2,\Sigma) = \Phi$. Since $\mathbb{Y} \in \mathcal{FFSC}(\mathbb{P},\Sigma)$, by Theorem 3.17 [20], (ϖ_1,Σ), $(\varpi_2,\Sigma) \in \mathcal{FFSC}(\mathbb{P},\Sigma)$. By $\mathcal{FFS}$-normality of $(\mathbb{P},\Gamma,\Sigma)$, $\exists$ $\mathcal{FFSS}$s, (μ_1,Σ), $(\mu_2,\Sigma) \in \Gamma$ such that $(\varpi_1,\Sigma) \subset (\mu_1,\Sigma_1)$, $(\varpi_2,\Sigma_2) \subset (\mu_2,\Sigma)$ and $(\mu_1,\Sigma_1)\cap(\mu_2,\Sigma_2) = \Phi$. Put $(\mu_1,\Sigma)_{\mathbb{Y}} = (\mu_1,\Sigma)\cap\mathbb{Y}$ and $(\mu_2,\Sigma)_{\mathbb{Y}} = (\mu_2,\Sigma)\cap\mathbb{Y}$. Then $(\mu_1,\Sigma)_{\mathbb{Y}},(\mu_2,\Sigma)_{\mathbb{Y}} \in \Gamma_{\mathbb{Y}}$. clearly we have $(\varpi_1,\Sigma) \subset (\mu_1,\Sigma) \Rightarrow (\varpi_1,\Sigma)\cap\mathbb{Y} \subset (\mu_1,\Sigma)\cap\mathbb{Y} \Rightarrow (\varpi_1,\Sigma) \subset (\mu_1,\Sigma)_{\mathbb{Y}}$ and $(\varpi_2,\Sigma) \subset (\mu_2,\Sigma) \Rightarrow (\varpi_2,\Sigma)\cap\mathbb{Y} \subset (\mu_2,\Sigma)\cap\mathbb{Y} \Rightarrow (\varpi_2,\Sigma) \subset (\mu_2,\Sigma)_{\mathbb{Y}}$. Moreover, $(\mu_1,\Sigma)_{\mathbb{Y}}\cap(\mu_2,\Sigma)_{\mathbb{Y}} = ((\mu_1,\Sigma)\cap\mathbb{Y})\cap((\mu_2,\Sigma)\cap\mathbb{Y}) = ((\mu_1,\Sigma)\cap(\mu_2,\Sigma))\cap\mathbb{Y} = \Phi\cap\mathbb{Y} = \Phi$. Hence $(\mathbb{Y},\Gamma_{\mathbb{Y}},\Sigma)$ $\mathcal{FFS}$-normal.

Theorem 3.24. *A $\mathcal{FFSTS}(\mathbb{P},\Gamma,\Sigma)$ is $\mathcal{FFS}$-normal if and only if for every $(\xi,\Sigma) \in \mathcal{FFSC}(\mathbb{P},\Sigma)$ and each $(\sigma,\Sigma) \in \Gamma$ containing (ξ,Σ) $\exists$, $(\rho,\Sigma) \in \Gamma$ such that $(\xi,\Sigma) \subset (\rho,\Sigma) \subset Cl(\rho,\Sigma) \subset (\sigma,\Sigma)$.*

Theorem 3.25. *Let $(\mathbb{P},\Gamma_1,\Sigma)$ and $(\mathbb{Q},\Gamma_2,\Omega)$ be two $\mathcal{FFSTS}$s and $f_{\psi\varphi} : (\mathbb{P},\Gamma_1,\Sigma) \to (\mathbb{Q},\Gamma_2,\Omega)$ is a bijective $\mathcal{FFS}$-continuous and $\mathcal{FFS}$-open mapping. If $(\mathbb{P},\Gamma_1,\Sigma)$ is $\mathcal{FFS}$-normal then $(\mathbb{Q},\Gamma_2,\Omega)$ is $\mathcal{FFS}$-normal.*

