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COMMON FIXED-POINTS OF RATIONAL CONTRACTIONS IN SUPERMETRIC SPACES

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Abstract: In this study, we prove a common fixed-point theorem for generalized rational-type contraction in supermetric space. Our findings expand the contractions of metric spaces to a supermetric space through Kannan's contraction, Reich's contraction, and Dass-Gupta's rational contraction. These theorems also extend to the supermetric context and generalize many interesting results from metric fixed-point theory. Additionally, we provide an example to elucidate our theorems.

Keywords and Phrases: Fixed point, iterative methods, contraction, rational contraction, super-metric space.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

A fixed point of a function is a point that doesn't move when the function is applied to it. In many branches of mathematics and its applications, including numerical analysis, optimization, and the study of dynamical systems, fixed points play a crucial role. They frequently depict equilibrium states of systems or solutions to equations. Finding a fixed point in an iterative process, for instance, can be comparable to finding a solution to the equation that is being iterated in the context of numerical equation-solving techniques.

A key finding in the theory of metric spaces is the Banach Contraction Principle [3], sometimes referred to as the contraction mapping theorem. It gives the circumstances in which there is a unique fixed point for a mapping from a metric space to itself. This idea is fundamental to many branches of mathematics and its applications, such as functional analysis, numerical techniques, analysis, and optimization. It offers a strong tool for proving convergence in iterative algorithms and ensures the existence and uniqueness of solutions to certain equations and problems. The literature then extensively generalized the Banach contraction principle (see [1-16]). It is widely used in applied and pure mathematics alike.

In 1968, Kannan [9] developed a modified version of this theory and removed the continuity requirement. The first important variation of Banach's remarkable finding on the metric fixed-point theory is Kannan's fixed-point theorem. There are various ways to generalize Kannan's theorem. Dass and Gupta [6] presented the Rational Contraction, which is a generalization of the Banach Contraction Mapping Principle. By using rational functions as the contraction condition rather than constants, it expands the concept of contraction maps to a more generic context. The traditional contraction mapping principle is made broader by the Dass-Gupta Rational Contraction condition, which permits the contraction factor to change based on the points being mapped. In certain applications, this enables a more flexible foundation. Similar to mappings satisfying the Banach Contraction Mapping Principle, the existence and uniqueness of fixed points for mappings satisfying the Dass-Gupta Rational Contraction condition can be determined by taking advantage of the rational function's properties as well as the underlying metric space's completeness. Super-metric space was introduced by Fulga and Karapinar [11]. In this framework, we were able to derive various fixed-point theorems, and we think this approach could help relieve the congestion and squeeze issues previously mentioned. Zamfirescu [16], obtained a very interesting fixed point theorem on complete metric spaces by combining the results of Banach [3], Kannan [9], and Chatterjea [4].

In super metric space, we establish some common fixed-point theorems related to rational contraction. Our results extend the metric space contractions via Kannan's contraction, Reich's contraction, and Dass-Gupta's rational contraction to a supermetric space. Furthermore, we present an example to illustrate our theorems.

2. Preliminaries

Determining the supermetric is the first step in this section.

Definition 2.1. (see [11]) *Consider $\mathcal{U}$ to be a non-empty set. A function $d : \mathcal{U} \times \mathcal{U} \to [0, +\infty)$ is considered a super metric if it fulfills the subsequent axioms:*

(s1). $d(\xi, \zeta) = 0 \Rightarrow \xi = \zeta$*;*

(s2). $d(\xi, \zeta) = d(\zeta, \xi), \forall \xi \in \zeta\mathcal{U}$*;*

(s3). There exists $\mathfrak{s} \geq 1$ such that for every $\zeta \in \mathcal{U}$, there exist distinct sequences $\{\xi_r\}, \{\zeta_r\} \subset \mathcal{U}$, with $d(\xi_r, \zeta_r) \to 0$ when $r \to \infty$, such that

$$\lim_{r\to\infty} \sup d(\zeta_r, \zeta) \leq \mathfrak{s} \lim_{r\to\infty} \sup d(\xi_r, \zeta).$$

The tripled $(\mathcal{U}, d, \mathfrak{s})$ is called a super metric space.

Definition 2.2. (see [11]) *A sequence $\{\xi_r\}$ on a super metric space $(\mathcal{U}, d, \mathfrak{s})$*

1) converges to $\xi \in \mathcal{U}$ if $\lim_{r\to\infty}(\xi_r, \xi) = 0$;

2) is a Cauchy sequence in $\mathcal{U}$ if $\lim_{r\to\infty} \sup\{d(\xi_r, \xi_j) : j > r\} = 0$.

Proposition 2.3. (see [11]) *The limit of a convergent sequence is unique on a super metric space.*

Definition 2.4. (see [11]) *A super-metric space $(\mathcal{U}, d, \mathfrak{s})$ is called complete if each Cauchy sequence is convergent in $\mathcal{U}$.*

Theorem 2.5. (see [11]) *Let $(\mathcal{U}, d, \mathfrak{s})$ be a complete super-metric space and let $\mathcal{J} : \mathcal{U} \to \mathcal{U}$ be a mapping. Suppose that $0 < \mathfrak{c} < 1$ such that*

$$d(\mathcal{J}\xi, \mathcal{J}\zeta) \leq \mathfrak{c}\mathfrak{d}(\xi, \zeta)$$

for all $(\xi, \zeta) \in \mathcal{U}$. Then, $\mathcal{J}$ has a unique fixed point in $\mathcal{U}$.

