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ANALYSIS OF CLIFFORD-WAVELET TRANSFORM IN $Cl_{(3,1)}$

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Abstract: Clifford-wavelet transform in L^2-spaces is defined in space-time algebra $Cl_{(3,1)}$ of Minkowski space with orthonormal vector basis. The properties of Clifford-wavelet transform are established. Plancherel's theorem and reproducing kernel is demonstrated. The inversion formula for Clifford-wavelet transform is established. The study is supported with examples and applications from Mathematical Physics.

Keywords and Phrases: Clifford-wavelet transform, similitude group, multivector, inversion formula.

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1. Introduction

In [12, 19] authors discussed the development and progress of Geometric Algebra. [2] have shown how continuous Clifford $Cl_{3,0}$ valued admissible wavelets were constructed using the similitude group $SIM(3)$, a subgroup of the affine group of $\mathbb{R}^3$. In 2006, [25] constructed the Clifford algebra-valued admissible wavelets, which were associated to more than 2-dimensional euclidean groups with dilations. Admissibility conditions, reconstruction formula and Plancherel's theorem were established in [13]. In [6] authors have considered Clifford-valued functions defined on $\mathbb{R}^n$ with representation in square integrable group. In the study of two-dimensional quaternion wavelet transform [10], authors have introduced continuous quaternion wavelet transform (CQWT) and established the admissibility condition in terms of the (right-sided) quaternion Fourier transform, norm relation and inversion formula. In 2011, [4] has proposed two-dimensional continuous quaternion wavelet transform using the orthogonality of harmonic exponential functions and an alternative proof for inner product relation. In [18], Clifford multivectors have been studied for a new description of space and time. Authors have shown Clifford geometric algebra utilizing multivectors to represent space-time providing a compact mathematical representation and insights into the nature of time. Author in [14] has studied admissibility condition in terms of a Cl_n Clifford-Fourier transform, dilation, translation, rotation covariance, reproducing kernel, and inversion formula of Clifford-wavelet transform. The extensions to wider classes of integral transform like Clifford algebra versions of wavelets shown in [7]. [8] authors have explored Clifford algebra to provide a natural alternative to Minkowski formulation with new insights into the nature of time. Clifford (Geometric) algebra wavelet transform was introduced and continuous Cl_n valued admissible wavelets were constructed using the similitude group $SIM(n)$ [20].

In the present study, authors have developed Clifford wavelet transform in space-time algebra $Cl(3,1)$ where as earlier researchers have worked in $Cl(2,0)$ and $Cl(3,0)$. Applications in Fermionic field and Klein-Gordon equation in $Cl(3,1)$ have been demonstrated in the concluding section.

2. Clifford geometric algebra $Cl_{(3,1)}$ of $\mathbb{R}^{(3,1)}$

The multiplication rules for an orthonormal base of inner-product vector-space explaining geometric product from [20]. Consider orthonormal vector basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^{(3,1)}$ with 2^4-dimensional basis considering: $\{1, e_1, e_2, e_3, e_4, e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}, e_{123}, e_{134}, e_{234}, e_{124}, e_{1234}\}$, where 1 is the real scalar identity element (grade 0), e_l basis vectors (grade 1), $e_{kl} = e_k e_l$; basis bivectors (grade 2), $e_{klm} = e_k e_l e_m$; basis tri-vectors (grade 3) and $e_{1234} = e_1 e_2 e_3 e_4 = i_4$ unit oriented pseudo-scalars 4 (grade 4) indicating the highest grade blade element in $Cl_{(3,1)}$. The generalized representation for (grade 2) basis vector in Clifford algebra is represented as

$$e_{kl} = \begin{cases} -e_l e_k; k < l \end{cases} \tag{2.1}$$

where k, l=1, 2, 3, 4.
For all values of $k = l$, we get

$$e_k^2 = \begin{cases} -1; k = 1 \\ 1; k = 2, 3, 4. \end{cases} \tag{2.2}$$

In $Cl_{(3,1)}$, every multivector M is a linear combination of 4-grade elements expressed as:

$$\begin{aligned} M = \sum_A \alpha_A e_A = &\underbrace{\alpha_0}_{\text{scalar part}} + \underbrace{\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4}_{\text{vector part}} \\ &+ \underbrace{\alpha_{12} e_{12} + \alpha_{13} e_{13} + \alpha_{14} e_{14} + \alpha_{23} e_{23} + \alpha_{24} e_{24} + \alpha_{34} e_{34}}_{\text{bi-vector part}} \\ &\underbrace{\alpha_{123} e_{123} + \alpha_{134} e_{134} + \alpha_{234} e_{234} + \alpha_{124} e_{124}}_{\text{tri-vector part}} + \underbrace{\alpha_{1234} e_{1234}}_{\text{quadra-vector part}} . \end{aligned} \tag{2.3}$$

where $A = \{0, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 134, 234, 124, 1234\}$ and $\alpha_A \in \mathbb{R}$. Then (2.3) can be represented as:

$$M = \langle M \rangle + \langle M \rangle_1 + \langle M \rangle_2 + \langle M \rangle_3 + \langle M \rangle_4. \tag{2.4}$$

The reverse of M is defined by the anti-automorphism [17]

$$\tilde{M} = \langle M \rangle + \langle M \rangle_1 - \langle M \rangle_2 - \langle M \rangle_3 + \langle M \rangle_4. \tag{2.5}$$

