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CHARACTERIZATIONS OF CONFORMAL η-EINSTEIN SOLITONS ON LP-KENMOTSU 3-MANIFOLDS

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Abstract: In this manuscript, Existence of conformal η-Einstein solitons on LP-Kenmotsu manifold is discussed. We have studied conformal η-Einstein solitons on 3-dimensional LP-Kenmotsu manifold where the Ricci tensors are Coddazi type and cyclic parallel under certain restriction of the Ricci tensor. We have also discussed second order parallel symmetric tensors admitting conformal η-Einstein solitons on 3-dimensional LP-Kenmotsu manifolds. We also use torse-forming vector fields in addition to conformal η-Einstein solitons on 3-dimensional LP-Kenmotsu manifolds. Finally, in 3-dimensional LP-Kenmotsu manifold, we have a non-trivial example.

Keywords and Phrases: Conformal η-Einstein solitons, LP-Kenmotsu manifold, codazzi type Ricci tensor, Second order parallel symmetric tensors.

2020 Mathematics Subject Classification: Primary 53C15, Secondary 53C25.

1. Introduction

The concept of Lorentzian para-Sasakian manifold (LP-Sasakian manifold) are introduced by K. Motsumoto [10]. Mihai and Rosca [12] defined the equvalent

concept independently and they found several important results on this manifold. In addition to this LP-Sasakian manifolds had been studied by Matsumoto and Mihai [11], Mihai, Shaikh and De [13], Venkatesha and Bagewadi [24], Venkatesha, Pradeep Kumar and Bagewadi [25, 26] and obtained several results of these manifolds.
The Ricci flow is an evolution equation for metrics on a Riemannian manifold is given as

$$\frac{\partial}{\partial t}\tilde{g}(t) = -2\overset{\star}{\mathcal{S}},$$

where $\overset{\star}{\mathcal{S}}$ is the Ricci tensor of Riemannian metric $\tilde{g}(t)$. A Ricci soliton emerges as the limit of the solutions of the Ricci flow [7, 23]. A Pesudo-Riemannian metric $\tilde{g}$, defined on a manifold $\mathcal{M}^n$, is called a Ricci soliton, such that

$$\frac{1}{2}\mathfrak{L}_{\mathcal{V}_*}\tilde{g} + \overset{\star}{\mathcal{S}} = \lambda_1\tilde{g}.$$

where $\mathfrak{L}_{\mathcal{V}_*}$ denotes the Lie-derivative along the vector field $\mathcal{V}_*$, $\overset{\star}{\mathcal{S}}$ is the Ricci tensor of $\tilde{g}$ and λ_1 is a constant. The Ricci soliton considered to be decreasing, state or growing depending on whether λ_1 is negative, zero or positive. Several geometers have investigated Ricci soliton [8, 21].

The concept of Einstein soliton was developed by G. Catino and L. Mazzieri [2] in 2016, which initiate self-similar solutions to the Einstein flow, it is provided by

$$\frac{\partial\tilde{g}}{\partial t} = -2(\overset{\star}{\mathcal{S}} - \frac{\sigma}{2}\tilde{g}),$$

where $\overset{\star}{\mathcal{S}}$ is the Ricci tensor, $\tilde{g}$ is a Riemannian metric and σ is the scalar curvature.

The equation of the η-Einstein soliton [1] is given as

$$\mathfrak{L}_{\tilde{\xi}}\tilde{g} + 2\overset{\star}{\mathcal{S}} + (2\lambda_1 - \sigma)\tilde{g} + 2\mu_1\tilde{\eta}\otimes\tilde{\eta} = 0. \tag{1.1}$$

For $\mu_1 = 0$, the data $(\tilde{g}, \tilde{\xi}, \lambda_1)$ is called Einstein soliton.

The notion of conformal η-Ricci soliton introduced by M. D. Siddiqui [19] is defined as

$$\mathfrak{L}_{\tilde{\xi}}\tilde{g} + 2\overset{\star}{\mathcal{S}} + \left[2\lambda_1 - \left(p + \frac{2}{n}\right)\right]\tilde{g} + 2\mu_1\tilde{\eta}\otimes\tilde{\eta} = 0, \tag{1.2}$$

where λ_1, μ_1 are constants, p is a scalar non-dynamical field (time dependent scalar field) and n is the dimension of manifold. For $\mu_1 = 0$, conformal η-Ricci soliton becomes conformal Ricci soliton and studied by many geometers [5, 15, 16, 22]. S. Roy, S. Dey and A. Bhattacharyya [17] introduced conformal Einstein soliton, which is given as

$$\mathfrak{L}_{\mathcal{V}_*}\tilde{g} + 2\overset{\star}{\mathcal{S}} + \left[2\lambda_1 - \sigma + \left(p + \frac{2}{n}\right)\right]\tilde{g} = 0. \tag{1.3}$$

Furthermore, an n-dimensional Riemannian manifold $\mathcal{M}^n$ is known as conformal η-Einstein soliton [2] if

$$\mathfrak{L}_{\tilde{\xi}}\tilde{g} + 2\overset{\star}{\mathcal{S}} + \left[2\lambda_1 - \sigma + \left(p + \frac{2}{n}\right)\right]\tilde{g} + 2\mu_1\tilde{\eta}\otimes\tilde{\eta} = 0. \tag{1.4}$$

Einstein solitons are considered by many authors in different contents [6, 9, 18].