Theorem 2.6. (see [11]) *Let $(\mathcal{U}, \mathfrak{d}, \mathfrak{s})$ be a complete super metric space and $\mathcal{J} : \mathcal{U} \to \mathcal{U}$ be a mapping, such that there exists $\mathfrak{c} \in [0, 1)$ and that*

$$d(\mathcal{J}\xi, \mathcal{J}\zeta) \leq c \max\left\{ d(\xi, \zeta) \frac{d(\xi, \mathcal{J}\xi) d(\zeta, \mathcal{J}\zeta)}{1 + d(\xi, \zeta)} \right\}$$

Then, $\mathcal{J}$ has a unique fixed point.

2. Main results

Our primary finding is as follows.

Theorem 3.1. *Let $(\mathcal{U}, d, \mathfrak{s})$ be a complete super-metric space and let $\mathcal{L}$, $\mathcal{J}$ be self-mappings of $\mathcal{U}$. Assume that there exists a real number $\mathfrak{c} \in [0, 1)$ such that*

$$d(\mathcal{L}\xi, \mathcal{J}\zeta) \leq \mathfrak{c} \max\left\{d(\xi, \zeta), d(\xi, \mathcal{L}\xi), d(\zeta, \mathcal{J}\zeta), \frac{d(\xi, \mathcal{L}\xi)d(\zeta, \mathcal{J}\xi)}{1 + d(\xi, \zeta)}\right\} \tag{3.1}$$

for all $\xi, \zeta \in \mathcal{U}$. Then, $\mathcal{L}$ and $\mathcal{J}$ have a unique common fixed point in $\mathcal{U}$.

Proof. Let $\xi_0 \in \mathcal{U}$ and we define the class of iterative sequences $\{\xi_r\}$ such that $\xi_{r+1} = \mathcal{L}\xi_r, \xi_{r+2} = \mathcal{J}\xi_{r+1}$ for all $r \in \mathbb{N}$. Without loss of generality, we assume that $\xi_{r+2} \neq \mathcal{J}\xi_{r+1}$ for each nonnegative integer r. Indeed, if there exists a nonnegative integer r_0 such that $\xi_{r_0+2} = \mathcal{J}\xi_{r_0+1}$, then our proof of the theorem proceeds as follows. Thus, from (3.1), we have

$$\begin{aligned}
0 < d(\xi_{r+1}, \xi_{r+2}) &= d(\mathcal{L}\xi_r, \mathcal{J}\xi_{r+1}) \\
&\leq \mathfrak{c} \max\left\{d(\xi_r, \xi_{r+1}), d(\xi_r, \mathcal{L}\xi_r), d(\xi_{r+1}, \mathcal{J}\xi_{r+1}), \frac{d(\xi_r, \mathcal{L}\xi_r)d(\xi_{r+1}, \mathcal{J}\xi_{r+1})}{1 + d(\xi_r, \xi_{r+1})}\right\} \\
&\leq \mathfrak{c} \max\left\{d(\xi_r, \xi_{r+1}), d(\xi_r, \xi_{r+1}), d(\xi_{r+1}, \xi_{r+2}), \frac{d(\xi_r, \xi_{r+1})d(\xi_{r+1}, \xi_{r+2})}{1 + d(\xi_r, \xi_{r+1})}\right\} \\
&\leq \mathfrak{c} \max\left\{d(\xi_r, \xi_{r+1}), d(\xi_{r+1}, \xi_{r+2})\right\}.
\end{aligned}$$

If $\max\{d(\xi_r, \xi_{r+1}), d(\xi_{r+1}, \xi_{r+2})\} = d(\xi_{r+1}, \xi_{r+2})$, then we get a contradiction

$$d(\xi_{r+1}, \xi_{r+2}) \leq \mathfrak{c} d(\xi_{r+1}, \xi_{r+2}) < d(\xi_{r+1}, \xi_{r+2}).$$

It follows that $\max\{d(\xi_r, \xi_{r+1}), d(\xi_{r+1}, \xi_{r+2})\} = d(\xi_r, \xi_{r+1})$. Thus, we have

$$\begin{aligned}
0 < d(\xi_{r+1}, \xi_{r+2}) &\leq \mathfrak{c} d(\xi_r, \xi_{r+1}) \\
&\leq \mathfrak{c}^2 d(\xi_{r-1}, \xi_r) \\
&\leq ... \leq \mathfrak{c}^{\mathfrak{r}+1} d(\xi_0, \xi_1).
\end{aligned} \tag{3.2}$$