Proof. Let $(\mathbb{P},\Gamma_1,\Sigma)$ be $\mathcal{FFS}$-normal and $f_{\psi\varphi}$ is a $\mathcal{FFS}$-continuous and $\mathcal{FFS}$-

open mapping from $(\mathbb{P}, \Gamma_1, \Sigma)$ onto a $\mathcal{FFSTS}(\mathbb{Q}, \Gamma_2, \Omega)$. Let $(\xi, \Omega) \in \mathcal{FFSC}(\mathbb{Q}, \Omega)$ and $(\sigma, \Omega) \in \Gamma_2$ such that, $(\xi, \Omega) \subset (\sigma, \Omega)$. Then $f_{\psi\varphi}^{-1}(\xi, \Omega) \in \mathcal{FFSC}(\mathbb{P}, \Sigma)$ contained in the $\mathcal{FFS}$-open set $f_{pu}^{-1}(\sigma, \Omega)$. Since $(\mathbb{P}, \Gamma_1, \Sigma)$ is $\mathcal{FFS}$-normal, by Theorem 3.24, $\exists$ $(\mu, \Sigma) \in \Gamma_1$ such that $f_{pu}^{-1}(\xi, \Omega) \subseteq (\mu, \Sigma) \subset \mathrm{Cl}(\mu, \Sigma) \subset f_{pu}^{-1}(\sigma, \Omega)$. Since$f_{\psi\varphi}$ is onto, we get, $(\xi, \Omega) \subseteq f_{\psi\varphi}(\mu, \Sigma) \subset f_{pu}(Cl(\mu, \Sigma)) \subset (\sigma, \Omega)$. Now by Lemma 2.7 this reduces to $(\xi, \Omega) \subset f_{pu}(\mu, \Sigma) \subset Cl(f_{pu}(\mu, \Sigma)) \subset (\sigma, \Omega)$. Hence by Theorem 3.24, $(\mathbb{Q}, \Gamma_2, \Omega)$ is $\mathcal{FFS}$-normal, since $f_{\psi\varphi}(\mu, \Sigma) \in \Gamma_2$.

Definition 3.26. *A $\mathcal{FFSTS}$ $(\mathbb{P}, \Gamma, \Sigma)$ is called $\mathcal{FFS}$-T_4, if it is $\mathcal{FFS}$-normal and $\mathcal{FFS}$-T_1.*

Theorem 3.27. *Every $\mathcal{FFS} - T_4$ space is $\mathcal{FFS} - T_3$.*

4. Fermatean fuzzy soft compactness

In this section we define Fermatean fuzzy soft compactness and explores its study in $\mathcal{FFSTS}$s.

Definition 4.1. *Let$(\mathbb{P}, \Gamma, \Sigma)$ be a $\mathcal{FFSTS}$. A collection $\mho$ of $\mathcal{FFSS}$s is called a cover of a $\mathcal{FFSS}$ (ξ, Σ) if $(\xi, \Sigma) \subset \bigcup_{\kappa \in \Lambda}\{(\xi_\kappa, \Sigma) : (\xi_\kappa, \Sigma) \in \mho\}$. It is a $\mathcal{FFS}$-open cover if every member of $\mho$ is a $\mathcal{FFSO}$. A subfamily of $\mho$ that is also a cover is referred to as a subcover of $\mho$.*

Definition 4.2. *Let $(\mathbb{P}, \Gamma, \Sigma)$ is a $\mathcal{FFSTS}$ and $(\xi, \Sigma) \in \mathcal{FSS}(\mathbb{P}, \Sigma)$. Then (ξ, Σ) called Fermatean fuzzy soft compact ($\mathcal{FFS}$-compact), if each $\mathcal{FFS}$-open cover of (ξ, Σ) has a finite subcover. Also $\mathcal{FFSTS}$ $(\mathbb{P}, \Gamma, \Sigma)$ is FFS compact if each $\mathcal{FFS}$-open cover of $\tilde{\mathbb{P}}$ has a finite subcover.*

Example 4.3. A finite $\mathcal{FFSTS}$ $(\mathbb{P}, \Gamma, \Sigma)$ is $\mathcal{FFS}$-compact.