In this manuscript, we study conformal η-Einstein solitons on LP-Kenmotsu manifold. We arrange our work in the following manner. In Section 2, we give the definition of LP-Kenmotsu manifolds. Section 3 deals with the study of conformal η-Einstein solitons on LP-Kenmotsu 3-manifold. The properties of second order parallel symmetric tensors are studied in Section 4. In Section 5, we have studied the nature of conformal η-Einstein solitons on LP-Kenmotsu 3-manifold whose vector field is torse-forming. In Section 6, we have contrived conformal η-Einstein solitons in LP-Kenmotsu 3-manifold in terms of Codazzi type and cyclic parallel Ricci tensor and characterized the nature of manifold. Finally, we construct some non-trival example to prove the existence of conformal η-Einstein solitons on LP-Kenmotsu manifolds in Section 7.

2. Lorentzian Para-Kenmotsu manifolds

Let $\mathcal{M}^n$ be an n-dimensional Lorentzian almost paracontact metric manifold. If it is equipped with structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$, where $\tilde{\xi}$ is a unit vector field , $\tilde{\varphi}$ is a $(1,1)$-tensor field , $\tilde{\eta}$ is a 1-form on $\mathcal{M}^n$ and $\tilde{g}$ is a Lorentzian metric, satisfying

$$\tilde{\varphi}^2 = I + \tilde{\eta}\otimes\tilde{\xi}, \quad \tilde{\eta}\circ\tilde{\varphi} = 0, \quad \tilde{\eta}(\tilde{\xi}) = -1, \quad \tilde{\varphi}(\tilde{\xi}) = 0, \tag{2.1}$$

$$\tilde{g}(\tilde{\varphi}\mathcal{G}_1, \tilde{\varphi}\mathcal{G}_2) = \tilde{g}(\mathcal{G}_1, \mathcal{G}_2) + \tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2), \tag{2.2}$$

$$\tilde{\varphi}(\mathcal{G}_1, \mathcal{G}_2) = \tilde{\varphi}(\mathcal{G}_2, \mathcal{G}_1) = \tilde{g}(\mathcal{G}_1, \tilde{\varphi}\mathcal{G}_2), \tag{2.3}$$

for all $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{X}\mathcal{M}$. A Lorentzian almost paracontact manifold $\mathcal{M}^n$ is called Lorentzian para-Kenmotsu manifold [14] if

$$(\overset{*}{\nabla}_{\mathcal{G}_1}\tilde{\varphi})\mathcal{G}_2 = -\tilde{g}(\tilde{\varphi}\mathcal{G}_1, \mathcal{G}_2)\tilde{\xi} - \tilde{\eta}(\mathcal{G}_2)\tilde{\varphi}\mathcal{G}_1, \tag{2.4}$$

$$\overset{*}{\nabla}_{\mathcal{G}_1}\tilde{\xi} = -\mathcal{G}_1 - \tilde{\eta}(\mathcal{G}_1)\tilde{\xi}, \tag{2.5}$$

$$(\overset{*}{\nabla}_{\mathcal{G}_1}\tilde{\eta})\mathcal{G}_2 = -\tilde{g}(\mathcal{G}_1, \mathcal{G}_2) - \tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2). \tag{2.6}$$

In LP-Kenmotsu manifold $\mathcal{M}^n$, the following relations holds:

$$\overset{\star}{\mathcal{R}}(\mathcal{G}_1, \mathcal{G}_2)\tilde{\xi} = \tilde{\eta}(\mathcal{G}_2)\mathcal{G}_1 - \tilde{\eta}(\mathcal{G}_1)\mathcal{G}_2, \tag{2.7}$$

$$\tilde{\eta}(\overset{\star}{\mathcal{R}}(\mathcal{G}_1, \mathcal{G}_2)\mathcal{G}_3) = \tilde{g}(\mathcal{G}_2, \mathcal{G}_3)\tilde{\eta}(\mathcal{G}_1) - \tilde{g}(\mathcal{G}_1, \mathcal{G}_3)\tilde{\eta}(\mathcal{G}_2), \tag{2.8}$$

$$\overset{\star}{\mathcal{R}}(\tilde{\xi}, \mathcal{G}_1)\mathcal{G}_2 = \tilde{g}(\mathcal{G}_1, \mathcal{G}_2)\tilde{\xi} - \tilde{\eta}(\mathcal{G}_2)\mathcal{G}_1, \tag{2.9}$$

$$\overset{\star}{\mathcal{R}}(\tilde{\xi}, \mathcal{G}_1)\tilde{\xi} = \mathcal{G}_1 + \tilde{\eta}(\mathcal{G}_1)\tilde{\xi}, \tag{2.10}$$

$$\overset{\star}{\mathcal{S}}(\mathcal{G}_1, \tilde{\xi}) = 2\tilde{\eta}(\mathcal{G}_1), \tag{2.11}$$

$$\overset{\star}{\mathcal{Q}}\tilde{\xi} = 2\tilde{\xi}, \tag{2.12}$$

$$\overset{\star}{\mathcal{S}}(\tilde{\varphi}\mathcal{G}_1, \tilde{\varphi}\mathcal{G}_2) = \overset{\star}{\mathcal{S}}(\mathcal{G}_1, \mathcal{G}_2) + 2\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2), \tag{2.13}$$

for all $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \in \mathfrak{X}\mathcal{M}$; where $\overset{\star}{\mathcal{R}}$ is the curvature tensor, $\overset{\star}{\mathcal{S}}$ is the Ricci tensor and $\overset{\star}{\mathcal{Q}}$ is the Ricci operator.