On the other hand, one writes

$$\begin{aligned}
0 < d(\xi_{r+1}, \xi_r) &= d(\mathcal{L}\xi_r, \mathcal{J}\xi_{r-1}) \\
&\leq \mathfrak{c} \max\left\{d(\xi_r, \xi_{r-1}), d(\xi_r, \mathcal{L}\xi_r), d(\xi_{r-1}, \mathcal{J}\xi_{r-1}), \frac{d(\xi_r, \mathcal{L}\xi_r)d(\xi_{r-1}, \mathcal{J}\xi_{r-1})}{1 + d(\xi_r, \xi_{r-1})}\right\} \\
&\leq \mathfrak{c} \max\left\{d(\xi_r, \xi_{r-1}), d(\xi_r, \xi_{r+1}), d(\xi_{r-1}, \xi_r), \frac{d(\xi_r, \xi_{r+1})d(\xi_{r-1}, \xi_r)}{1 + d(\xi_r, \xi_{r-1})}\right\} \\
&\leq \mathfrak{c} \max\left\{d(\xi_r, \xi_{r-1}), d(\xi_r, \xi_{r+1})\right\}.
\end{aligned}$$

If $\max\{d(\xi_r, \xi_{r-1}), d(\xi_r, \xi_{r+1})\} = d(\xi_r, \xi_{r+1})$, then we get a contradiction

$$d(\xi_{r+1}, \xi_r) \leq \mathfrak{c} d(\xi_r, \xi_{r+1}) < d(\xi_{r+1}, \xi_r).$$

It follows that $\max\{d(\xi_r, \xi_{r-1}), d(\xi_r, \xi_{r+1})\} = d(\xi_r, \xi_{r-1})$. Thus, we have

$$\begin{aligned} 0 < d(\xi_r, \xi_{r+1}) &\leq \mathfrak{c} d(\xi_r, \xi_{r-1}) \\ &\leq \mathfrak{c}^2 d(\xi_{r-1}, \xi_{r-2}) \\ &\leq \ldots \leq \mathfrak{c}^{\mathfrak{r}} d(\xi_0, \xi_1). \end{aligned} \tag{3.3}$$

By appealing to (3.2) and (3.3), we find that

$$0 < d(\xi_r, \xi_{r+1}) \leq c^r d(\xi_0, \xi_1) \tag{3.4}$$

Taking the limit r tends to infinity in inequality (3.4), we get

$$\lim_{r \to \infty} d(\xi_r, \xi_{r+1}) = 0. \tag{3.5}$$

In what follows, we want to show that the sequence $\{\xi_r\}$ is a Cauchy sequence. Now suppose that $r, j \in \mathbb{N}$ with $r > j$.Then from inequality (3.5) and using $(s3)$, we get

$$\lim_{r \to \infty} \sup d(\xi_r, \xi_{r+2}) \leq \mathfrak{s} \lim_{r \to \infty} \sup d(\xi_{r+1}, \xi_{r+2}) \leq \mathfrak{s} \lim_{r \to \infty} \sup\{c^{r+1} d(\xi_0, \xi_1)\} \tag{3.6}$$

Hence,$\lim_{r \to \infty} \sup d(\xi_r, \xi_{r+2}) = 0$. Similarly, we have

$$\lim_{r \to \infty} \sup d(\xi_r, \xi_{r+3}) \leq \mathfrak{s} \lim_{r \to \infty} \sup d(\xi_{r+2}, \xi_{r+3}) \leq \mathfrak{s} \lim_{r \to \infty} \sup\{c^{r+2} d(\xi_0, \xi_1)\} \tag{3.7}$$

Inductively, one can conclude that $\lim\limits_{r \to \infty} \sup d(\xi_r, \xi_j) : r > j = 0$. Thus, $\{\xi_r\}$ is a Cauchy sequence in a complete super-metric space $(\mathcal{U}, d, \mathfrak{s})$, the sequence $\{\xi_r\}$ converges to $\xi^* \in \mathcal{U}$ and then $\lim\limits_{r \to \infty} d(\xi_r, \xi^*) = 0$. Further, we show that ξ^* is the fixed point of $\mathcal{L}$ and $\mathcal{J}$ If not, $\xi^* \neq \mathcal{L}\xi^* \neq \mathcal{J}\xi^*$ and then $d(\xi^*, \mathcal{L}\xi^*) > 0$ and $d(\xi^*, \mathcal{J}\xi^*) > 0$. Note that

$$\begin{aligned} 0 &< d(\xi_{r+2}, \mathcal{L}\xi^*) = d(\mathcal{L}\xi^*, \xi_{r+2}) = d(\mathcal{L}\xi^*, \mathcal{J}\xi_{r+1}) \\ &\leq c \max\left\{ d(\xi^*, \xi_{r+1}), d(\xi^*, \mathcal{L}\xi^*), d(\xi_{r+1}, \mathcal{J}\xi_{r+1}), \frac{d(\xi^*, \mathcal{L}\xi^*) d(\xi_{r+1}, \mathcal{J}\xi_{r+1})}{1 + d(\xi^*, \xi_{r+1})} \right\} \\ &= c \max\left\{ d(\xi^*, \xi_{r+1}), d(\xi^*, \mathcal{L}\xi^*), d(\xi_{r+1}, \xi_{r+2}), \frac{d(\xi^*, \mathcal{L}\xi^*) d(\xi_{r+1}, \xi_{r+2})}{1 + d(\xi^*, \xi_{r+1})} \right\} \end{aligned}$$

