J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol.4, No.1 (2015), pp. 85-94

Truncated bilateral hypergeometric summation theorems motivated by the work of Verma

M. I. Qureshi[1] and Kaleem A. Quraishi[2]
[1]Department of Applied Sciences and Humanities,
Faculty of Engineering and Technology,
Jamia Millia Islamia(A Central University), New Delhi-110025(India)
[2]Mathematics Section, Mewat Engineering College (Waqf),
Palla, Nuh, Mewat, Haryana-122107(India)
E-mails: miqureshi_delhi@yahoo.co.in; kaleemspn@yahoo.co.in

(Dedicated to Prof. R. P. Agarwal, June 15, 1925-February 9, 2008)

Abstract: In this paper some results on truncated bilateral hypergeometric series of positive unit argument are obtained by using series rearrangement technique and theory of polynomial equations, subject to certain conditions.
Keywords and Phrases: Pochhammer symbol; Unilateral, bilateral, truncated and non terminating series
2010 AMS Subject Classifications: 33- Special Functions, Primary 33C99; Secondary 33C20

1. Introduction

Truncated Unilateral Generalized Hypergeometric Series

$$ {}_AF_B\left[\begin{array}{lcc} a_1, a_2, \ldots, a_A & ; & \\ & & z \\ b_1, b_2, \ldots, b_B & ; & \end{array}\right] \text{to } (N+1) \text{ terms} $$

$$ = {}_AF_B\left[\begin{array}{lcc} a_1, a_2, \ldots, a_A & ; & \\ & & z \\ b_1, b_2, \ldots, b_B & ; & \end{array}\right]_N = \sum_{k=0}^{N} \frac{\prod_{j=1}^{A}(a_j)_k \; z^k}{\prod_{j=1}^{B}(b_j)_k \; k!} \tag{1.1} $$

where numerator and denominator parameters are neither zero nor negative integers and $A,\ B$ are non-negative integers. When $N \to \infty$ then (1.1) reduces to non-terminating unilateral generalized hypergeometric series and Pochhammer's symbol $(c)_k$ is given by $(c)_k = \prod_{j=0}^{k-1}(c+j)$

Non Terminating Bilateral Generalized Hypergeometric Series

(Dirichlet series or Laurent series)[3,pp.180-182(6.1.1.2,6.1.1.3,6.1.2.3)]
The values of parameters $a_1, a_2, \ldots, a_A,\ b_1, b_2, b_3, \ldots, b_A$ are adjusted in such a manner that each term in the expansion of (1.2) is well defined then

$$ {}_AH_A\left[\begin{array}{ll} a_1,\ a_2,\ \ldots,\ a_A & ; \\ & z \\ b_1,\ b_2,\ \ldots,\ b_A & ; \end{array}\right] = \sum_{r=-\infty}^{\infty} \frac{(a_1)_r (a_2)_r \cdots (a_A)_r\ z^r}{(b_1)_r (b_2)_r \cdots (b_A)_r} \tag{1.2}$$

Truncated Bilateral Generalized Hypergeometric Series

In 1996, R. P. Agarwal[1,p.19(13)] gave the following definition

$$ {}_AH_A\left[\begin{array}{ll} a_1,\ a_2,\ \ldots,\ a_A & ; \\ & z \\ b_1,\ b_2,\ \ldots,\ b_A & ; \end{array}\right]_N^M = \sum_{r=-M}^{N} \frac{\prod_{j=1}^{A}(a_j)_r\ z^r}{\prod_{j=1}^{A}(b_j)_r} \tag{1.3}$$

where $A, M, N \in \{1, 2, 3, \ldots\}$.
When $M, N \to \infty$ in (1.3), we get non terminating bilateral generalized hypergeometric series (1.2).

In our analysis, the symbol $S_r(g_1, g_2, g_3, \ldots, g_B)$ represents the sum of all possible combinations of the products of parameters taken "r" at a time from the set of "B" parameters $\{g_1, g_2, g_3, \ldots, g_B\}$.