Definition 2.1. *A 3-dimensional LP-Kenmotsu manifold $\mathcal{M}^n$ is said to be an η-Einstein manifold if its Ricci tensor $\overset{\star}{\mathcal{S}}$ is of the form*

$$\overset{\star}{\mathcal{S}}(\mathcal{G}_1, \mathcal{G}_2) = a\tilde{g}(\mathcal{G}_1, \mathcal{G}_2) + b\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2) \tag{2.14}$$

for all $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{X}\mathcal{M}$ and smooth functions a, b on the manifold $\mathcal{M}^n$.

3. Conformal η-Einstein Soliton on 3-dimensional LP-Kenmotsu manifolds

Let us consider a LP-Kenmotsu 3-manifold $\mathcal{M}^n$ equipped with conformal η-Einstein soliton $(\tilde{g}, \tilde{\xi}, \lambda_1, \mu_1)$, then (1.4) can be written as

$$\begin{aligned}(\mathfrak{L}_{\tilde{\xi}}\tilde{g})(\mathcal{G}_1, \mathcal{G}_2) + 2\overset{\star}{\mathcal{S}}(\mathcal{G}_1, \mathcal{G}_2) + \left[2\lambda_1 - \sigma + \left(p + \frac{2}{n}\right)\right]\tilde{g}(\mathcal{G}_1, \mathcal{G}_2) \\ + 2\mu_1\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2) = 0,\end{aligned} \tag{3.1}$$

for all $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{X}\mathcal{M}$. We know that

$$(\mathfrak{L}_{\tilde{\xi}}\tilde{g})(\mathcal{G}_1, \mathcal{G}_2) = \tilde{g}(\overset{*}{\nabla}_{\mathcal{G}_1}\tilde{\xi}, \mathcal{G}_2) + \tilde{g}(\mathcal{G}_1, \overset{*}{\nabla}_{\mathcal{G}_2}\tilde{\xi}). \tag{3.2}$$

Using equation (2.5), we get

$$(\mathfrak{L}_{\tilde{\xi}}\tilde{g})(\mathcal{G}_1, \mathcal{G}_2) = -2[\tilde{g}(\mathcal{G}_1, \mathcal{G}_2) + \tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2)]. \tag{3.3}$$

In view of (3.1) and (3.3), we get

$$\overset{\star}{\mathcal{S}}(\mathcal{G}_1, \mathcal{G}_2) = \left[\frac{\sigma}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda_1 + 1\right] \tilde{g}(\mathcal{G}_1, \mathcal{G}_2) - (\mu_1 - 1)\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2). \tag{3.4}$$

This shows that the manifold $\mathcal{M}^n$ is an η-Einstein manifold.
Replacing $\mathcal{G}_1$ by $\tilde{\xi}$, we find that

$$\overset{\star}{\mathcal{S}}(\mathcal{G}_1, \tilde{\xi}) = \left[\frac{\sigma}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda_1 + \mu_1\right] \tilde{\eta}(\mathcal{G}_1). \tag{3.5}$$

Using identity (2.11) in (3.5), we get

$$\sigma = \left(p + \frac{2}{n}\right) + 4 + 2\lambda_1 - 2\mu_1. \tag{3.6}$$

Contracting Eq. (3.4) along $\mathcal{G}_1$ and $\mathcal{G}_2$, we get

$$\sigma = 3\left(p + \frac{2}{n}\right) + 6\lambda_1 - 4 - 2\mu_1. \tag{3.7}$$

Now combining equations (3.6) and (3.7), we have

$$\lambda_1 = 2 - \left(\frac{p}{2} + \frac{1}{n}\right). \tag{3.8}$$

Thus the above discussion leads to the following theorem:

Theorem 3.1. *If a LP-Kenmotsu* 3*-manifolds* $\mathcal{M}^n$ *bearing a conformal* η*-Einstein soliton* $(\tilde{g}, \tilde{\xi}, \lambda_1, \mu_1)$ *then the manifold becomes an* η*-Einstein manifold and scalar curvature* $\sigma = 3\left(p + \frac{2}{n}\right) + 6\lambda_1 - 4 - 2\mu_1$*. Furthermore, the soliton is shrinking, steady or expanding according as;* $\left(\frac{p}{2} + \frac{1}{n}\right) < 2$*,* $\left(\frac{p}{2} + \frac{1}{n}\right) = 2$ *or* $\left(\frac{p}{2} + \frac{1}{n}\right) > 2$*.*
Now, we suppose a LP-Kenmotsu 3-manifold $\mathcal{M}^n$ that admits a conformal η-Einstein soliton $(\tilde{g}, \mathcal{V}_*, \lambda_1, \mu_1)$ such that $\mathcal{V}_*$ is parallel to $\tilde{\xi}$ i.e. $\mathcal{V}_* = b\tilde{\xi}$, where b is a function on $\mathcal{M}^n$, then the equation (1.4) yields

$$\begin{aligned} &b\tilde{g}(\overset{*}{\nabla}_{\mathcal{G}_1}\tilde{\xi}, \mathcal{G}_2) + (\mathcal{G}_1 b)\tilde{\eta}(\mathcal{G}_2) + b\tilde{g}(\mathcal{G}_1, \overset{*}{\nabla}_{\mathcal{G}_2}\tilde{\xi}) + (\mathcal{G}_2 b)\tilde{\eta}(\mathcal{G}_1) + 2\overset{\star}{\mathcal{S}}(\mathcal{G}_1, \mathcal{G}_2) \\ &+ \left[2\lambda_1 - \sigma + \left(p + \frac{2}{n}\right)\right] \tilde{g}(\mathcal{G}_1, \mathcal{G}_2) + 2\mu_1\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2) = 0. \end{aligned} \tag{3.9}$$