Taking $r \to \infty$, we derive $\lim_{r\to\infty} \sup d(\xi_{r+2}, \mathcal{L}\xi^*) \le cd(\xi^*, \mathcal{L}\xi^*)$. Thus, we have,

$$0 < d(\xi^*, \mathcal{L}\xi^*) \le \lim_{r\to\infty} \sup d(\xi_{r+2}, \mathcal{L}\xi^*) \le cd(\xi^*, \mathcal{L}\xi^*) \tag{3.8}$$

and one can conclude that $d(\xi^*, \mathcal{L}\xi^*) = 0$, which implies that $\mathcal{L}\xi^* = \xi^*$. On the other hand,

$$\begin{aligned}0 < d(\xi_{r+2}, \mathcal{J}\xi^*) &= d(\mathcal{L}\xi_{r+1}, \mathcal{J}\xi^*)\\ &\le c\max\left\{d(\xi_{r+1}, \xi^*), d(\xi_{r+1}, \mathcal{L}\xi_{r+1}), d(\xi^*, \mathcal{J}\xi^*), \frac{d(\xi_{r+1}, \mathcal{L}\xi_{r+1})d(\xi^*, \mathcal{J}\xi^*)}{1 + d(\xi_{r+1}, \xi^*)}\right\}\\ &= c\max\left\{d(\xi_{r+1}, \xi^*), d(\xi_{r+1}, \xi_{r+2}), d(\xi^*, \mathcal{J}\xi^*), \frac{d(\xi_{r+1}, \xi_{r+2})d(\xi^*, \mathcal{J}\xi^*)}{1 + d(\xi_{r+1}, \xi^*)}\right\}\end{aligned}$$

Taking $r \to \infty$, we derive $\lim_{r\to\infty} \sup d(\xi_{r+2}, \mathcal{J}\xi^*) \le cd(\xi^*, \mathcal{J}\xi^*)$. Thus, we have,

$$0 < d(\xi^*, \mathcal{J}\xi^*) \le \lim_{r\to\infty} \sup d(\xi_{r+2}, \mathcal{J}\xi^*) \le cd(\xi^*, \mathcal{J}\xi^*). \tag{3.9}$$

and one can conclude that $d(\xi^*, \mathcal{L}\xi^*) = 0$, which implies that $\mathcal{J}\xi^* = \xi^*$. Hence, ξ^* is a common fixed point of $\mathcal{L}$ and $\mathcal{J}$. We shall now prove the uniqueness of ξ^*. Suppose there exists another point $\zeta^* \in \mathcal{U}$ such that $\mathcal{L}\zeta^* = \mathcal{J}\zeta^* = \zeta^*$. Then, by inequality (3.1), we have

$$\begin{aligned}d(\mathcal{L}\xi^*, \mathcal{J}\zeta^*) &\le c\max\left\{d(\xi^*, \zeta^*), d(\xi^*, \mathcal{L}\xi^*), d(\zeta^*, \mathcal{J}\zeta^*), \frac{d(\xi^*, \mathcal{L}\xi^*)d(\zeta^*, \mathcal{J}\zeta^*)}{1 + d(\xi^*, \zeta^*)}\right\}\\ &\le cd(\xi^*, \zeta^*) < d(\xi^*, \zeta^*)\end{aligned} \tag{3.10}$$

which is a contradiction.
If we take $\mathcal{L} = \mathcal{J}$ in condition (3.1), we obtain the following Corollary.

Corollary 3.2. *Let $(\mathcal{U}, d, \mathfrak{s})$ be a complete super-metric space and let $\mathcal{L}$ be a self-mapping of $\mathcal{U}$. Assume that there exists a real number $\mathfrak{c} \in [0, 1)$ such that*

$$d(\mathcal{L}\xi, \mathcal{L}\zeta) \le c\max\left\{k_1 d(\xi, \zeta), d(\xi, \mathcal{L}\xi), d(\zeta, \mathcal{L}\zeta), \frac{d(\xi, \mathcal{L}\xi)d(\zeta, \mathcal{L}\zeta)}{1 + d(\xi, \zeta)}\right\} \tag{3.11}$$

for all $\xi, \zeta \in \mathcal{U}$. Then, $\mathcal{L}$ has a unique fixed point in $\mathcal{U}$.

Corollary 3.3. *Let $(\mathcal{U}, d, \mathfrak{s})$ be a complete super-metric space and let $\mathcal{L}, \mathcal{J}$ be a self-mapping of $\mathcal{U}$. Assume that there exist real numbers $k, k_j \in [0, 1), (j \in \{1, 2, 3, 4\})$ satisfying $0 \le k_1 + k_2 + k_3 + k_4 < 1$, $k = \max\{k_2, k_3\}$ such that any one of the*

following contractive condition is hold:
1) $d(\mathcal{L}\xi, \mathcal{J}\zeta) \le k_1 d(\xi, \zeta)$;
2) $d(\mathcal{L}\xi, \mathcal{J}\zeta) \le k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{J}\zeta)$;
3) $d(\mathcal{L}\xi, \mathcal{J}\zeta) \le k \max\{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{J}\zeta)\}$;
4) $d(\mathcal{L}\xi, \mathcal{J}\zeta) \le k_1 d(\xi, \zeta) + k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{J}\zeta)$;
5) $d(\mathcal{L}\xi, \mathcal{J}\zeta) \le k_1 d(\xi, \zeta) + k_4 \dfrac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{J}\zeta)}{1 + d(\xi, \zeta)}$;
6) $d(\mathcal{L}\xi, \mathcal{J}\zeta) \le k_1 d(\xi, \zeta) + k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{J}\zeta) + k_4 \dfrac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{J}\zeta)}{1 + d(\xi, \zeta)}$;
for all $\xi, \zeta \in \mathcal{U}$. *Then,* $\mathcal{L}$ *and* $\mathcal{J}$ *has a unique fixed point in* $\mathcal{U}$.