For the sake of convenience $S_r(a_0, a_1, a_2, \ldots, a_A)$, $S_r(b_1, b_2, \ldots, b_A)$, $S_r(b_0, 1-b_1, 1-b_2, \ldots, 1-b_A)$ and $S_r(1-a_1, 1-a_2, \ldots, 1-a_A)$ are denoted by $S_r(a_0, (a_A))$, $S_r((b_A))$, $S_r(b_0, 1-(b_A))$ and $S_r(1-(a_A))$ respectively.

Summation Theorems for Truncated Unilateral Hypergeometric Series

First theorem[2,pp.274-275(3.3.1)-(3.3.6)]

$$ {}_{A+3}F_{A+2}\left[\begin{array}{ll} a_0,\ a_1,\ a_2,\ a_3,\ \cdots,\ a_A,\ 1-\beta,\ 1-\gamma & ; \\ & 1 \\ 1+b_1,\ 1+b_2,\ 1+b_3,\ \cdots,\ 1+b_A,\ -\beta,\ -\gamma & ; \end{array}\right]_N $$

$$ = \frac{(1+a_0)_N\ (1+a_1)_N\ (1+a_2)_N\ (1+a_3)_N\ \cdots\ (1+a_A)_N}{N!\ (1+b_1)_N\ (1+b_2)_N\ (1+b_3)_N\ \cdots\ (1+b_A)_N} \tag{1.4}$$

where β, γ and D are given by

$$\beta = \frac{-S_A(a_0,(a_A)) + S_A((b_A)) + \sqrt{D}}{2\,\{S_{A-1}(a_0,(a_A)) - S_{A-1}((b_A))\}} \tag{1.5}$$

$$\gamma = \frac{-S_A(a_0,(a_A)) + S_A((b_A)) - \sqrt{D}}{2\,\{S_{A-1}(a_0,(a_A)) - S_{A-1}((b_A))\}} \tag{1.6}$$

$$D = \{S_A(a_0,(a_A)) - S_A((b_A))\}^2 - 4\{S_{A+1}(a_0,(a_A))\}\{S_{A-1}(a_0,(a_A)) - S_{A-1}((b_A))\} \tag{1.7}$$

subject to "$(A-2)$" number of actual conditions in compact notation, given by (1.8)

$$S_r(a_0,(a_A)) = S_r((b_A)),\;\; A \in \{3,4,5,\cdots\} \text{ and } r = 1,2,3,\cdots,(A-2) \tag{1.8}$$

and

$$S_{A-1}(a_0,(a_A)) \neq S_{A-1}((b_A));\;\; S_{A+1}(a_0,(a_A)) \neq 0,\;\; N \in \{1,2,3,\cdots\} \tag{1.9}$$

Second theorem[2,pp.275-277(3.4.1)-(3.4.8)]

$$ {}_{A+4}F_{A+3}\left[\begin{array}{ll} a_0,\, a_1,\, a_2,\, a_3,\, \cdots,\, a_A,\, 1-\delta,\; 1-\theta,\;\; 1-\psi & ; \\ & \quad 1 \\ 1+b_1,\, 1+b_2,\, 1+b_3,\, \cdots,\, 1+b_A,\, -\delta,\, -\theta,\, -\psi & ; \end{array}\right]_N$$

$$= \frac{(1+a_0)_N\;(1+a_1)_N\;(1+a_2)_N\;(1+a_3)_N\;\cdots\;(1+a_A)_N}{N!\;(1+b_1)_N\;(1+b_2)_N\;(1+b_3)_N\;\cdots\;(1+b_A)_N} \tag{1.10}$$

where δ, θ and ψ are given by

$$\delta = -\frac{\{S_{A-1}(a_0,(a_A)) - S_{A-1}((b_A))\}}{3\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}} + \frac{(-1-i\sqrt{3})\;\lambda^{\frac{1}{3}}}{6(2)^{\frac{1}{3}}\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}} + $$
$$+ \frac{(1-i\sqrt{3})\Omega}{3(2)^{\frac{2}{3}}\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}\;\lambda^{\frac{1}{3}}} \tag{1.11}$$