Using equation (2.5) in the above equation (3.9) to get

$$\begin{aligned}\left[2\lambda_1 - 2b - \sigma + \left(p + \frac{2}{n}\right)\right]\tilde{g}(\mathcal{G}_1, \mathcal{G}_2) + (\mathcal{G}_1 b)\tilde{\eta}(\mathcal{G}_2)\\ + (\mathcal{G}_2 b)\tilde{\eta}(\mathcal{G}_1) + 2\overset{\star}{\mathcal{S}}(\mathcal{G}_1, \mathcal{G}_2) + 2(\mu_1 - b)\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2) = 0.\end{aligned} \tag{3.10}$$

Now, taking $\mathcal{G}_2 = \tilde{\xi}$ into the identity (3.10) we have

$$\left[2\lambda_1 - \sigma + \left(p + \frac{2}{n}\right) - 2\mu_1\right]\tilde{\eta}(\mathcal{G}_1) + (\tilde{\xi} b)\tilde{\eta}(\mathcal{G}_1) - (\mathcal{G}_1 b) + 2\overset{\star}{\mathcal{S}}(\mathcal{G}_1, \tilde{\xi}) = 0. \tag{3.11}$$

Again taking $\mathcal{G}_1 = \tilde{\xi}$ in (3.11) and using equation (2.11) we have

$$\tilde{\xi} b = \frac{\sigma}{2} - \lambda_1 - \left(\frac{p}{2} + \frac{1}{n}\right) + \mu_1 + 2. \tag{3.12}$$

Using (3.12) in equation (3.11), we have

$$db = \left[\frac{\sigma}{2} - \lambda_1 - \left(\frac{p}{2} + \frac{1}{n}\right) + \mu_1 + 2\right]\tilde{\eta}. \tag{3.13}$$

Applying exterior derivative in equation (3.13), we obtain

$$\left[\frac{\sigma}{2} - \lambda_1 - \left(\frac{p}{2} + \frac{1}{n}\right) + \mu_1 + 2\right] d\tilde{\eta} = 0. \tag{3.14}$$

Since $d\tilde{\eta} \neq 0$, we have

$$\sigma = 2\lambda_1 + \left(p + \frac{2}{n}\right) - 2\mu_1 - 4. \tag{3.15}$$

In view of (3.15) and (3.13), gives $db = 0$ i.e. the function b is constant, then (3.10) reduces to

$$\begin{aligned}\overset{\star}{\mathcal{S}}(\mathcal{G}_1, \mathcal{G}_2) = \left[\frac{\sigma}{2} - \lambda_1 + b - \left(\frac{p}{2} + \frac{1}{n}\right)\right]\tilde{g}(\mathcal{G}_1, \mathcal{G}_2)\\ + (b - \mu_1)\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2).\end{aligned} \tag{3.16}$$

Thus, we leads to the following theorem:

Theorem 3.2. *If $\mathcal{M}^n$ be a LP-Kenmotsu manifolds admitting a conformal η-Einstein soliton $(\tilde{g}, \mathcal{V}_*, \lambda_1, \mu_1)$ such that $\mathcal{V}_*$ is pointwise collinear with $\tilde{\xi}$, then $\mathcal{V}_*$ is constant multiple of $\tilde{\xi}$ and the $\mathcal{M}^n$ becomes an η-Einstein manifold of constant scalar curvature $\sigma = 2\lambda_1 + \left(p + \frac{2}{n}\right) - 2\mu_1 - 4$.*

4. Second order parallel symmetric tensors and conformal η-Einstein solitons on 3-dimensional LP-Kenmotsu manifolds

Definition 4.1. [3] *A symmetric tensor h of second order is said to be a parallel tensor if ∇h=0, where ∇ is a covariant differentiation operator with respect to the metric tensor $\tilde{g}$.*

Let h is a symmetric tensor field of type $(0,2)$ which we suppose to be parallel with respect to ∇, i.e. ∇h=0. Taking the Ricci commutation identity [20], we have

$$\nabla^2 h(\mathcal{G}_1,\mathcal{G}_2;\mathcal{G}_3,\mathcal{G}_4) - \nabla^2 h(\mathcal{G}_1,\mathcal{G}_2;\mathcal{G}_4,\mathcal{G}_3) = 0, \tag{4.1}$$

for all $\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_3,\mathcal{G}_4 \in \mathfrak{X}(\mathcal{M})$. From the above equation we get

$$h(\overset{\star}{\mathcal{R}}(\mathcal{G}_1,\mathcal{G}_2)\mathcal{G}_3,\mathcal{G}_4) + h(\mathcal{G}_3,\overset{\star}{\mathcal{R}}(\mathcal{G}_1,\mathcal{G}_2)\mathcal{G}_4) = 0. \tag{4.2}$$

Putting $\mathcal{G}_1 = \mathcal{G}_3 = \mathcal{G}_4 = \tilde{\xi}$ in (4.2) and using (2.10), we have

$$h(\mathcal{G}_2,\tilde{\xi}) + \tilde{\eta}(\mathcal{G}_2)h(\tilde{\xi},\tilde{\xi}) = 0. \tag{4.3}$$