Proof. Set $\mathfrak{c} = \max\limits_{j \in \{1,2,3,4\}} \{k_j, k\}$. Then

$$\begin{aligned} d(\mathcal{L}\xi, \mathcal{J}\zeta) &\le k_1 d(\xi, \zeta) \\ &\le k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{J}\zeta) \\ &\le k_1 d(\xi, \zeta) + k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{J}\zeta) \\ &\le k_1 d(\xi, \zeta) + k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{J}\zeta) + k_4 \frac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{J}\zeta)}{1 + d(\xi, \zeta)} \\ &\le \mathfrak{c} \max\left\{ d(\xi, \zeta), d(\xi, \mathcal{L}\xi), d(\zeta, \mathcal{J}\zeta), \frac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{J}\zeta)}{1 + d(\xi, \zeta)} \right\} \end{aligned}$$

Also,

$$\begin{aligned} d(\mathcal{L}\xi, \mathcal{J}\zeta) &\le k_1 d(\xi, \zeta) + k_5 \frac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{J}\zeta)}{1 + d(\xi, \zeta)} \\ &\le \mathfrak{c} \max\left\{ d(\xi, \zeta), d(\xi, \mathcal{L}\xi), d(\zeta, \mathcal{J}\zeta), \frac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{J}\zeta)}{1 + d(\xi, \zeta)} \right\} \end{aligned}$$

and

$$\begin{aligned} d(\mathcal{L}\xi, \mathcal{J}\zeta) &\le k \max\{d(\xi, \mathcal{L}\xi), d(\zeta, \mathcal{J}\zeta)\} \\ &\le \mathfrak{c} \max\left\{ d(\xi, \zeta), d(\xi, \mathcal{L}\xi), d(\zeta, \mathcal{J}\zeta), \frac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{J}\zeta)}{1 + d(\xi, \zeta)} \right\} \end{aligned}$$

From Theorem 3.1, $\mathcal{L}$ and $\mathcal{J}$ have a unique common fixed point in $\mathcal{U}$.

Corollary 3.4. *Let* $(\mathcal{U}, d, \mathfrak{s})$ *be a complete super-metric space and let* $\mathcal{L}$ *be a self-mapping of* $\mathcal{U}$. *Assume that there exist real numbers* $k, k_j \in [0, 1), (j \in \{1, 2, 3, 4\})$ *satisfying* $0 \le k_1 + k_2 + k_3 + k_4 < 1$, $k = \max\{k_2, k_3\}$ *such that any one of the*

following contractive condition is hold:
1) $d(\mathcal{L}\xi, \mathcal{L}\zeta) \le k_1 d(\xi, \zeta)$;
2) $d(\mathcal{L}\xi, \mathcal{L}\zeta) \le k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{L}\zeta)$;
3) $d(\mathcal{L}\xi, \mathcal{L}\zeta) \le k \max\{d(\xi, \mathcal{L}\xi), d(\zeta, \mathcal{L}\zeta)\}$;
4) $d(\mathcal{L}\xi, \mathcal{L}\zeta) \le k_1 d(\xi, \zeta) + k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{L}\zeta)$;
5) $d(\mathcal{L}\xi, \mathcal{L}\zeta) \le k_1 d(\xi, \zeta) + k_4 \dfrac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{L}\zeta)}{1 + d(\xi, \zeta)}$;
6) $d(\mathcal{L}\xi, \mathcal{L}\zeta) \le k_1 d(\xi, \zeta) + k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{L}\zeta) + k_4 \dfrac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{L}\zeta)}{1 + d(\xi, \zeta)}$;
for all $\xi, \zeta \in \mathcal{U}$*. Then,* $\mathcal{L}$ *has a unique fixed point in* $\mathcal{U}$*.*

Proof. Set $\mathfrak{c} = \max_{j \in \{1,2,3,4\}} \{k_j, k\}$. Then

$$\begin{aligned} d(\mathcal{L}\xi, \mathcal{L}\zeta) &\le k_1 d(\xi, \zeta) \\ &\le k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{L}\zeta) \\ &\le k_1 d(\xi, \zeta) + k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{L}\zeta) \\ &\le k_1 d(\xi, \zeta) + k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{L}\zeta) + k_4 \frac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{L}\zeta)}{1 + d(\xi, \zeta)} \\ &\le \mathfrak{c} \max\left\{ k_1 d(\xi, \zeta), d(\xi, \mathcal{L}\xi), d(\zeta, \mathcal{L}\zeta), \frac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{L}\zeta)}{1 + d(\xi, \zeta)} \right\}. \end{aligned}$$