$$\theta = -\frac{\{S_{A-1}(a_0,(a_A)) - S_{A-1}((b_A))\}}{3\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}} + \frac{(-1+i\sqrt{3})\;\lambda^{\frac{1}{3}}}{6(2)^{\frac{1}{3}}\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}} + $$
$$+ \frac{(1+i\sqrt{3})\Omega}{3(2)^{\frac{2}{3}}\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}\;\lambda^{\frac{1}{3}}} \tag{1.12}$$

$$\psi = -\frac{\{S_{A-1}(a_0,(a_A)) - S_{A-1}((b_A))\}}{3\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}} + \frac{\lambda^{\frac{1}{3}}}{3(2)^{\frac{1}{3}}\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}} - $$
$$ - \frac{(2)^{\frac{1}{3}}\Omega}{3\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}\ \lambda^{\frac{1}{3}}} \tag{1.13}$$

where

$$\lambda = [-2\{S_{A-1}(a_0,(a_A)) - S_{A-1}((b_A))\}^3 + 9\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}\times$$
$$\times\{S_{A-1}(a_0,(a_A)) - S_{A-1}((b_A))\}\{S_A(a_0,(a_A)) - S_A((b_A))\}$$
$$-27\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}^2\{S_{A+1}(a_0,(a_A))\}] + \sqrt{R} \tag{1.14}$$

and

$$R = 4[3\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}\{S_A(a_0,(a_A)) - S_A((b_A))\}$$
$$-\{S_{A-1}(a_0,(a_A)) - S_{A-1}((b_A))\}^2]^3 +$$
$$+[-2\{S_{A-1}(a_0,(a_A)) - S_{A-1}((b_A))\}^3 + 9\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}\times$$
$$\times\{S_{A-1}(a_0,(a_A)) - S_{A-1}((b_A))\}\{S_A(a_0,(a_A)) - S_A((b_A))\}$$
$$-27\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}^2\{S_{A+1}(a_0,(a_A))\}]^2 \tag{1.15}$$

subject to "$(A-3)$" number of actual conditions in compact notation, given by (1.16)

$$S_r(a_0,(a_A)) = S_r((b_A)),\ A \in \{4,5,6,\cdots\} \text{ and } r = 1,2,3,\cdots,(A-3) \tag{1.16}$$

and

$$S_{A-2}(a_0,(a_A)) \neq S_{A-2}((b_A));\ S_{A+1}(a_0,(a_A)) \neq 0,\ N \in \{1,2,3,\cdots\} \tag{1.17}$$

We define the values of Ω as follows

$$\Omega = [3\{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}\{S_A(a_0,(a_A)) - S_A((b_A))\} -$$
$$-\{S_{A-1}(a_0(a_A)) - S_{A-1}((b_A))\}^2]$$

Third theorem[2,pp.278-279(3.6.1)-(3.6.4)]

$${}_{A+B+1}F_{A+B}\left[\begin{array}{lc} a_0,\, a_1,\, a_2,\, \ldots,\, a_A,\, 1-\lambda_1, 1-\lambda_2,\, \ldots,\, 1-\lambda_B \quad ; & \\ & 1 \\ 1+b_1,\, 1+b_2,\, \ldots,\, 1+b_A,\, -\lambda_1, -\lambda_2,\, \ldots,\, -\lambda_B \quad ; & \end{array}\right]_N$$

$$= \frac{(1+a_0)_N\ (1+a_1)_N\ (1+a_2)_N\ \cdots\ (1+a_A)_N}{N!\ (1+b_1)_N\ (1+b_2)_N\ \cdots\ (1+b_A)_N} \tag{1.18}$$

subject to "$(A-B)$" number of actual conditions in compact notation given by (1.19)

$$S_r(a_0,(a_A)) = S_r((b_A)),\ A \in \{B+1, B+2, B+3, \dots\} \text{ and } r = 1,2,3,\dots,(A-B) \tag{1.19}$$

and

$$S_{A-B+1}(a_0,(a_A)) \neq S_{A-B+1}((b_A)),\ S_{A+1}(a_0,(a_A)) \neq 0;\ 0 < B < A \tag{1.20}$$

where $\lambda_1,\ \lambda_2,\ \dots,\ \lambda_B$ are the roots(neither zero nor positive integers) of the following polynomial equation

$$[\{S_{A-B+1}(a_0,(a_A)) - S_{A-B+1}((b_A))\}m^B + \{S_{A-B+2}(a_0,(a_A)) - S_{A-B+2}((b_A))\}m^{B-1} + \\ + \cdots + \{S_{A-2}(a_0,(a_A)) - S_{A-2}((b_A))\}m^3 + \{S_{A-1}(a_0,(a_A)) - S_{A-1}((b_A))\}m^2 + \\ + \{S_A(a_0,(a_A)) - S_A((b_A))\}m + \{S_{A+1}(a_0,(a_A))\}] = 0 \tag{1.21}$$