Theorem 4.4. *Let $(\mathbb{P}, \Gamma_1, \Sigma)$ and $(\mathbb{P}, \Gamma_2, \Sigma)$ be two $\mathcal{FFSTS}$s and $\Gamma_1 \subset \Gamma_2$. If $(\mathbb{P}, \Gamma_2, \Sigma)$ is $\mathcal{FFS}$-compact then so is $(\mathbb{P}, \Gamma_1, \Sigma)$.*

Theorem 4.5. *Let $(\mathbb{P}, \Gamma_1, \Sigma)$ and $(\mathbb{Q}, \Gamma_2, \Omega)$ be two $\mathcal{FFSTS}$s and $f_{\psi\varphi} : (\mathbb{P}, \Gamma_1, \Sigma) \rightarrow (\mathbb{Q}, \Gamma_1, \Omega)$ is a surjective $\mathcal{FFS}$-continuous mapping. If $(\mathbb{P}, \Gamma_1, \Sigma)$ is $\mathcal{FFS}$-compact then $(\mathbb{Q}, \Gamma_2, \Omega)$ is $\mathcal{FFS}$-compact.*

Proof. Let $\mho = \{(\mu_\kappa, \Omega) : \kappa \in \Lambda\}$ be a $\mathcal{FFS}$-open cover of $\mathbb{Q}$. Since $f_{\psi\varphi}$ is $\mathcal{FFS}$-continuous $f_{\psi\varphi}^{-1}(\mu_\kappa, \Omega) \in \Gamma_1$, $\forall \kappa \in \Lambda$. It follows that$\{f_{\psi\varphi}^{-1}(\mu_\kappa, \Omega) : \kappa \in \Lambda\}$ is a $\mathcal{FFS}$-open cover of $\mathbb{P}$. Since $\mathbb{P}$ is compact $\exists$ a finite number of indices $\kappa_1, \kappa_2, ..., \kappa_n$ such that

$$\mathbb{P} \subset f_{\psi\varphi}^{-1}(\mu_{\kappa_1}, \Omega) \bigcup f_{\psi\varphi}^{-1}(\mu_{\kappa_2}, \Omega) ... \bigcup f_{\psi\varphi}^{-1}(\mu_{\kappa_n}, \Omega).$$

$$\Rightarrow f_{\psi\varphi}(\mathbb{P}) \subset f_{\psi\varphi}(f_{\psi\varphi}^{-1}(\mu_{\kappa_1}, \Omega)) \bigcup f_{\psi\varphi}(f_{\psi\varphi}^{-1}(\mu_{\kappa_2}, \Omega)) ... \bigcup f_{\psi\varphi}(f_{\psi\varphi}^{-1}(\mu_{\kappa_n}, \Omega)).$$

$$\Rightarrow \mathbb{Q} \subset (\mu_{\kappa_1}, \Omega) \bigcup (\mu_{\kappa_2}, \Omega) ... \bigcup (\mu_{\kappa_n}, \Omega).$$

because $f_{\psi\varphi}$ is surjective. Hence $(\mathbb{Q}, \Gamma_2, \Omega)$ is $\mathcal{FFS}$-compact.

Theorem 4.6. *If a $\mathcal{FFSTS}$ $(\mathbb{P}, \Gamma, \Sigma)$ is $\mathcal{FFS}$-compact and $(\xi, \Sigma) \in \mathcal{FFSC}(\mathbb{P}, \Sigma)$. Then (ξ, Σ) is $\mathcal{FFS}$-compact.*

Proof. Let $\mho = \{(\nu_\kappa, \Sigma) : \kappa \in \Lambda\}$ be a $\mathcal{FFS}$-open cover of (ξ, Σ). Then $\mho \cup (\xi, \Sigma)^c$ is a $\mathcal{FFS}$-open cover of $\mathbb{P}$ because $(\xi, \Sigma) \in \mathcal{FFSC}(\mathbb{P}, \Sigma)$. Since $(\mathbb{P}, \Gamma, \Sigma)$ is $\mathcal{FFS}$-compact, $\exists$ finite number of indices $\kappa_1, \kappa_2 ... \kappa_n$ such that $\tilde{\mathbb{P}} \subset (\nu_{\kappa_1}, \Sigma) \cup (\nu_{\kappa_2}, \Sigma) ... \cup (\nu_{\kappa_n}, \Sigma) \cup (\xi, \Sigma)^c$. It implies that $(\xi, \Sigma) \subset (\nu_{\kappa_1}, \Sigma) \cup (\nu_{\kappa_2}, \Sigma) ... \cup (\nu_{\kappa_n}, \Sigma)$. Hence (ξ, Σ) is $\mathcal{FFS}$-compact.