A multivector-valued function $f : \mathbb{R}^{(3,1)} \to Cl_{(3,1)}$; let $\mathbf{x} \in \mathbb{R}^{(3,1)}$ be a multivector variable, then $f(\mathbf{x})$ can be decomposed as:

$$\begin{aligned} f(\mathbf{x}) = \textstyle\sum_A f_A(\mathbf{x}) e_A = f_0(\mathbf{x}) + f_1(\mathbf{x}) e_1 + f_2(\mathbf{x}) e_2 + f_3(\mathbf{x}) e_3 + f_4(\mathbf{x}) e_4 + \\ f_{12}(\mathbf{x}) e_{12} + f_{13}(\mathbf{x}) e_{13} + f_{14}(\mathbf{x}) e_{14} + f_{23}(\mathbf{x}) e_{23} + f_{24}(\mathbf{x}) e_{24} + f_{34}(\mathbf{x}) e_{34} + \\ f_{123}(\mathbf{x}) e_{123} + f_{134}(\mathbf{x}) e_{134} + f_{234}(\mathbf{x}) e_{234} + f_{124}(\mathbf{x}) e_{124} + f_{1234}(\mathbf{x}) e_{1234}. \end{aligned} \tag{2.6}$$

From [15], the volume-time Fourier transform can be applied to multivector valued functions in the space-time algebra $f : \mathbb{R}^{(3,1)} \to Cl_{(3,1)}$

$$\hat{f}(\omega) = F\{f\}(\omega) = \int\limits_{\mathbb{R}^{(3,1)}} e^{-e_1 \omega_1} f(\mathbf{x})\, e^{-i_4 \vec{x}.\vec{\omega}}\, d^4\mathbf{x}. \tag{2.7}$$

Definition 2.1. *The inner product of $f, g : \mathbb{R}^{(3,1)} \to Cl_{(3,1)}$ is defined as* [15]*:*

$$\langle f, g \rangle_{L^2(\mathbb{R}^{(3,1)}, Cl_{(3,1)})} = \int\limits_{\mathbb{R}^{(3,1)}} f(\mathbf{x})\, \widetilde{g(\mathbf{x})} d^4\mathbf{x}. \tag{2.8}$$

Definition 2.2. *For $f, g : \mathbb{R}^{(3,1)} \to Cl_{(3,1)}$ the norm is defined as* [17]*:*

$$\|f\|^2{}_{L^2(\mathbb{R}^{(3,1)}, Cl_{(3,1)})} = \left\langle (f, f)_{L^2(\mathbb{R}^{(3,1)}, Cl_{(3,1)})} \right\rangle. \tag{2.9}$$

Definition 2.3. *Plancherel's theorem is given from* [13] *as*

$$\left\langle f_1(\mathbf{x}), \widetilde{f_2(\mathbf{x})} \right\rangle_{L^2(\mathbb{R}^{(3,1)}, Cl_{(3,1)})} = \frac{1}{(2\pi)^4} \left\langle \hat{f}_1(\mathbf{x}), \widetilde{\hat{f}_2(\mathbf{x})} \right\rangle_{L^2(\mathbb{R}^{(3,1)}, Cl_{(3,1)})}. \tag{2.10}$$

3. Main results

Representing $SO(4)$ as similitude group of $\mathbb{R}^{(3,1)}$ using rotors $R \in Cl^{+}_{(3,1)}$ is as follows:

$$SO(4) = \left\{ r_{\Theta}(\mathbf{x}) = \widetilde{R}\mathbf{x}R \right\} \tag{3.1}$$

where

$$R = R_{\alpha}R_{\beta}R_{\gamma}R_{\lambda}R_{\theta}R_{\phi} \tag{3.2}$$

Considering

$$R_{\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e_{11}\cos\alpha & e_{21}\sin\alpha \\ 0 & 0 & -e_{12}\sin\alpha & -e_{22}\cos\alpha \end{pmatrix}$$

$$R_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e_{11}\cos\beta & 0 & e_{21}\sin\beta \\ 0 & 0 & 1 & 0 \\ 0 & -e_{12}\sin\beta & 0 & -e_{22}\cos\beta \end{pmatrix}$$

$$R_{\gamma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e_{11}\cos\gamma & e_{21}\sin\gamma & 0 \\ 0 & -e_{12}\sin\gamma & -e_{22}\cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{\lambda} = \begin{pmatrix} e_{11}\cos\lambda & e_{21}\sin\lambda & 0 & 0 \\ -e_{12}\sin\lambda & -e_{22}\cos\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{\theta} = \begin{pmatrix} e_{11}\cos\theta & 0 & e_{21}\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -e_{12}\sin\theta & 0 & -e_{22}\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_{\phi} = \begin{pmatrix} e_{11}\cos\phi & 0 & 0 & e_{21}\sin\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -e_{12}\sin\phi & 0 & 0 & -e_{22}\cos\phi \end{pmatrix}$$

and

$$\widetilde{R}_{\alpha=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e_{11}\cos\alpha & -e_{21}\sin\alpha \\ 0 & 0 & e_{12}\sin\alpha & -e_{22}\cos\alpha \end{pmatrix}$$