Motivated by the work of Verma [4, pp. 233-234(3.4, 3.5, 3.6, 3.7)], in next sections we shall derive more summation theorems, using series iteration technique.

Since Pochhammer's symbol is associated with Gamma function and Gamma function is undefined for zero and negative integers therefore numerator and denominator parameters are adjusted in such a way that each term of following results is completely well defined and meaningful then we have following summation theorems.

2. First summation theorem for truncated bilateral hypergeometric series

If the values of the parameters $a_1,\ a_2,\dots,\ a_A,\ b_1,\ b_2,\dots,\ b_A$ and $\zeta,\ \sigma$ are neither zero nor integers, then

$${}_{A+2}H_{A+2}\left[\begin{array}{lc} a_1,\ a_2,\ \dots,\ a_A,\ 1-\zeta,\ 1-\sigma & ; \\ & \quad 1 \\ 1+b_1,\ 1+b_2,\ 1+b_3,\ \dots,\ 1+b_A,\ -\zeta,\ -\sigma & ; \end{array}\right]_N^M$$

$$= \frac{\{S_{A-1}(1,(a_A)) - S_{A-1}((b_A)) - S_A(1,(a_A)) + S_A((b_A)) + S_{A+1}(1,(a_A))\}}{\{S_{A+1}(1,(a_A))\}} \times$$

$$\times \frac{(2)_{M-1} \prod_{j=1}^{A}(b_j) \prod_{j=1}^{A}(2-b_j)_{M-1}}{(M-1)! \prod_{j=1}^{A}(-1+a_j) \prod_{j=1}^{A}(2-a_j)_{M-1}} + \frac{(2)_N \prod_{j=1}^{A}(1+a_j)_N}{N! \prod_{j=1}^{A}(1+b_j)_N} \tag{2.1}$$

under the "$(A-2)$" number of actual conditions given by

$$S_r(1,(a_A)) = S_r((b_A)) \text{ or equivalently } S_r(1,1-(b_A)) = S_r(1-(a_A)) \tag{2.2}$$

$$S_{A-1}(1,(a_A)) \neq S_{A-1}((b_A)) \text{ and } S_{A-1}(1,1-(b_A)) \neq S_{A-1}(1-(a_A)) \tag{2.3}$$

$$S_{A+1}(1,(a_A)) \neq 0 \text{ and } S_{A+1}(1,1-(b_A)) \neq 0 \tag{2.4}$$

$$M \in \{2,3,4,\dots\},\ A \in \{3,4,5,\dots\},\ N \in \{1,2,3,\dots\},\ r = 1,2,3,\dots,(A-2) \tag{2.5}$$

where ζ and σ are the roots of the quadratic equation

$$\{S_{A-1}(1,(a_A)) - S_{A-1}((b_A))\}m^2 + \{S_A(1,(a_A)) - S_A((b_A))\}m + \{S_{A+1}(1,(a_A))\} = 0 \tag{2.6}$$

Proof of Theorem (2.1): Suppose ζ^* and σ^* are the roots of the quadratic equation

$$\{S_{A-1}(1,1-(b_A)) - S_{A-1}(1-(a_A))\}m^2 + \{S_A(1,1-(b_A)) - S_A(1-(a_A))\}m+$$
$$+\{S_{A+1}(1,1-(b_A))\} = 0 \tag{2.7}$$

By taking suitable pairs of the roots of (2.6) and (2.7), different positive integral values of A, non-integral values of parameters $a_1, a_2, \dots, a_A, b_1, b_2, \dots, b_A$, satisfying the "$(A-2)$" number of conditions given by (2.2) and using Mathematica 6.0 we can verify that $-\zeta - \zeta^* = 1$ and $-\sigma - \sigma^* = 1$.