Theorem 4.7. *Every $\mathcal{FFS}$-compact set in a $\mathcal{FFS} - T_2$ space is $\mathcal{FFS}$-closed .*

Proof. Let (ν, Σ) be a $\mathcal{SFS}$ compact set in a $\mathcal{FFS} - T_2$ space $(\mathbb{P}, \Gamma, \Sigma)$ and a $\mathcal{FFS}$-point $\epsilon_\alpha \in (\nu, \Sigma)^c$. Then $\forall$ $\epsilon_\beta \in (\nu, \Sigma)$ we have $\epsilon_\alpha \neq \epsilon_\beta$. Therefore $\exists$ $(\xi_{\epsilon_\beta}, \Sigma), (\mu_{\epsilon_\beta}, \Sigma) \in \Gamma$ such that$\epsilon_\alpha \in (\xi_{\epsilon_\beta}, \Sigma)$, $\epsilon_\beta \in (\mu_{\epsilon_\beta}, \Sigma)$ and $(\xi_{\epsilon_\beta}, \Sigma) \cap (\mu_{\epsilon_\beta}, \Sigma) = \Phi$. Thus $\{(\mu_{\epsilon_\beta}, \Sigma) : \epsilon_\beta \in (\nu, \Sigma)\}$ is a $\mathcal{SFS}$ open cover of (ν, Σ). Since (ν, Σ) is compact $\exists$ finite number of indices $\beta_1, \beta_2 ... \beta_n$ such that $(\nu, \Sigma) \subset \cup_{i=1}^{i=n} \{(\mu_{\epsilon_{\beta_i}}, \Sigma)\}$. Put $(\lambda, \Sigma) \cap_{i=1}^{i=n} \{(\xi_{\epsilon_{\beta_i}}, \Sigma)\}$. Then $(\lambda, \Sigma) \in \Gamma$ and $\epsilon_\alpha \in (\lambda, \Sigma) \subset (\nu, \Sigma)^c$. Hence, $(\nu, \Sigma)^c \in \Gamma$ and $(\nu, \Sigma) \in \mathcal{FFSC}(\mathbb{P}, \Sigma)$.

Theorem 4.8. *Every $\mathcal{FFS}$-compact and $\mathcal{FFS} - T_2$ $\mathcal{FFSTS}$ is $\mathcal{FFS}$-normal.*

Proof. Easy and left to the readers.

Theorem 4.9. *Let $(\mathbb{P}, \Gamma_1, \Sigma)$ and $(\mathbb{Q}, \Gamma_2, \Omega)$ be two $\mathcal{FFSTS}$s and $f_{\psi\varphi} : (\mathbb{P}, \Gamma_1, \Sigma) \to (\mathbb{Q}, \Gamma_2, \Omega)$ is a surjective $\mathcal{FFS}$-continuous mapping. If $(\mathbb{P}, \Gamma_1, \Sigma)$ is $\mathcal{FFS}$-compact and $(\mathbb{Q}, \Gamma_2, \Omega)$ is $\mathcal{FFS} - T_2$, then $f_{\psi\varphi}$ is $\mathcal{FFS}$-closed.*

Proof. Let $(\nu, \Sigma) \in \mathcal{FFSC}(\mathbb{P}, \Sigma)$. From Theorem 4.6 we have $\nu, \Sigma)$ is $\mathcal{FFS}$-compact. Since $f_{\psi\varphi}$ is $\mathcal{FFS}$ continuous, by Theorem 4.5, $f_{\psi\varphi}(\nu, \Sigma)$ is $\mathcal{FFS}$-compact in $(\mathbb{Q}, \Gamma_2, \Omega)$. Therefore by Theorem 4.7, $f_{\psi\varphi}(\nu, \Sigma) \in \mathcal{FFSC}(\mathbb{Q}, \Omega)$ because $(\mathbb{Q}, \Gamma_2, \Omega)$ is $\mathcal{FFS} - T_2$. Hence $f_{\psi\varphi}$ is $\mathcal{FFS}$-closed.