$$\widetilde{R}_{\beta=}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e_{11}\cos\beta & 0 & -e_{21}\sin\beta \\ 0 & 0 & 1 & 0 \\ 0 & e_{12}\sin\beta & 0 & -e_{22}\cos\beta \end{pmatrix}$$

$$\widetilde{R}_{\gamma=}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e_{11}\cos\gamma & -e_{21}\sin\gamma & 0 \\ 0 & e_{12}\sin\gamma & -e_{22}\cos\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\widetilde{R}_{\lambda=}\begin{pmatrix} e_{11}\cos\lambda & -e_{21}\sin\lambda & 0 & 0 \\ e_{12}\sin\lambda & -e_{22}\cos\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\widetilde{R}_{\theta=}\begin{pmatrix} e_{11}\cos\theta & 0 & -e_{21}\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ e_{12}\sin\theta & 0 & -e_{22}\cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\widetilde{R}_{\phi=}\begin{pmatrix} e_{11}\cos\phi & 0 & 0 & -e_{21}\sin\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ e_{12}\sin\phi & 0 & 0 & -e_{22}\cos\phi \end{pmatrix}$$

where

$$\widetilde{R} = \widetilde{R}_{\alpha}\widetilde{R}_{\beta}\widetilde{R}_{\gamma}\widetilde{R}_{\lambda}\widetilde{R}_{\theta}\widetilde{R}_{\phi}. \tag{3.3}$$

Any $SO(4)$ has an unique Euler angle representation with rotors for $\Theta = (\alpha, \beta, \gamma, \lambda, \theta, \phi)$ with $\alpha, \beta, \gamma, \lambda, \theta, \phi \in [0, 2\pi]$ and $\widetilde{R}R = R\widetilde{R} = 1$.

The representation [21] defined is consistent with the group action on $\mathbb{R}^{(3,1)}$ as follows:

$$(a, r_{\Theta}(\mathbf{x}), \mathbf{b}) : \mathbb{R}^{(3,1)} \to \mathbb{R}^{(3,1)}, \mathbf{x} \to a\,\widetilde{R}\,(\Theta)\,\mathbf{x}R\,(\Theta) + \mathbf{b} \tag{3.4}$$

where $(a, r_{\Theta}(\mathbf{x}), \mathbf{b})$ can be represented as $(a, \Theta, \mathbf{b})$.

Also

$$G = \mathbb{R}^{+} \times SO(4) \otimes \mathbb{R}^{(3,1)} = \left\{(a, r_{\Theta}(\mathbf{x}), \mathbf{b}) : a \in \mathbb{R}^{+}, r_{\Theta}(\mathbf{x}) \in SO(4), \mathbf{b} \in \mathbb{R}^{(3,1)}\right\} \tag{3.5}$$

Moreover from [5], we represent $SO(4)$ of $\mathbb{R}^{(3,1)}$ by rotors R (3.2) and $\widetilde{R}$ (3.3) in the spin group from (3.1)

$$Spin(4) = \left\{R \in Cl^{+}_{(3,1)}, \widetilde{R}R = R\widetilde{R} = 1\right\} \tag{3.6}$$

$$SO(4) = \left\{r_{\Theta}(\mathbf{x}) = \widetilde{R}\mathbf{x}R, R \in Spin(4)\right\}. \tag{3.7}$$

Also note that $SO(4) = Spin(4)/\{\pm 1\}$, where $SO(4)$ is the special orthogonal group of $\mathbb{R}^{(3,1)}$. Note that the group G includes dilation, rotation, time parameter and translation.
The left Haar measure on G is given by:

$$d\boldsymbol{\lambda}\,(a, \Theta,\ \mathbf{b}) = d\boldsymbol{\mu}\,(a, \Theta)\, d^4\mathbf{b} \tag{3.8}$$

where $d\boldsymbol{\mu}\,(a, \Theta) = \frac{dad\Theta}{a^5}$ for $d\Theta$ is the Haar measure on $SO(4)$ in [13].

Definition 3.1. ***Clifford-wavelet*** *with respect to the mother Clifford-wavelet* $\psi \in L^2\left(\mathbb{R}^{(3,1)}; Cl_{(3,1)}\right)$ *as analogous to* [9]*:*

$$U_{a,\Theta,\boldsymbol{b}} : L^2\left(\mathbb{R}^{(3,1)}; Cl_{(3,1)}\right) \to L^2\left(G; Cl_{(3,1)}\right). \tag{3.9}$$

$$\psi\,(\boldsymbol{x}) \to U_{a,\Theta,\boldsymbol{b}}\ \psi\,(\boldsymbol{x}) = \psi_{a,\Theta,\boldsymbol{b}}\,(\boldsymbol{x}). \tag{3.10}$$

$$\psi_{a,\Theta,\boldsymbol{b}}\,(\boldsymbol{x}) = \frac{1}{a^2}\psi\left(r_\Theta^{-1}\left(\frac{\boldsymbol{x}-\boldsymbol{b}}{a}\right)\right). \tag{3.11}$$

The family of wavelets $\psi_{a,\Theta,\boldsymbol{b}}$ *is called daughter Clifford-wavelet* [13] *with* $a \in \mathbb{R}^+$*-dilation,* Θ*- rotation and* $\boldsymbol{b} \in \mathbb{R}^{(3,1)}$*- translation vector parameters.*