Also,

$$\begin{aligned} d(\mathcal{L}\xi, \mathcal{L}\zeta) &\le k_1 d(\xi, \zeta) + k_4 \frac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{L}\zeta)}{1 + d(\xi, \zeta)} \\ &\le \mathfrak{c} \max\left\{ d(\xi, \zeta), d(\xi, \mathcal{L}\xi), d(\zeta, \mathcal{J}\zeta), \frac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{J}\zeta)}{1 + d(\xi, \zeta)} \right\} \end{aligned}$$

and

$$\begin{aligned} d(\mathcal{L}\xi, \mathcal{L}\zeta) &\le k \max\{d(\xi, \zeta), d(\xi, \mathcal{L}\xi)\} \\ &\le \mathfrak{c} \max\left\{ d(\xi, \zeta), d(\xi, \mathcal{L}\xi), d(\zeta, \mathcal{J}\zeta), \frac{d(\xi, \mathcal{L}\xi) d(\zeta, \mathcal{J}\zeta)}{1 + d(\xi, \zeta)} \right\}. \end{aligned}$$

From Corollary 3.2, $\mathcal{L}$ has a unique common fixed point in $\mathcal{U}$.

4. Example

We give an example that satisfies the conditions of Theorem 3.1.

Example 4.1. Let $\mathfrak{s} = 1$, and $d : [0, 1] \times [0, 1] \to [0, +\infty)$ be defined as follows:

$d(\xi,\zeta) = \xi\zeta$ for all $\xi \neq \zeta$, and $\xi,\zeta \in (0,1)$;
$d(\xi,\zeta) = 0$ for all $\xi = \zeta$, and $\xi,\zeta \in [0,1]$;
$d(0,\zeta) = d(\zeta,0) = \zeta$ for all $\zeta \in (0,1]$;
$d(1,\zeta) = d(\zeta,1) = 1 - \dfrac{\zeta}{2}$ for all $\zeta \in [0,1)$.
First, we claim that d is a super-metric on $[0,1]$. We will concentrate on (s3) because (s1) and (s2) are simple to confirm. For any $\zeta \in (0,1)$, we can choose the sequences $\{\xi_r\}, \{\zeta_r\} \subset [0,1]$, where $\xi_r = \dfrac{r^2+1}{r^2+2}$ and $\zeta_r = \dfrac{r+1}{r^2+2}$, for any $n \in \mathbb{N}$.

$$\lim_{r\to\infty} \xi_r = \lim_{r\to\infty} \frac{r^2+1}{r^2+2} = \lim_{r\to\infty} \frac{1+\frac{1}{r^2}}{1+\frac{2}{r^2}} = 1$$

and

$$\lim_{r\to\infty} \zeta_r = \lim_{r\to\infty} \frac{r+1}{r^2+2} = \lim_{r\to\infty} \frac{1+\frac{1}{r}}{r(1+\frac{2}{r^2})} = 0.$$

Also,

$$\lim_{r\to\infty} d(\xi_r,\zeta_r) = \lim_{r\to\infty} \xi_r\zeta_r = \lim_{r\to\infty} \frac{r^2+1}{r^2+2}\frac{r+1}{r^2+2} = \lim_{r\to\infty} \frac{1+\frac{1}{r^2}}{1+\frac{2}{r^2}} \lim_{r\to\infty} \frac{1+\frac{1}{r}}{r(1+\frac{2}{r^2})} = 0.$$

Thus,

$$\lim_{r\to\infty} \sup d(\xi_r,\zeta) = \lim_{r\to\infty} \sup \xi_r\zeta = \lim_{r\to\infty} \sup \left\{ \left(\frac{r^2+1}{r^2+2}\right)\zeta \right\} = \zeta \lim_{r\to\infty} \sup \left(\frac{r^2+1}{r^2+2}\right) = \zeta,$$

$$\lim_{r\to\infty} \sup d(\zeta_r,\zeta) = \lim_{r\to\infty} \sup \zeta_r\zeta = \lim_{r\to\infty} \sup \left\{ \left(\frac{r+1}{r^2+2}\right)\zeta \right\} = \zeta \lim_{r\to\infty} \sup \left(\frac{r+1}{r^2+2}\right) = 0,$$

Therefore,

$$\lim_{r\to\infty} \sup d(\zeta_r,\zeta) = 0 < \zeta = \mathfrak{s} \lim_{r\to\infty} \sup d(\xi_r,\zeta)$$

and (s3) holds.
If $\zeta = 0$, using the same sequences, we get

$$\lim_{r\to\infty} \sup d(\xi_r,\zeta) = \lim_{r\to\infty} \sup \xi_r = \lim_{r\to\infty} \sup \frac{r^2+1}{r^2+2} = 1,$$

$$\lim_{r\to\infty} \sup d(\zeta_r,\zeta) = \lim_{r\to\infty} \sup \zeta_r = \lim_{r\to\infty} \sup \frac{r+1}{r^2+2} = 0,$$