Differentiating (4.3) covariantly along $\mathcal{G}_1$, we have

$$\nabla_{\mathcal{G}_1}(h(\mathcal{G}_2,\tilde{\xi})) + \nabla_{\mathcal{G}_1}(\tilde{\eta}(\mathcal{G}_2)h(\tilde{\xi},\tilde{\xi})) = 0. \tag{4.4}$$

Expanding the equation (4.4) and using (2.1), (2.2) and (2.5), we get

$$h(\mathcal{G}_1,\mathcal{G}_2) = -\tilde{g}(\mathcal{G}_1,\mathcal{G}_2)h(\tilde{\xi},\tilde{\xi}), \tag{4.5}$$

for all $\mathcal{G}_1,\mathcal{G}_2 \in \mathfrak{X}\mathcal{M}$.
Let us consider,

$$h = \mathfrak{L}_{\tilde{\xi}}\tilde{g} + 2\overset{\star}{\mathcal{S}} + 2\mu_1\tilde{\eta}\otimes\tilde{\eta}. \tag{4.6}$$

From (3.3) and (3.4), we get

$$h(\tilde{\xi},\tilde{\xi}) = 2\lambda_1 + \left(p + \frac{2}{n}\right) - \sigma. \tag{4.7}$$

Using (4.6) and (4.7), equation (4.5) reduces to

$$\begin{aligned}(\mathfrak{L}_{\tilde{\xi}}\tilde{g})(\mathcal{G}_1,\mathcal{G}_2) + 2\overset{\star}{\mathcal{S}}(\mathcal{G}_1,\mathcal{G}_2) + \left[2\lambda_1 + \left(p + \frac{2}{n}\right) - \sigma\right]\tilde{g}(\mathcal{G}_1,\mathcal{G}_2)\\ + 2\mu_1\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2) = 0.\end{aligned} \tag{4.8}$$

which is the Conformal η-Einstein soliton. Hence we arrive the following theorem:

Theorem 4.2. *If in a 3-dimensional LP-Kenmotsu manifolds admitting a symmetric tensor field* $h = \mathfrak{L}_{\tilde{\xi}}\tilde{g} + 2\overset{\star}{\mathcal{S}} + 2\mu_1\tilde{\eta}\otimes\tilde{\eta}$ *is parallel endowed with the Levi-Civita connection associated to* $\tilde{g}$*, then* $(\tilde{g}, \tilde{\xi}, \lambda_1)$ *yields a conformal* η*-Einstein soliton.*

5. Conformal η-Einstein soliton on 3-dimensional LP-Kenmotsu manifolds endowed with Torse-Forming vector field

Definition 5.1. *A vector field* $\mathcal{V}_*$ *on* 3*-dimensional LP-Kenmotsu manifolds is a torse-forming vector fields* [27] *if*

$$\nabla_{\mathcal{V}_*} = f_*\mathcal{G}_1 + \gamma(\mathcal{G}_1)\mathcal{V}_*. \tag{5.1}$$

$\forall\ \mathcal{G}_1 \in \mathfrak{X}\mathcal{M}$*, where* f_* *is a smooth function and* γ *is a* 1*-form.*

Now let us consider $(\tilde{g}, \tilde{\xi}, \lambda_1, \mu_1)$ be a conformal η-Einstein soliton on a LP-Kenmotsu 3-manifolds and suppose that the Reeb vector field $\tilde{\xi}$ of the manifold is a torse-forming vector field. Then $\tilde{\xi}$ being a torse-forming vector field, by definition 5.1 we have

$$\nabla_{\mathcal{G}_1}\tilde{\xi} = f_*\mathcal{G}_1 + \gamma(\mathcal{G}_1)\tilde{\xi}. \tag{5.2}$$

Taking inner product with $\tilde{\xi}$ in (2.5), we find

$$\tilde{g}(\nabla_{\mathcal{G}_1}\tilde{\xi}, \tilde{\xi}) = 0. \tag{5.3}$$

Taking inner product with $\tilde{\xi}$ in (5.2), we can write

$$\tilde{g}(\nabla_{\mathcal{G}_1}\tilde{\xi}, \tilde{\xi}) = f_*\tilde{\eta}(\mathcal{G}_1) - \gamma(\mathcal{G}_1). \tag{5.4}$$

Combining equations (5.3) and (5.4) to get $\gamma = f_*\tilde{\eta}$.

Putting the value of γ in (5.2), we have

$$\nabla_{\mathcal{G}_1}\tilde{\xi} = f_*(\mathcal{G}_1 + \tilde{\eta}(\mathcal{G}_1)\tilde{\xi}. \tag{5.5}$$

Now, using (3.2) and (5.5), we have

$$(\mathfrak{L}_{\tilde{\xi}}\tilde{g})(\mathcal{G}_1, \mathcal{G}_2) = 2f_*[\tilde{g}(\mathcal{G}_1, \mathcal{G}_2) + \tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2)]. \tag{5.6}$$

In view of (5.6), (1.2) reduces to

$$\overset{\star}{\mathcal{S}}(\mathcal{G}_1, \mathcal{G}_2) = \left[\frac{\sigma}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda_1 - f_*\right]\tilde{g}(\mathcal{G}_1, \mathcal{G}_2) - (\mu_1 + f_*)\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2). \tag{5.7}$$

Thus the manifold is an η-Einstein manifold.
On taking $\mathcal{G}_2 = \tilde{\xi}$ in (5.7), using (2.1) and (2.11), we find

$$\sigma = 2\lambda_1 + \left(p + \frac{2}{n}\right) - 2\mu_1 + 4. \tag{5.8}$$

Contracting (5.7), we obtain

$$\sigma = 6\lambda_1 + 6\left(\frac{p}{2} + \frac{1}{n}\right) - 2\mu_1 + 4f_*. \tag{5.9}$$