Theorem 3.2. ***Fourier of Wavelet:*** *Fourier transform on Clifford-wavelet function in* $Cl_{(3,1)}$*, can be represented in the form of*

$$F\left\{\psi_{a,\Theta,\mathbf{b}}\right\}(\boldsymbol{\omega}) = a^2 e^{-i_4\,\mathbf{b}.\vec{w}}\hat{\Psi}\left(a r_\Theta^{-1}\boldsymbol{\omega}\right). \tag{3.12}$$

Proof. Substituting (3.11) using two-sided Clifford Fourier transform [1], we get

$$F\left\{\psi_{a,\Theta,\mathbf{b}}\right\}(\boldsymbol{\omega}) = \int_{\mathbb{R}^{(3,1)}} \frac{1}{a^2}\, e^{-e_1 w_1}\psi\left(r_\Theta^{-1}\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right)\right) e^{-i_4\vec{x}.\vec{w}} d^4\mathbf{x}. \tag{3.13}$$

Considering $\frac{\mathbf{x}-\mathbf{b}}{a} = \mathbf{y}$ and solving we get,
$F\left\{\psi_{a,\Theta,\mathbf{b}}\right\}(\boldsymbol{\omega})$

$$\begin{aligned}
&= \int_{\mathbb{R}^{(3,1)}} \frac{1}{a^2} e^{-e_1 w_1}\psi\left(r_\Theta^{-1}\mathbf{y}\right) e^{-i_4\,(a\mathbf{y}+\mathbf{b}-x_1 e_1).\vec{w}} a^4 d^4\mathbf{y} \\
&= e^{-i_4\,\mathbf{b}.\vec{w}} \int_{\mathbb{R}^{(3,1)}} a^2 e^{-e_1 w_1}\psi\left(r_\Theta^{-1}\mathbf{y}\right) e^{-i_4\,ay.\vec{w}} e^{i_4\,x_1 e_1.\vec{w}} d^4\mathbf{y} \\
&= a^2 e^{-i_4\,\mathbf{b}.\vec{w}} \left\{\hat{\psi}\left(a r_\Theta^{-1}\mathbf{y}\ \right) e^{-x_1 e_1.\,\vec{w}}\right\} \\
&= a^2 e^{-i_4\,\mathbf{b}.\vec{w}}\hat{\Psi}\left(a r_\Theta^{-1}\mathbf{y}\ \right)
\end{aligned}$$

where

$$\hat{\Psi}\left(ar_{\Theta}^{-1}\mathbf{y}\right) = \hat{\psi}\left(ar_{\Theta}^{-1}\mathbf{y}\right)e^{-x_1 e_1 . \vec{w}}. \tag{3.14}$$

Hence established a relation for Fourier of Clifford-wavelet in $Cl_{(3,1)}$; Clifford-Fourier domain.

Theorem 3.3. *The normalization constant ensures that the norm of* $\|\psi_{a,\Theta,\boldsymbol{b}}\|_{L^2\left(\mathbb{R}^4, Cl_{(3,1)}\right)}$ *is independent of* $'a'$ *as stated:*

$$\|\psi_{a,\Theta,\mathbf{b}}\|_{L^2\left(\mathbb{R}^4, Cl_{(3,1)}\right)} = \|\psi\|_{L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)}. \tag{3.15}$$

Proof. Using (2.8) and (2.9), we get

$$\|\psi_{a,\Theta,\mathbf{b}}\|_{L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)} = \int\limits_{\mathbb{R}^{(3,1)}} \sum_{A} \frac{1}{a^4}\psi_A^2\left(r_{\Theta}^{-1}\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right)\right)d^4\mathbf{x}.$$

Thus

$$\|\psi_{a,\Theta,\mathbf{b}}\|_{L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)} = \frac{1}{a^4}\int\limits_{\mathbb{R}^{(3,1)}} \sum_{A} \psi_A^2 a^4 |r_{\Theta}^{-1}| d^4\mathbf{z}.$$

$$= \int\limits_{\mathbb{R}^{(3,1)}} \sum_{A} \psi_A^2\left(\mathbf{z}\ \right)d^4\mathbf{z}.$$

Put $r_{\Theta}^{-1}\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right) = \mathbf{z}$ and $d^4\mathbf{x} = a^4|r_{\Theta}|d^4\mathbf{z}$.
Hence the proof.

Definition 3.4. ***Wavelet transform in*** $\boldsymbol{Cl_{(3,1)}}$***:*** *Let* $f \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ *and* $\psi \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ *be a Clifford-mother-wavelet, then Clifford-wavelet transform is defined by*

$$\begin{aligned} W_{\psi}f\left(a,\Theta,\mathbf{b}\right) &= \int\limits_{\mathbb{R}^{(3,1)}} f\left(\mathbf{x}\right)\widetilde{\psi_{a,\Theta,\mathbf{b}}}\left(\mathbf{x}\right)\, d^4\mathbf{x} \\ &= \langle f, \psi\left(a,\Theta,\mathbf{b}\right)\rangle_{L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)} \\ &= \frac{1}{a^2}\int\limits_{\mathbb{R}^{(3,1)}} f\left(\mathbf{x}\ \right)\widetilde{\psi\left(r_{\Theta}^{-1}\left(\frac{\mathbf{x}-\mathbf{b}}{a}\right)\right)}d^4\mathbf{x}. \end{aligned} \tag{3.16}$$