Therefore,

$$\lim_{r\to\infty} \sup d(\zeta_r,\zeta) = 0 < 1 = \mathfrak{s} \lim_{r\to\infty} \sup d(\xi_r,\zeta),$$

and again (s3) holds.
If $\zeta = 1$, choosing $\xi_r = \dfrac{r+1}{r^2+2}$ and $\zeta_r = \dfrac{r+2}{r+3}$, for any $n \in \mathbb{N}$. We obtain

$$\lim_{r\to\infty} \xi_r = \lim_{r\to\infty} \frac{r+1}{r^2+2} = 0 \quad \text{and} \quad \lim_{r\to\infty} \zeta_r = \lim_{r\to\infty} \frac{r+2}{r+3} = 1.$$

Then, we have

$$\lim_{r\to\infty} d(\xi_r, \zeta_r) = \lim_{r\to\infty} \xi_r \zeta_r = \lim_{r\to\infty} \frac{r+1}{r^2+2}\frac{r+2}{r+3} = 0.$$

Thus,

$$\lim_{r\to\infty} \sup d(\xi_r, \zeta) = \lim_{r\to\infty} \sup \left(1 - \frac{\xi_r}{2}\right) = \lim_{r\to\infty} \sup \left(1 - \frac{r+1}{2(r^2+2)}\right)$$

$$= \lim_{r\to\infty} \sup \frac{2r^2 - r + 3}{2(r^2+2)} = 1,$$

and

$$\lim_{r\to\infty} \sup d(\zeta_r, \zeta) = \lim_{r\to\infty} \sup \left(1 - \frac{\zeta_r}{2}\right) = \lim_{r\to\infty} \sup \left(1 - \frac{r+2}{2(r+3)}\right)$$

$$= \lim_{r\to\infty} \sup \left(\frac{r+4}{2(r+3)}\right) = \frac{1}{2}$$

Therefore,

$$\lim_{r\to\infty} \sup d(\zeta_r, \zeta) = \frac{1}{2} < 1 = \mathfrak{s} \lim_{r\to\infty} \sup d(\xi_r, \zeta),$$

and again (s3) holds. Hence, d defines a super-metric on $[0,1]$. Define mappings $\mathcal{L}, \mathcal{J} : [0,1] \to [0,1]$ as

$$\mathcal{L}\xi = \frac{\xi}{4}, \ \text{ if } \ \xi \in [0,1) \ \text{ and } \ \mathcal{L}\xi = \frac{1}{16}, \ \text{ if } \ \xi = 1,$$

$$\mathcal{J}\xi = \frac{\xi}{2}, \ \text{ if } \ \xi \in [0,1) \ \text{ and } \ \mathcal{J}\xi = \frac{1}{8}, \ \text{ if } \ \xi = 1.$$

Taking $k_1 = \frac{1}{2}, k_2 = \frac{1}{9}, k_3 = \frac{1}{9}, k_4 = \frac{1}{9}, k = \frac{1}{9}$ then $\mathfrak{c} = \max_{j\in\{1,2,3,4\}} \{k_j, k\} = \frac{1}{2}$.
We consider the following cases:

1. If $\xi, \zeta \in (0,1)$, we have

$$\begin{aligned} d(\mathcal{L}\xi, \mathcal{J}\zeta) &= d\left(\frac{\xi}{4}, \frac{\zeta}{2}\right) = \frac{\xi\zeta}{8} \leq \frac{1}{2}\xi\zeta + \frac{1}{9}\frac{\xi^2}{4} + \frac{1}{9}\frac{\zeta^2}{2} + \frac{1}{9}\frac{\xi^2\zeta^2}{(8+\xi\zeta)} \\ &\leq k_1 d(\xi,\zeta) + k_2 d(\xi,\mathcal{L}\xi) + k_3 d(\zeta,\mathcal{J}\zeta) + k_4 \frac{d(\xi,\mathcal{L}\xi)d(\zeta,\mathcal{J}\zeta)}{1+d(\xi,\zeta)} \\ &\leq \mathfrak{c}\max\left\{d(\xi,\zeta), d(\xi,\mathcal{L}\xi), d(\zeta,\mathcal{J}\zeta), \frac{d(\xi,\mathcal{L}\xi)d(\zeta,\mathcal{J}\zeta)}{1+d(\xi,\zeta)}\right\}. \end{aligned}$$

2. If $\xi = 0, \zeta \in (0,1)$, we have

$$\begin{aligned} d(\mathcal{L}\xi, \mathcal{J}\zeta) &= d(\mathcal{L}0, \mathcal{J}\zeta) = d\left(0, \frac{\zeta}{2}\right) = \frac{\zeta}{2} \leq \frac{1}{2}\zeta + \frac{1}{9}0 + \frac{1}{9}\frac{\zeta^2}{2} + \frac{1}{9}0 \\ &\leq k_1 d(\xi,\zeta) + k_2 d(\xi,\mathcal{L}\xi) + k_3 d(\zeta,\mathcal{J}\zeta) + k_4 \frac{d(\xi,\mathcal{L}\xi)d(\zeta,\mathcal{J}\zeta)}{1+d(\xi,\zeta)} \\ &\leq \mathfrak{c}\max\left\{d(\xi,\zeta), d(\xi,\mathcal{L}\xi), d(\zeta,\mathcal{J}\zeta), \frac{d(\xi,\mathcal{L}\xi)d(\zeta,\mathcal{J}\zeta)}{1+d(\xi,\zeta)}\right\}. \end{aligned}$$