Suppose left hand side of (2.1) is denoted by Ω, then

$$\Omega = \sum_{r=-M}^{N} \frac{(a_1)_r(a_2)_r \cdots (a_A)_r(1-\zeta)_r(1-\sigma)_r}{(1+b_1)_r(1+b_2)_r \cdots (1+b_A)_r(-\zeta)_r(-\sigma)_r}$$

$$= \sum_{r=1}^{M} \frac{(a_1)_{-r}(a_2)_{-r} \cdots (a_A)_{-r}(1-\zeta)_{-r}(1-\sigma)_{-r}}{(1+b_1)_{-r}(1+b_2)_{-r} \cdots (1+b_A)_{-r}(-\zeta)_{-r}(-\sigma)_{-r}} +$$

$$+ \sum_{r=0}^{N} \frac{(1)_r(a_1)_r(a_2)_r \cdots (a_A)_r(1-\zeta)_r(1-\sigma)_r}{(1+b_1)_r(1+b_2)_r \cdots (1+b_A)_r(-\zeta)_r(-\sigma)_r\ r!}$$

$$= \frac{(1+\zeta)(1+\sigma)\prod_{i=1}^{A}(b_i)}{\zeta\sigma\prod_{i=1}^{A}(-1+a_i)} \times$$

$$\times {}_{A+3}F_{A+2}\left[\begin{matrix} 1,\, 1-b_1,\, 1-b_2,\, \ldots,\, 1-b_A,\, 2+\zeta,\, 2+\sigma \;; \\ 2-a_1,\; 2-a_2,\; \ldots,\; 2-a_A,\; 1+\zeta,\, 1+\sigma \;; \end{matrix}\; 1\right]_{M-1} +$$

$$+{}_{A+3}F_{A+2}\left[\begin{matrix} 1,\; a_1,\; a_2,\; \ldots,\; a_A,\; 1-\zeta,\, 1-\sigma \;; \\ 1+b_1,\; 1+b_2,\; \ldots,\; 1+b_A,\; -\zeta,\, -\sigma \;; \end{matrix}\; 1\right]_{N} \tag{2.8}$$

$$= \frac{(1+\zeta)(1+\sigma)\prod_{i=1}^{A}(b_i)}{\zeta\sigma\prod_{i=1}^{A}(-1+a_i)} \times$$

$$\times {}_{A+3}F_{A+2}\left[\begin{matrix} 1,\, 1-b_1,\, 1-b_2,\, \ldots,\, 1-b_A,\, 1-\zeta^*,\, 1-\sigma^* \;; \\ 2-a_1,\; 2-a_2,\; \ldots,\; 2-a_A,\; -\zeta^*,\, -\sigma^* \;; \end{matrix}\; 1\right]_{M-1} +$$

$$+{}_{A+3}F_{A+2}\left[\begin{matrix} 1,\; a_1,\; a_2,\; \ldots,\; a_A,\; 1-\zeta,\, 1-\sigma \;; \\ 1+b_1,\; 1+b_2,\; \ldots,\; 1+b_A,\; -\zeta,\, -\sigma \;; \end{matrix}\; 1\right]_{N} \tag{2.9}$$

Now applying authors summation theorem (1.4), we get the right hand side of (2.1).

3. Second summation theorem for truncated bilateral hypergeometric series

If the values of the parameters $a_1, a_2, \ldots, a_A$, $b_1, b_2, \ldots, b_A$ and ξ, μ, ρ are neither zero nor integers, then

$${}_{A+3}H_{A+3}\left[\begin{matrix} a_1,\, a_2,\, \ldots,\, a_A,\, 1-\xi,\, 1-\mu,\; 1-\rho \;; \\ 1+b_1,\, 1+b_2,\, \ldots,\, 1+b_A,\, -\xi,\, -\mu,\, -\rho \;; \end{matrix}\; 1\right]_{N}^{M}$$