Definition 4.10. *A family Ψ of $\mathcal{FFSS}$s has the finite intersection property if the intersection of the members of each finite subfamily of Ψ is not the null $\mathcal{FFSS}$.*

Theorem 4.11. *A $\mathcal{FFSTS}$ $(\mathbb{P}, \Gamma, \Sigma)$ is $\mathcal{FFS}$-compact if and only if every family of $\mathcal{FFS}$-closed sets with the finite intersection property has a non null intersection.*

Proof. Necessity: Suppose $(\mathbb{P}, \Gamma, \Sigma)$ is $\mathcal{FFS}$-compact and Ψ be any family of $\mathcal{FFS}$-closed sets of $\mathbb{P}$ such that $\bigcap \{(\mu_\kappa, \Sigma) : (\mu_\kappa, \Sigma) \in \Psi, \kappa \in \Lambda\} = \Phi$. Consider $\mho = \{(\mu_\kappa, \Sigma)^c : (\mu_\kappa, \Sigma) \in \Psi, \kappa \in \Lambda\}$. Then $\mho$ is a $\mathcal{FFS}$-open cover of $\tilde{\mathbb{P}}$. Since $(\mathbb{P}, \Gamma, \Sigma)$ is $\mathcal{FFS}$-compact $\exists$ a finite subcovering $\{(\mu_{\kappa_1}, \Sigma)^c, (\mu_{\kappa_2}, \Sigma)^c, ..., (\mu_{\kappa_n}, \Sigma)^c\}$. It follows that $\bigcap_{i=1}^{i=n} \{(\mu_{\kappa_i}, \Sigma)\} = (\bigcup_{i=1}^{i=n} \{(\mu_{\kappa_i}, \Sigma)^c\})^c = \tilde{\mathbb{P}}^c = \Phi$ and Ψ cannot have finite intersection property.

Sufficiency: Suppose contrary that $\mathcal{FFSTS}$ $(\mathbb{P}, \Gamma, \Sigma)$ is not $\mathcal{FFS}$-compact. Then there is $\mathcal{FFS}$-open cover which does not has a finite subcover. Let $\mho = \{(\mu_\kappa, \Sigma) : \kappa \in \Lambda\}$ be a $\mathcal{FFS}$-open cover of $\mathbb{P}$. So $\bigcup_{i=1}^{i=n}\{(\mu_{\kappa_i}, \Sigma)\} \neq \tilde{\mathbb{P}}$. Therefore $\bigcap_{i=1}^{i=n}\{(\mu_{\kappa_i}, \Sigma)^c\} \neq \Phi$. Thus $\{(\mu_\kappa, \Sigma)^c\}$ is a family of $\mathcal{FFS}$ have finite intersection property. By hypothesis $\bigcap(\mu_\kappa, \Sigma)^c \neq \Phi$ and so $\bigcup(\mu_\kappa, \Sigma) \neq \tilde{\mathbb{P}}$. This is a contradiction. Hence $\mathcal{FFSTS}$ $(\mathbb{P}, \Gamma, \Sigma)$ is $\mathcal{FFS}$-compact.

References

[1] Al-shami, T. M., Ibrahim, H. Z., Azzam, A. A., EL-Maghrabi, A. I., SR-fuzzy sets and their applications to weighted aggregated operators in decision-making, J. Funct. Space, 1 (2022), 1-14.

[2] Al-shami, T. M., Mhemdi, A., Generalized Frame for Orthopair Fuzzy Sets: (m, n)-Fuzzy Sets and Their Applications to Multi-Criteria Decision-Making Methods, Information, 14(56) (2023), 1-21.

[3] Al-shami, T. M., Carlos, J., Alcantud R., Mhemdi, A., New generalization of fuzzy soft sets: (a, b)-Fuzzy soft sets, AIMS Mathematics, 8(2) (2023), 2995–3025.

[4] Al-shami, T. M., Mhemdi, A., A weak form of soft α-open sets and its applications via soft topologies, AIMS Mathematics, 8(5) (2023), 11373–11396.

[5] Al-shami, T. M., Hosny, R. A., Abu-Gdairi, R., Arar, M., A novel approach to study soft preopen sets inspired by classical topologies, Journal of Intelligent and Fuzzy Systems, 45(4) (2023), 6339-6350.

[6] Al-shami T. M., Ibrahim, H. Z., Mhemdi, A.,Abu-Gdairi, R., nth- Power Root Fuzzy Sets and Its Topology, International Journal of Fuzzy Logic and Intelligent Systems, 22(4) (2022), 350-365.