3. If $\xi = 0, \zeta = 0$ or $\xi = 1, \zeta = 1$, we have

$$\begin{aligned} d(\mathcal{L}\xi, \mathcal{J}\zeta) &= 0 \leq \frac{1}{2}d(\xi,\zeta) + \frac{1}{9}d(\xi,\mathcal{L}\xi) + \frac{1}{9}d(\zeta,\mathcal{J}\zeta) + \frac{1}{9}\frac{d(\xi,\zeta\xi)d(\zeta,\mathcal{J}\zeta)}{1+d(\xi,\zeta)} \\ &\leq k_1 d(\xi,\zeta) + k_2 d(\xi,\mathcal{L}\xi) + k_3 d(\zeta,\mathcal{J}\zeta) + k_4 \frac{d(\xi,\mathcal{L}\xi)d(\zeta,\mathcal{J}\zeta)}{1+d(\xi,\zeta)} \\ &\leq \mathfrak{c}\max\left\{d(\xi,\zeta), d(\xi,\mathcal{L}\xi), d(\zeta,\mathcal{J}\zeta), \frac{d(\xi,\mathcal{L}\xi)d(\zeta,\mathcal{J}\zeta)}{1+d(\xi,\zeta)}\right\}. \end{aligned}$$

4. If $\xi = 0, \zeta = 1$, we have

$$\begin{aligned} d(\mathcal{L}\xi, \mathcal{J}\zeta) &= d(\mathcal{L}0, \mathcal{J}1) = d\left(0, \frac{1}{8}\right) = \frac{1}{8} \\ &\leq \frac{1}{2}(1) + \frac{1}{9}(0) + \frac{1}{9}\frac{1}{8} + \frac{1}{9}\frac{(0)\left(\frac{1}{8}\right)}{1+1} \\ &\leq k_1 d(\xi,\zeta) + k_2 d(\xi,\mathcal{L}\xi) + k_3 d(\zeta,\mathcal{J}\zeta) + k_4 \frac{d(\xi,\mathcal{L}\xi)d(\zeta,\mathcal{J}\zeta)}{1+d(\xi,\zeta)} \\ &\leq \mathfrak{c}\max\left\{d(\xi,\zeta), d(\xi,\mathcal{L}\xi), d(\zeta,\mathcal{J}\zeta), \frac{d(\xi,\mathcal{L}\xi)d(\zeta,\mathcal{J}\zeta)}{1+d(\xi,\zeta)}\right\}. \end{aligned}$$

5. If $\xi = 1, \zeta \in (0,1)$, we have

$$d(\mathcal{L}\xi, \mathcal{J}\zeta) = d(\mathcal{L}1, \mathcal{J}\zeta) = d\left(\frac{1}{16}, \frac{\zeta}{2}\right) = \frac{\zeta}{32} \leq \frac{1}{2}\zeta + \frac{1}{9}\frac{1}{16} + \frac{1}{9}\frac{\zeta^2}{2} + \frac{1}{9}\frac{\frac{\zeta^2}{32}}{1+\zeta}$$
$$\leq k_1 d(\xi,\zeta) + k_2 d(\xi, \mathcal{L}\xi) + k_3 d(\zeta, \mathcal{J}\zeta) + k_4 \frac{d(\xi,\mathcal{L}\xi)d(\zeta,\mathcal{J}\zeta)}{1+d(\xi,\zeta)}$$
$$\leq \mathfrak{c} \max\left\{d(\xi,\zeta), d(\xi,\mathcal{L}\xi), d(\zeta,\mathcal{J}\zeta), \frac{d(\xi,\mathcal{L}\xi)d(\zeta,\mathcal{J}\zeta)}{1+d(\xi,\zeta)}\right\}.$$

Therefore, we conclude that $\mathcal{L}$ and $\mathcal{J}$ have a unique common fixed point $0 \in [0,1]$.

5. Conclusions

Extending the findings of Kannan, Reich, and Dass Gupta, fixed point theorem 3.1 shows the strength and adaptability of fixed-point theory in metric spaces while encompassing a larger class of contractive mappings. The generalization of the Kannan, Reich, and Dass Gupta fixed point results provides a robust foundation for numerous future research directions and practical applications. By extending these theorems to more general spaces and conditions, and by exploring their implications in various fields, researchers can uncover new theoretical insights and develop practical tools for solving complex real-world problems. The future scope is vast and promising, highlighting the central role of fixed-point theory in mathematics and its interdisciplinary potential. Many of the common fixed-point results have been rediscovered or have overlapped with existing results in the recent decades concerning the metric fixed point theory; comparable versions have also been produced because of some erroneous assumptions. These circumstances are mostly caused by the theory's use of constricted supermetric spaces. This paper discusses the uniqueness and existence of the common fixed point of specific operators. Expanding the metric fixed point theory could be a great opportunity to use the concept of the supermetric. Some typical fixed-point theorems for supermetrics were presented in this study. We think that clearing the congestion of the metric fixed point theory will come first in a thorough analysis of generalized rational-type contraction.

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