$$= \left\{ \frac{-S_{A-2}(1,(a_A)) + S_{A-2}((b_A)) + S_{A-1}(1,(a_A))}{\{S_{A+1}(1,(a_A))\}} - \right.$$

$$\left. - \frac{S_{A-1}((b_A)) + S_A(1,(a_A)) - S_A((b_A)) - S_{A+1}(1,(a_A))\}}{\{S_{A+1}(1,(a_A))\}} \right\} \times$$

$$\times \frac{(2)_{M-1}\prod_{j=1}^{A}(b_j)\prod_{j=1}^{A}(2-b_j)_{M-1}}{(M-1)!\prod_{j=1}^{A}(-1+a_j)\prod_{j=1}^{A}(2-a_j)_{M-1}} + \frac{(2)_N\prod_{j=1}^{A}(1+a_j)_N}{N!\prod_{j=1}^{A}(1+b_j)_N} \tag{3.1}$$

where ξ, μ and ρ are given by

$$\xi = -\frac{\{S_{A-1}(1,(a_A)) - S_{A-1}((b_A))\}}{3\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}} + \frac{(-1-i\sqrt{3})\ \lambda_1^{\frac{1}{3}}}{6(2)^{\frac{1}{3}}\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}} + $$
$$+\frac{(1-i\sqrt{3})\Omega_1}{3(2)^{\frac{2}{3}}\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}\ \lambda_1^{\frac{1}{3}}} \tag{3.2}$$

$$\mu = -\frac{\{S_{A-1}(1,(a_A)) - S_{A-1}((b_A))\}}{3\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}} + \frac{(-1+i\sqrt{3})\ \lambda_1^{\frac{1}{3}}}{6(2)^{\frac{1}{3}}\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}} + $$
$$+\frac{(1+i\sqrt{3})\Omega_1}{3(2)^{\frac{2}{3}}\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}\ \lambda_1^{\frac{1}{3}}} \tag{3.3}$$

$$\rho = -\frac{\{S_{A-1}(1,(a_A)) - S_{A-1}((b_A))\}}{3\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}} + \frac{\lambda_1^{\frac{1}{3}}}{3(2)^{\frac{1}{3}}\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}} - $$
$$-\frac{(2)^{\frac{1}{3}}\Omega_1}{3\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}\ \lambda_1^{\frac{1}{3}}} \tag{3.4}$$

where

$$\lambda_1 = [-2\{S_{A-1}(1,(a_A)) - S_{A-1}((b_A))\}^3 + 9\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}\times$$
$$\{S_{A-1}(1,(a_A)) - S_{A-1}((b_A))\}\{S_A(1,(a_A)) - S_A((b_A))\}$$
$$-27\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}^2 S_{A+1}(1,(a_A))] + \sqrt{R_1} \tag{3.5}$$

and

$$R_1 = 4[3\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}\{S_A(1,(a_A)) - S_A((b_A))\} - \{S_{A-1}(1,(a_A)) -$$
$$-S_{A-1}((b_A))\}^2]^3 + [-\ 2\{S_{A-1}(1,(a_A))\ -\ S_{A-1}((b_A))\}^3 + 9$$
$$\{S_{A-2}(1,(a_A))\ -\ S_{A-2}((b_A))\}\{S_{A-1}(1,(a_A))\ -\ S_{A-1}((b_A))\}\times$$
$$\times\{S_A(1,(a_A))\ -\ S_A((b_A))\} - 27\{S_{A-2}(1,(a_A))\ -\ S_{A-2}((b_A))\}^2\ \{S_{A+1}(1,(a_A))\}]^2 \tag{3.6}$$

subject to the "$(A-3)$" number of conditions in compact notation given by

$$S_r(1,(a_A)) = S_r((b_A)) \text{ or equivalently } S_r(1, 1-(b_A)) = S_r(1-(a_A)) \tag{3.7}$$

$$S_{A-2}(1,(a_A)) \neq S_{A-2}((b_A)) \text{ and } S_{A-2}(1,1-(b_A)) \neq S_{A-2}(1-(a_A)) \tag{3.8}$$

$$S_{A+1}(1,(a_A)) \neq 0 \text{ and } S_{A+1}(1,1-(b_A)) \neq 0 \tag{3.9}$$

$$A \in \{4,5,6,\dots\},\ r = 1,2,\dots,(A-3),\ M \in \{2,3,4,\dots\},\ N \in \{1,2,3,\dots\} \tag{3.10}$$

$$\Omega_1 = [3\{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}\{S_A(1,(a_A)) - S_A((b_A))\} - \\ -\{S_{A-1}(1,(a_A)) - S_{A-1}((b_A))\}^2] \tag{3.11}$$