Theorem 3.5. *The Clifford Fourier transform of Clifford-wavelet is represented as:*

$$W_{\psi}f\left(a,\Theta,\mathbf{b}\right) = \frac{1}{(2\pi)^4}\int\limits_{\mathbb{R}^{(3,1)}} \widehat{f}\left(\boldsymbol{\omega}\right)e^{-i_4\,\mathbf{b}.\boldsymbol{\omega}}\, a^2\left\{\widehat{\Psi}\,\widetilde{\left(ar_{\Theta}^{-1}\boldsymbol{\omega}\right)}\right\}d^4\boldsymbol{\omega}. \tag{3.17}$$

Proof. From (3.14) and [22], we have

$$W_{\psi} f(a,\Theta,\mathbf{b}) = \langle f, \psi_{a,\Theta,\mathbf{b}} \rangle_{L^2\left(\mathbb{R}^{(3,1)};Cl_{(3,1)}\right)} = \frac{1}{(2\pi)^4} \left\langle \hat{f}, \hat{\Psi}_{a,\Theta,\mathbf{b}} \right\rangle \tag{3.18}$$

$$= \frac{1}{(2\pi)^4} \int_{\mathbb{R}^{(3,1)}} \hat{f}(\boldsymbol{\omega}) \, \widetilde{\widehat{\Psi}_{a,\Theta,\mathbf{b}}}(\boldsymbol{\omega}) \, d^4\boldsymbol{\omega}. \tag{3.19}$$

Hence from (3.17), the proof follows.

Remark 3.6. ***Admissibility:*** *Analogous to the classical wavelet* [22]*, an admissibility Clifford-valued mother wavelet* $\psi \in L^2\left(\mathbb{R}^{(3,1)}; Cl_{(3,1)}\right)$ *satisfies:*

$$\int_{\mathbb{R}^{(3,1)}} \psi(\boldsymbol{x}) d^4\boldsymbol{x} = \int_{\mathbb{R}^{(3,1)}} \psi_A(\boldsymbol{x}) e_A d^4\boldsymbol{x} = 0 \tag{3.20}$$

where $\psi_A(\boldsymbol{x})$ *is real-valued wavelet; A is considered from section 2.*
The admissibility constant for $Cl_{(3,1)}$ is written from [16]:

$$\mathbf{C}_{\psi'} = \int_{\mathbb{R}^{(3,1)}} \frac{\widetilde{\widehat{\Psi}}(\zeta)\, \widehat{\Psi}(\zeta)}{|\zeta|^4} d^4\zeta. \tag{3.21}$$

$$\langle \mathbf{C}_{\psi} \rangle = \int_{\mathbb{R}^{(3,1)}} \frac{\widetilde{\widehat{\Psi}}(\zeta)\, \widehat{\Psi}(\zeta)}{|\zeta|^4} d^4\zeta = \| |\zeta|^{-2} \widehat{\Psi}(\zeta) \|_{L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)}. \tag{3.22}$$

$\mathbf{C}_{\psi} = \tilde{\mathbf{C}}_{\psi}$ as in [22] and from (2.4) and (2.5), we get $\mathbf{C}_{\psi} = \langle \mathbf{C}_{\psi} \rangle + \langle \mathbf{C}_{\psi} \rangle_1$ with positive scalar part $\langle \mathbf{C}_{\psi} \rangle > 0$.

$$\langle \mathbf{C}_{\psi} \rangle = \int_{\mathbb{R}^{(3,1)}} \left[\left\langle \widehat{\Psi}(\zeta) \right\rangle^2 + \left\langle \widehat{\Psi}(\zeta) \right\rangle_1^2 - \left\langle \widehat{\Psi}(\zeta) \right\rangle_2^2 - \left\langle \widehat{\Psi}(\zeta) \right\rangle_3^2 + \left\langle \widehat{\Psi}(\zeta) \right\rangle_4^2 \right] \frac{1}{\zeta^4} d^4\zeta.$$

And the vector part is given by:

$$\langle \mathbf{C}_{\psi} \rangle_1 = \int_{\mathbb{R}^{(3,1)}} \left\langle \widetilde{\widehat{\Psi}(\zeta)} \widehat{\Psi}(\zeta) \frac{1}{\zeta^4} d^4\zeta \right\rangle_1$$

$$= \int_{\mathbb{R}^{(3,1)}} \left[\left\langle \widehat{\Psi}(\zeta) \right\rangle \left\langle \widehat{\Psi}(\zeta) \right\rangle_1 + \left\langle \widehat{\Psi}(\zeta) \right\rangle_1 \left\langle \widehat{\Psi}(\zeta) \right\rangle_2 - \left\langle \widehat{\Psi}(\zeta) \right\rangle_2 \left\langle \widehat{\Psi}(\zeta) \right\rangle_3 - \left\langle \widehat{\Psi}(\zeta) \right\rangle_3 \left\langle \widehat{\Psi}(\zeta) \right\rangle_4 \right] \frac{1}{\zeta^4} d^4\zeta.$$

The inverse of $\mathbf{C}_{\psi}$ is given by:

$$\mathbf{C}_{\psi}^{-1} = \frac{\langle \mathbf{C}_{\psi} \rangle - \langle \mathbf{C}_{\psi} \rangle_1}{\langle \mathbf{C}_{\psi} \rangle^2 - \langle \mathbf{C}_{\psi} \rangle_1^2}. \tag{3.23}$$

The inverse exists if and only if $\langle \mathbf{C}_\psi \rangle^2 \neq \langle \mathbf{C}_\psi \rangle_1^2$.