From (5.8) and (5.9), we get

$$\lambda_1 = 1 - f_* - \left(\frac{p}{2} + \frac{1}{n}\right). \tag{5.10}$$

Thus we have the following theorem:

Theorem 5.2. *Let $(\tilde{g}, \tilde{\xi}, \lambda_1, \mu_1)$ be a conformal η-Einstein soliton on LP-Kenmotsu 3-manifolds $\mathcal{M}^n$ with torse-forming vector field $\tilde{\xi}$. Then the manifold is an η-Einstein manifold and the soliton is shrinking, steady or expanding according as $f_* > 1 - \left(\frac{p}{2} - \frac{1}{n}\right)$, $f_* = 1 - \left(\frac{p}{2} - \frac{1}{n}\right)$ or $f_* < 1 - \left(\frac{p}{2} - \frac{1}{n}\right)$.*

6. Conformal η-Einstein soliton on 3-dimensional LP-Kenmotsu manifolds with cyclic parallel Ricci tensor

Definition 6.1. [4] *A LP-Kenmotsu manifolds is said to be Codazzi type Ricci tensor if its Ricci tensor $\overset{\star}{\mathcal{S}}$ is non-zero and satisfying the following relation*

$$(\overset{*}{\nabla}_{\mathcal{G}_1}\overset{\star}{\mathcal{S}})(\mathcal{G}_2, \mathcal{G}_3) = (\overset{*}{\nabla}_{\mathcal{G}_2}\overset{\star}{\mathcal{S}})(\mathcal{G}_1, \mathcal{G}_3), \quad \forall \quad \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \in \mathfrak{X}\mathcal{M}. \tag{6.1}$$

Taking covariant derivative of (3.4) and using (2.6), we yields

$$\begin{aligned}(\overset{*}{\nabla}_{\mathcal{G}_1}\overset{\star}{\mathcal{S}})(\mathcal{G}_2, \mathcal{G}_3) &= (\mu_1 - 1)[\tilde{g}(\mathcal{G}_1, \mathcal{G}_2)\tilde{\eta}(\mathcal{G}_3) \\ &+ \tilde{g}(\mathcal{G}_1, \mathcal{G}_3))\tilde{\eta}(\mathcal{G}_2) + 2\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2)\tilde{\eta}(\mathcal{G}_3)].\end{aligned} \tag{6.2}$$

Also, we have

$$\begin{aligned}(\overset{*}{\nabla}_{\mathcal{G}_2}\overset{\star}{\mathcal{S}})(\mathcal{G}_1, \mathcal{G}_3) &= (\mu_1 - 1)[\tilde{g}(\mathcal{G}_2, \mathcal{G}_1)\tilde{\eta}(\mathcal{G}_3) \\ &+ \tilde{g}(\mathcal{G}_2, \mathcal{G}_3)\tilde{\eta}(\mathcal{G}_1) + 2\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2)\tilde{\eta}(\mathcal{G}_3)].\end{aligned} \tag{6.3}$$

If the Ricci tensor $\overset{\star}{\mathcal{S}}$ of the Codazzi type, then from (6.1), (6.2) and (6.3), we have

$$(\mu_1 - 1)[\tilde{g}(\mathcal{G}_1, \mathcal{G}_3)\tilde{\eta}(\mathcal{G}_2) - \tilde{g}(\mathcal{G}_2, \mathcal{G}_3)\tilde{\eta}(\mathcal{G}_1)] = 0. \tag{6.4}$$

It follows that $\mu_1 = 1$ [since $\tilde{g}(\mathcal{G}_1, \mathcal{G}_3)\tilde{\eta}(\mathcal{G}_2) - \tilde{g}(\mathcal{G}_2, \mathcal{G}_3)\tilde{\eta}(\mathcal{G}_1) \neq 0$], then from (3.4), we find

$$\overset{\star}{\mathcal{S}}(\mathcal{G}_1, \mathcal{G}_2) = \left[\frac{\sigma}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda_1 + 1\right]\tilde{g}(\mathcal{G}_1, \mathcal{G}_2). \tag{6.5}$$

On contracting (6.5), we get

$$\sigma = 6\lambda_1 + 3\left(p + \frac{2}{n}\right) - 6. \tag{6.6}$$

Hence, in view of the identity (6.5) and (6.6), we have the following theorem:

Theorem 6.2. *In an 3-dimensional LP-Kenmotsu manifolds bearing a conformal η-Einstein soliton $(\tilde{g}, \tilde{\xi}, \lambda_1, \mu_1)$, if the Ricci tensor of the manifold is of Codazzi type, then $\mathcal{M}^n$ becomes an Einstein manifold of constant scalar curvature $\sigma = 6\lambda_1 + 3\left(p + \frac{2}{n}\right) - 6$, provided $\mu_1 = 1$.*

Definition 6.3. [4] *A LP-Kenmotsu 3-manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor $\overset{\star}{\mathcal{S}}$ is non-zero and satisfies the following condition*

$$(\overset{*}{\nabla}_{\mathcal{G}_1}\overset{\star}{\mathcal{S}})(\mathcal{G}_2, \mathcal{G}_3) + (\overset{*}{\nabla}_{\mathcal{G}_2}\overset{\star}{\mathcal{S}})(\mathcal{G}_1, \mathcal{G}_3) + (\overset{*}{\nabla}_{\mathcal{G}_3}\overset{\star}{\mathcal{S}})(\mathcal{G}_1, \mathcal{G}_2) = 0, \quad \forall \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3 \in \mathfrak{X}\mathcal{M}. \tag{6.7}$$