If we proceed on the same parallel lines as discussed in section 2 and apply authors summation theorem (1.10)-(1.17), we can obtain (3.1)-(3.10).

4. Third summation theorem for truncated bilateral hypergeometric series

If the values of the parameters $a_1, a_2, \dots, a_A, b_1, b_2, \dots, b_A$ and $\Upsilon_1 \Upsilon_2, \dots, \Upsilon_B$ are neither zero nor integers, then

$$ {}_{A+B}H_{A+B}\left[\begin{array}{ll} a_1,\, a_2,\, \dots,\, a_A,\, 1-\Upsilon_1,\, 1-\Upsilon_2,\, \dots,\, 1-\Upsilon_B & ; \\ & 1 \\ 1+b_1,\, 1+b_2,\, \dots,\, 1+b_A,\, -\Upsilon_1,\, -\Upsilon_2,\, \dots,\, -\Upsilon_B & ; \end{array}\right]_N^M $$

$$= \frac{\left[\sum_{j=A-B+1}^{A}(-1)^{j+B-A-1}\{S_j(1,(a_A)) - S_j((b_A))\} + (-1)^B S_{A+1}(1,(a_A))\right]}{(-1)^B\{S_{A+1}(1,(a_A))\}} \times$$

$$\times \frac{(2)_{M-1}\prod_{j=1}^{A}(b_j)\prod_{j=1}^{A}(2-b_j)_{M-1}}{(M-1)!\prod_{j=1}^{A}(-1+a_j)\prod_{j=1}^{A}(2-a_j)_{M-1}} + \frac{(2)_N\prod_{j=1}^{A}(1+a_j)_N}{N!\prod_{j=1}^{A}(1+b_j)_N} \tag{4.1}$$

where $\Upsilon_1, \Upsilon_2, \Upsilon_3, \dots, \Upsilon_B$ are the non zero roots of the polynomial equation of degree B in m, given by

$$[\{S_{A-B+1}(1,(a_A)) - S_{A-B+1}((b_A))\}m^B + \{S_{A-B+2}(1,(a_A)) - S_{A-B+2}((b_A))\}m^{B-1} \\ + \dots + \{S_{A-3}(1,(a_A)) - S_{A-3}((b_A))\}m^4 + \{S_{A-2}(1,(a_A)) - S_{A-2}((b_A))\}m^3 + \\ +\{S_{A-1}(1,(a_A)) - S_{A-1}((b_A))\}m^2 + \{S_A(1,(a_A)) - \\ -S_A((b_A))\}m + \{S_{A+1}(1,(a_A))\}] = 0 \tag{4.2}$$

under the "$(A-B)$" number of conditions in compact notation given by

$$S_r(1,(a_A)) = S_r((b_A)) \text{ or equivalently } S_r(1,1-(b_A)) = S_r(1-(a_A)) \tag{4.3}$$

$$S_{A-B+1}(1,(a_A)) \neq S_{A-B+1}((b_A)) \text{ and } S_{A-B+1}(1,1-(b_A)) \neq S_{A-B+1}(1-(a_A)) \quad (4.4)$$

$$S_{A+1}(1,(a_A)) \neq 0 \text{ and } S_{A+1}(1,1-(b_A)) \neq 0 \quad (4.5)$$

$$A \in \{B+1, B+2, B+3, \dots\},$$

$$r = 1,2,\dots,(A-B),\ M \in \{2,3,4,\dots\},\ N \in \{1,2,3,\dots\} \quad (4.6)$$

If we proceed on the same parallel lines as discussed in section 3 and apply authors summation theorem (1.18)-(1.21), we can obtain (4.1)-(4.6).

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