4. Properties of Clifford-wavelet transform

Theorem 4.1. ***Left linearity:*** *Let $f, g \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ and $\psi \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ be a Clifford-mother-wavelet. The Clifford-wavelet transform W_ψ is a linear operator defined as* [11]*:*

$$W_\psi(\rho f + \sigma g)(a, \Theta, \boldsymbol{b}) = \rho W_\psi f(a, \Theta, \boldsymbol{b}) + \sigma W_\psi g(a, \Theta, \boldsymbol{b}) \tag{4.1}$$

with multivector constants $\rho, \sigma \in Cl_{(3,1)}$.

Proof. Considering the left-hand-side of (4.1) and (3.16), the proof is obvious.

Theorem 4.2. ***Translation covariance:*** *Let $\psi \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ be a Clifford-mother-wavelet and $W_\psi f(\boldsymbol{x})$ is translated by a constant $\boldsymbol{x}_0$, then*

$$\left[W_\psi f(. - \mathbf{x}_0)\right](a, \Theta, \mathbf{b}) = W_\psi f(a, \Theta, \ \mathbf{b} - \mathbf{x}_0). \tag{4.2}$$

Proof. Here the left-hand-side of (4.2), (3.16) and substituting $\mathbf{x} - \mathbf{x}_0 = \mathbf{y}$, we get

$$\left[W_\psi f(. - \mathbf{x}_0)\right](a, \Theta, \mathbf{b}) = \int_{\mathbb{R}^{(3,1)}} \frac{1}{a^2} f(\mathbf{y}) \psi\left(r_\Theta^{-1}\left(\widetilde{\frac{\mathbf{y} - (\mathbf{b} - \mathbf{x}_0)}{a}}\right)\right) d^4\mathbf{x}$$

$$= W_\psi f(a, \Theta, \mathbf{b} - \mathbf{x}_0)\, d^4\mathbf{x}.$$

Hence the proof as in (4.2).

Theorem 4.3. ***Dilation covariance:*** *Let $\psi \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ be a Clifford-mother-wavelet. If the 'c' is positive real constant, then*

$$\left[W_\psi f(.\, c)\right](a, \Theta, \mathbf{b}) = \frac{1}{c^2} W_\psi f(ac, \Theta, c\mathbf{b}). \tag{4.3}$$

Proof. Considering (4.3), (3.16), and using $\mathbf{y} = c\mathbf{x}$, we get

$$\left[W_\psi f(.\, c)\right](a, \Theta, \mathbf{b}) = \int_{\mathbb{R}^{(3,1)}} \frac{1}{c^4} \frac{1}{a^2} f(\mathbf{y})\, \psi\left(r_\Theta^{-1}\left(\widetilde{\frac{\mathbf{y} - \mathbf{b}c}{ac}}\right)\right) d^4\mathbf{y}$$

$$= \frac{1}{c^2} \int_{\mathbb{R}^{(3,1)}} \frac{1}{(ac)^2} f(\mathbf{y})\, \psi\left(r_\Theta^{-1}\left(\widetilde{\frac{\mathbf{y} - \mathbf{b}c}{ac}}\right)\right) d^4\mathbf{y}$$

$$= \frac{1}{c^2} W_\psi f(ac, \Theta, c\mathbf{b}).$$

Hence the proof.

Theorem 4.4. ***Rotation Covariance:*** *Let* $\psi \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ *be a Clifford-mother-wavelet. Let the rotations be represented by* r_θ *and* r_{θ_0}*, then*

$$[W_\psi f(r_{\theta_0}.)](a,\Theta,\boldsymbol{b}\ )] = W_\psi f\left(a,\Theta',r_{\theta_0}\boldsymbol{b}\right) = \int\limits_{\mathbb{R}^{(3,1)}} f(r_{\theta_0}\boldsymbol{x})\widetilde{\psi_{a,\Theta,\boldsymbol{b}}}(\boldsymbol{x})\ d^4\boldsymbol{x} \tag{4.4}$$

with rotors $r_{\theta'} = r_{\theta_0} r_\theta$.

Proof. From left-hand-side of (4.4), it is obtained as:

$[W_\psi f(r_{\theta_0}.)](a,\Theta,\mathbf{b}\ )$

$$= \int\limits_{\mathbb{R}^{(3,1)}} f(\mathbf{y})\left[\psi\left(r_\Theta^{-1}\widetilde{\left(\frac{r_{\Theta_0}^{-1}\mathbf{y}\text{-}\mathbf{b}}{a}\right)}\right)\right]\det{}^{-1}(r_\Theta)\,d^4\mathbf{y}$$

$$= \int\limits_{\mathbb{R}^{(3,1)}} f(\mathbf{y})\left[\psi\left((r_{\Theta_0}r_\Theta)^{-1}\widetilde{\left(\frac{\mathbf{y}-r_{\Theta_0}\mathbf{b}}{a}\right)}\right)\right]d^4\mathbf{y}$$

$$= W_\psi f\left(a,\Theta',r_{\Theta_0}\mathbf{b}\right).$$

Hence the proof.