Let an 3-dimensional LP-Kenmotsu manifold admitting a conformal η-Einstein soliton $(\tilde{g}, \tilde{\xi}, \lambda_1, \mu_1)$ and the manifold has cyclic parallel Ricci tensor, so (6.5) holds. Similarly from (6.2) and (6.3), we have

$$\begin{aligned}(\overset{*}{\nabla}_{\mathcal{G}_3}\overset{\star}{\mathcal{S}})(\mathcal{G}_1, \mathcal{G}_2) = (\mu_1 - 1)[&\tilde{g}(\mathcal{G}_3, \mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2) \\ &+ \tilde{g}(\mathcal{G}_3, \mathcal{G}_2)\tilde{\eta}(\mathcal{G}_1) + 2\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2)\tilde{\eta}(\mathcal{G}_3)].\end{aligned} \tag{6.8}$$

Using equations (6.2), (6.3) and (6.8) in (6.7), we conclude

$$\begin{aligned}(\mu_1 - 1)[&\tilde{g}(\mathcal{G}_1, \mathcal{G}_2)\tilde{\eta}(\mathcal{G}_3) + \tilde{g}(\mathcal{G}_2, \mathcal{G}_3)\tilde{\eta}(\mathcal{G}_1) \\ &+ \tilde{g}(\mathcal{G}_3, \mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2) + 3\tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2)\tilde{\eta}(\mathcal{G}_3)] = 0.\end{aligned} \tag{6.9}$$

Replacing $\mathcal{G}_3$ by $\tilde{\xi}$ in Eq. (6.9), we obtain

$$(\mu_1 - 1)\tilde{g}(\tilde{\varphi}\mathcal{G}_1, \tilde{\varphi}\mathcal{G}_2) = 0. \tag{6.10}$$

Hence $\mu_1 = 1$. (since $\tilde{g}(\tilde{\varphi}\mathcal{G}_1, \tilde{\varphi}\mathcal{G}_2) \neq 0$).
Putting $\mu_1 = 1$ in (3.4), we get

$$\overset{\star}{\mathcal{S}}(\mathcal{G}_1, \mathcal{G}_2) = \left[\frac{\sigma}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda_1 + 1\right] \tilde{g}(\mathcal{G}_1, \mathcal{G}_2). \tag{6.11}$$

Theorem 6.4. *In an 3-dimensional LP-Kenmotsu manifolds bearing a conformal η-Einstein soliton $(\tilde{g}, \tilde{\xi}, \lambda_1, \mu_1)$, if the manifold has a cyclic parallel Ricci tensor, then $\mathcal{M}^n$ becomes an Einstein manifold of constant scalar curvature $\sigma = 6\lambda_1 + 3\left(p + \frac{2}{n}\right) - 6$, if $\mu_1 = 1$.*

7. Example

Let us take a 3-dimensional manifold $\mathcal{M} = \{(r, s, t) \in \Re^3 : (r, s, t) \neq 0\}$, where (r, s, t) are the cartesian coordinates in $\Re^3$. Let $(\vartheta_1, \vartheta_2, \vartheta_3)$ be the orthogonal system of vector fields at each point of $\mathcal{M}$, defined as

$$\vartheta_1 = \vartheta^t \frac{\partial}{\partial s}, \;\; \vartheta_2 = \vartheta^t \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right), \;\; \vartheta_3 = \frac{\partial}{\partial t}$$

and

$$[\vartheta_1, \vartheta_2] = 0, \;\; [\vartheta_1, \vartheta_3] = -\vartheta_1, \;\; [\vartheta_2, \vartheta_3] = -\vartheta_2.$$

Let $\tilde{g}$ be the metric defined as follows

$$\tilde{g}(\vartheta_1, \vartheta_1) = \tilde{g}(\vartheta_2, \vartheta_2) = 1, \tilde{g}(\vartheta_3, \vartheta_3) = -1,$$
$$\tilde{g}(\vartheta_1, \vartheta_2) = \tilde{g}(\vartheta_1, \vartheta_3 = \tilde{g}(\vartheta_2, \vartheta_3) = 0.$$

Let $\tilde{\varphi}$ be the $(1, 1)$-tensor field defined by

$$\tilde{\varphi}\,\vartheta_1 = -\vartheta_1, \qquad \tilde{\varphi}\,\vartheta_2 = -\vartheta_2, \qquad \tilde{\varphi}\,\vartheta_3 = 0.$$