[7] Al-shami, Tareq M. , Arar, M., Abu-Gdairi, R., Ameen, Z. A., On weakly soft β-open sets and weakly soft β-continuity, Journal of Intelligent and Fuzzy Systems, 45(4) (2023), 6351-6363.

[8] Al-shami, T. M., Hosny, R. A., Mhemdi, A., Abu-Gdairi,R., Saleh, S., Weakly soft b-open sets and their usages via soft topologies : A novel approach, Journal of Intelligent and Fuzzy Systems, 45(5) (2023), 7727-7738.

[9] Al-shami, T. M., Mhemdi, A., Abu-Gdairi, R., A Novel Framework for Generalizations of Soft Open Sets and Its Applications via Soft Topologies, Mathematics, 11(840) (2023).

[10] Atanassov K. T., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1986), 87-96.

[11] Chang C. L., Fuzzy topological spaces, J. Math. Anal. Appl., 24 (1968), 182-190.

[12] Coker D., An introduction to Intuitionistic fuzzy topological spaces, Fuzzy sets and systems, 88 (1997), 81-89.

[13] Ibrahim H. Z., Fermatean Fuzzy topological spaces, J. Appl. Math. and Informatics, 40(1-2) (2022), 85-98.

[14] Li Z. and Cui R., On the topological structure of intuitionistic fuzzy soft sets, Ann. Fuzzy Math. Inform., 5(1) (2013), 229-239.

[15] Maji P. K., Biswas R. and Roy A. R., Fuzzy soft sets, J. Fuzzy Math., 9 (2001), 589-602.

[16] Maji P. K. , Biswas R. and Roy A. R., Intuitionistic fuzzy soft sets, J. Fuzzy Math., 9 (2001), 677-692.

[17] Maji P. K. , Biswas R. and Roy A. R., Soft Set Theory, Computers and Mathematics with Applications, 45(4-5) (2003), 555-562.

[18] Molodtsov D., Soft set theory-First results, Comput. Math. Appl., 37 (1999), 19-31.

[19] Olgun Z., Unver M. and Yard S., Pythagorean fuzzy topological spaces, Complex Intell. Syst., 5 (2019), 1-7.

[20] Prasad A. K. , Bajpai J. P. and Thakur S. S., Topological structure on Fermatean fuzzy soft sets, GANITA, 74(1) (2024), 227-237.

[21] Prasad A. K., Thakur M., Bajpai J. P., and Thakur S. S., Mappings on Fermatean fuzzy soft classes, South East Asian J. of Mathematics and Mathematical Sciences, 20(1) (2024), 197-210.

[22] Peng X., Yang Y., Song J. and Jiang Y., Pyhagorean fuzzy soft set and its application, Computer Engineering, 41(7) (2015), 224-229.

[23] Senapati T. and Yager R. R., Fermatean fuzzy sets, Journal of Ambient Intelligence and Humanized Computing, 11(2020), 663–674.

[24] Shabir M. and Naz M., On soft topological spaces, Comput. Math. Appl., 61 (2011), 1786-1799.

[25] Sivadas A. and John S. J., Fermatean Fuzzy soft sets and its applications, Computational Sciences - Modelling, Computing and Soft Computing, Communications in Computer and Information Science, In eds. A. Awasthi, S. J John, and S. Panda , Springer, Singapore, (2020). https://doi.org/10.1007/978-981-16-4772-7-16.

[26] Tanay B. and Kandemir K. B., Topological structure of fuzzy soft sets, Comput. Math. Appl., 61 (2011), 2952–2957.

[27] Yager R. R., Pythagorean fuzzy subsets, In : Editor, Pedrycz, W. Reformat, Marek Z.,(eds) Proceedings of the 2013 joint IFSA world congress and NAFIPS annual meeting (IFSA/NAFIPS) IEEE, Edmonton, Canada, (2013), 57-61.

[28] Yolcu A. and Taha Y. O., Some new results of pythagorean fuzzy soft topological spaces, TWMS J. App. and Eng. Math, 12(3) (2022), 1107-1122.

[29] Zadeh L. A., Fuzzy sets, Inf. and Control, 8 (1965), 338-353.

This page intentionally left blank.