Theorem 4.5. ***Derivative property:*** *Let* $\psi \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ *be a Clifford-mother-wavelet, then from* [24] *with small increment* h*, the first derivative of* $f(\mathbf{x})$ *w.r.t* x_1 *and* $\vec{x}$ *respectively follows:*

i) $W_\psi \partial_{x_1} f(\mathbf{x}) = \frac{d}{d\mathbf{x}} W_\psi f(\mathbf{x})$.

ii) $W_\psi \partial_{\vec{x}} f(\mathbf{x}) = -\left(\frac{\partial \psi_{a,\Theta,\mathbf{b}}(\mathbf{x})}{\partial \vec{x}} / \psi_{a,\Theta,\mathbf{b}}(\mathbf{x})\right)(c)\, W_\psi f(\mathbf{x})$.

Proof. From (3.16) we get

i) $W_\psi \partial_{x_1} f(\mathbf{x}) = \frac{1}{a^2} \int\limits_{\mathbb{R}^{(3,1)}} f_{x_1}\,\psi_{a,\Theta,\mathbf{b}}\ (\mathbf{x})\,d^4\mathbf{x}$

$$= \lim_{h\to 0}\frac{1}{h}\ \{W_\psi f(x_1+h,\vec{x}) - W_\psi f(\mathbf{x})\}$$

$$= \frac{d}{d\mathbf{x}} W_\psi f(\mathbf{x}).$$

ii) $W_{\psi}\partial_{\vec{x}}f(\mathbf{x})$

$$= -\frac{1}{a^2}\int\limits_{\mathbb{R}^{(3,1)}} f_{\vec{x}}\frac{\partial\psi_{a,\Theta,\mathbf{b}}}{\partial\vec{x}}\,(\mathbf{x})\,d^4\mathbf{x}$$

$$= -\frac{1}{a^2}\int\limits_{\mathbb{R}^{(3,1)}} f_{\vec{x}}\,\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})\,\frac{\frac{\partial\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})}{\partial\vec{x}}}{\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})}\,d^4\mathbf{x}$$

Hence the proof from [24].

Remark 4.6. $W_{\psi}\left(\partial_{x_1}+\partial_{\vec{x}}\right)f(\boldsymbol{x}) = W_{\psi}\Delta f(\boldsymbol{x})$.

5. Plancherel theorem

Theorem 5.1. *Clifford Fourier transform represented as* $\hat{f}_1(\boldsymbol{\omega})$ *and* $\hat{f}_2(\boldsymbol{\omega})$ *for* $f_1(\boldsymbol{x}),\ f_2(\boldsymbol{x}) \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ *respectively, then*

$$\left\langle f_1(\mathbf{x}), \tilde{f}_2(\mathbf{x})\right\rangle = \frac{1}{(2\pi)^4}\left\langle \hat{f}_1(\boldsymbol{\omega}), \tilde{\hat{f}}_2(\boldsymbol{\omega})\right\rangle. \tag{5.1}$$

Proof. Using (2.8), we get

$$\begin{aligned}
&\left\langle f_1(\mathbf{x}), \tilde{f}_2(\mathbf{x})\right\rangle \\
&\quad= \int\limits_{\mathbb{R}^{(3,1)}} f_1(\mathbf{x})\ \tilde{f}_2(\mathbf{x})\,d^4\mathbf{x} \\
&\quad= \frac{1}{(2\pi)^4}\int\limits_{\mathbb{R}^{(3,1)}}\left(\int\limits_{\mathbb{R}^{(3,1)}} \hat{f}_1(\boldsymbol{\omega})\,e^{i_4\boldsymbol{\omega}.\mathbf{x}}d^4\boldsymbol{\omega}\right)\tilde{f}_2(\mathbf{x})\,d^4\mathbf{x} \quad \text{(applying (2.7))} \\
&\quad= \frac{1}{(2\pi)^4}\int\limits_{\mathbb{R}^{(3,1)}} \hat{f}_1(\boldsymbol{\omega})\left(\int\limits_{\mathbb{R}^{(3,1)}} \tilde{f}_2(\mathbf{x})\,e^{-i_4\boldsymbol{\omega}.x}d^4\boldsymbol{\omega}\right)d^4\mathbf{x} \quad \text{(using (2.8))} \\
&\quad= \frac{1}{(2\pi)^4}\left\langle \hat{f}_1(\mathbf{x}), \tilde{\hat{f}}_2(\mathbf{x})\right\rangle.
\end{aligned}$$

In particular if $f_1(\mathbf{x}) = f_2(\mathbf{x}) = f(\mathbf{x})$, then Parseval theorem is obtained as follows:

$$\int\limits_{\mathbb{R}^{(3,1)}} \|f(\boldsymbol{\omega})\|^2\,d^4\mathbf{x} = \frac{1}{(2\pi)^4}\int\limits_{\mathbb{R}^{(3,1)}}\left\|\hat{f}(\boldsymbol{\omega})\right\|^2\,d^4\boldsymbol{\omega}. \tag{5.2}$$