Making use of the linearity of $\tilde{\varphi}$ and $\tilde{g}$ we have

$$\tilde{\eta}(\vartheta_3) = -1,$$

$$\tilde{\varphi}^2(\mathcal{G}_1) = -\mathcal{G}_1 + \tilde{\eta}(\mathcal{G}_1)\vartheta_3,$$
$$\tilde{\eta}(\mathcal{G}_1) = \tilde{g}(\mathcal{G}_1, \tilde{\xi}),$$
$$\tilde{g}(\tilde{\varphi}\mathcal{G}_1, \tilde{\varphi}\mathcal{G}_2) = \tilde{g}(\mathcal{G}_1, \mathcal{G}_2) - \tilde{\eta}(\mathcal{G}_1)\tilde{\eta}(\mathcal{G}_2),$$

for any $\mathcal{G}_1, \mathcal{G}_2 \in \mathfrak{X}\mathcal{M}$. Thus for $\vartheta_3 = \tilde{\xi}$ the structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ leads to a contact metric structure in $\Re^3$. We recall the Koszul's formula

$$2\tilde{g}(\nabla_{\mathcal{G}_1}\mathcal{G}_2, \mathcal{G}_3) = \mathcal{G}_1(\tilde{g}(\mathcal{G}_2, \mathcal{G}_3)) + \mathcal{G}_2(\tilde{g}(\mathcal{G}_3, \mathcal{G}_1)) - \mathcal{G}_3(\tilde{g}(\mathcal{G}_1, \mathcal{G}_2))$$
$$-\tilde{g}(\mathcal{G}_1, [\mathcal{G}_2, \mathcal{G}_3]) - \tilde{g}(\mathcal{G}_2, [\mathcal{G}_1, \mathcal{G}_3]) + \tilde{g}(\mathcal{G}_3, [\mathcal{G}_1, \mathcal{G}_2]).$$

Making use Koszul's formula we follows:

$$\begin{cases} \nabla_{\vartheta_1}\vartheta_1 = -\vartheta_3, & \nabla_{\vartheta_1}\vartheta_2 = 0, & \nabla_{\vartheta_1}\vartheta_3 = -\vartheta_1, \\ \nabla_{\vartheta_2}\vartheta_1 = 0, & \nabla_{\vartheta_2}\vartheta_2 = -\vartheta_3, & \nabla_{\vartheta_2}\vartheta_3 = -\vartheta_2, \\ \nabla_{\vartheta_3}\vartheta_1 = 0, & \nabla_{\vartheta_3}\vartheta_2 = 0, & \nabla_{\vartheta_3}\vartheta_3 = 0 \end{cases},$$

Thus from the above relations it follows that the manifold $\mathcal{M}^n$ is a LP-Kenmotsu 3-manifold.

The non-vanishing component of Riemannian curvature tensor as follows:

$$\begin{cases} \overset{\star}{\mathcal{R}}(\vartheta_1, \vartheta_2)\vartheta_3 = 0, & \overset{\star}{\mathcal{R}}(\vartheta_1, \vartheta_3)\vartheta_3 = -\vartheta_1, & \overset{\star}{\mathcal{R}}(\vartheta_3, \vartheta_2)\vartheta_2 = \vartheta_3, \\ \overset{\star}{\mathcal{R}}(\vartheta_3, \vartheta_1)\vartheta_1 = \vartheta_3, & \overset{\star}{\mathcal{R}}(\vartheta_2, \vartheta_1)\vartheta_1 = \vartheta_2, & \overset{\star}{\mathcal{R}}(\vartheta_2, \vartheta_3)\vartheta_3 = -\vartheta_2, \\ \overset{\star}{\mathcal{R}}(\vartheta_2, \vartheta_3)\vartheta_1 = 0, & \overset{\star}{\mathcal{R}}(\vartheta_1, \vartheta_2)\vartheta_2 = \vartheta_1, & \overset{\star}{\mathcal{R}}(\vartheta_3, \vartheta_1)\vartheta_2 = 0, \end{cases}$$

From the above expressions of the curvature tensor, we evaluate the value of the Ricci tensor as follows:

$$\overset{\star}{\mathcal{S}}(\vartheta_1, \vartheta_1) = \overset{\star}{\mathcal{S}}(\vartheta_2, \vartheta_2) = 2, \overset{\star}{\mathcal{S}}(\vartheta_3, \vartheta_3) = -2.$$

From equation (3.4), we have

$$\overset{\star}{\mathcal{S}}(\vartheta_3, \vartheta_3) = \left[\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda_1 + 1\right] - (\mu_1 - 1).$$

which implies that

$$\sigma = 2\lambda_1 + \left(\frac{p}{2} + \frac{1}{n}\right) - 2\mu_1 + 4.$$

Hence λ_1 and μ_1 satisfying equation (3.6).

Here, the scalar curvature of the manifold is calculated

$$\sigma = \sum_{i=1}^{3} \overset{\star}{\mathcal{S}}(e_i, e_i) = \overset{\star}{\mathcal{S}}(\vartheta_1, \vartheta_1) + \overset{\star}{\mathcal{S}}(\vartheta_2, \vartheta_2) - \overset{\star}{\mathcal{S}}(\vartheta_3, \vartheta_3) = 2.$$

Now, from (3.4), we get

$$\sum_{i=1}^{3} \overset{\star}{\mathcal{S}}(e_i, e_i) = \left[\frac{r}{2} - \left(\frac{p}{2} + \frac{1}{n}\right) - \lambda_1 + 1\right] \sum_{i=1}^{3} \tilde{g}(e_i, e_i) - (\mu_1 - 1) \sum_{i=1}^{3} \tilde{\eta}(e_i)\tilde{\eta}(e_i)$$

after some calculation, we get

$$\lambda_1 = 2 - p + \frac{\mu_1}{3}.$$

Now, from equation (3.8), we get $\mu_1 = -2$ and $\lambda_1 = \frac{4}{3} - p$. Thus the data $(\tilde{g}, \tilde{\xi}, \lambda_1, \mu_1)$ for $\mu_1 = -2$ and $\lambda_1 = -(p - \frac{4}{3})$ defines a conformal η-Einstein solitons on LP-Kenmotsu manifold $\mathcal{M}^n$.

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