Theorem 5.2. ***Inner product:*** *Let* $\psi \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ *be an admissible Clifford-mother- wavelet and* $f, g \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$. *Then*

$$\langle W_\psi f, W_\psi g\rangle_{L^2(G,Cl_{(3,1)})} = \left\langle f\,\mathbf{C}_\psi, g\right\rangle_{L^2(\mathbb{R}^{(3,1)},Cl_{(3,1)})}. \tag{5.3}$$

Proof. Considering left-hand-side of (5.3) and (2.8) and [13], we get

$$\langle W_\psi f, W_\psi g\rangle_{L^2(G,Cl_{(3,1)})} = \int\limits_G W_\psi f\left(a,\Theta,\mathbf{b}\right)\, W_\psi \widetilde{g\left(a,\Theta,\mathbf{b}\right)}\, d\boldsymbol{\mu}\, d^4\mathbf{b}$$

$$= \int\limits_{\mathbb{R}^+}\int\limits_{SO(4)} \frac{a^4}{(2\pi)^8}\Big\{\int\limits_{\mathbb{R}^{(3,1)}}\left(\int\limits_{\mathbb{R}^{(3,1)}} \hat{f}\left(\boldsymbol{\omega}\right)\, e^{i_4\boldsymbol{\omega}.\mathbf{b}}\hat{\Psi}\left(\widetilde{ar_\Theta^{-1}(\boldsymbol{\omega})}\right) d^4\boldsymbol{\omega}\right)$$

$$\times \widetilde{\left(\int\limits_{\mathbb{R}^{(3,1)}} \hat{g}\left(\boldsymbol{\omega}'\right)\, e^{i_4\boldsymbol{\omega}.\mathbf{b}}\hat{\Psi}\left(\widetilde{ar_\Theta^{-1}(\boldsymbol{\omega})}\right) d^4\boldsymbol{\omega}'\right)} d^4\mathbf{b}\,\Big\}\, d\boldsymbol{\mu}.$$

Thus

$$F_\Theta(\boldsymbol{\omega}) = \hat{f}\left(\boldsymbol{\omega}\right)\left[\hat{\Psi}\left(\widetilde{ar_\Theta^{-1}(\boldsymbol{\omega})}\right)\right]. \tag{5.4}$$

$$G_\Theta\left(\omega'\right) = \hat{g}\left(\omega'\right)\hat{\Psi}\left(\widetilde{ar_\Theta^{-1}\left(\omega\right)}\ \right). \tag{5.5}$$

We obtain,
$\langle W_\psi f, W_\psi g\rangle_{L^2(G,Cl_{(3,1)})}$.

$$= \int\limits_{\mathbb{R}^+}\int\limits_{SO(4)} \frac{a^4}{(2\pi)^8}\int\limits_{\mathbb{R}^{(3,1)}}\left(\int\limits_{\mathbb{R}^{(3,1)}} F_\Theta\left(\omega\right) e^{i_4\omega.\mathbf{b}} d^4\omega\right) G_\Theta\,\widetilde{\left(\omega\right) e^{i_4\omega.\mathbf{b}}} d^4\omega' d^4\mathbf{b} d\mu.$$

From (2.10), we get

$$\langle W_\psi f, W_\psi g\rangle_{L^2(G,Cl_{(3,1)})} = \frac{1}{(2\pi)^8}\int\limits_{\mathbb{R}^+} a^4 \int\limits_{SO(4)}\left\{\int\limits_{\mathbb{R}^{(3,1)}} \hat{F}_\Theta(-\mathbf{b})\widetilde{\hat{G}_\Theta(-\mathbf{b})} d^4\mathbf{b}\right\} d\boldsymbol{\mu}.$$

From Plancherel's theorem (2.10), it follows:

$$\langle W_\psi f, W_\psi g\rangle_{L^2(G,Cl_{(3,1)})} = \frac{a^4}{(2\pi)^4}\int\limits_{\mathbb{R}^+}\int\limits_{SO(4)}\left\{\int\limits_{\mathbb{R}^{(3,1)}} F_\Theta(\zeta)\{\widetilde{G_\Theta(\zeta)}\} d^4\zeta\right\} d\boldsymbol{\mu}.$$

From (5.4) and (5.5), we get
$\langle W_\psi f, W_\psi g\rangle_{L^2(G,Cl_{(3,1)})}$

$$= \frac{a^4}{(2\pi)^4} \int\limits_{\mathbb{R}^+} \int\limits_{SO(4)} \left\{ \int\limits_{\mathbb{R}^{(3,1)}} \hat{f}(\zeta) \left[\hat{\Psi}\left(\widetilde{ar_\Theta^{-1}(\zeta)}\right)\right] \widetilde{\hat{g}(\zeta)} \hat{\Psi}\left(ar_\Theta{}^{-1}(\zeta)\right) d^4\zeta \right\} d\boldsymbol{\mu}$$
$$= \frac{1}{(2\pi)^4} \int\limits_{\mathbb{R}^{(3,1)}} \hat{f}(\zeta) \left\{ \int\limits_{\mathbb{R}^+} \int\limits_{SO(4)} a^4 \left[\hat{\Psi}\left(\widetilde{ar_\Theta{}^{-1}(\zeta)}\right)\right] \hat{\Psi}\left(ar_\Theta{}^{-1}(\zeta)\right) d\boldsymbol{\mu} \right\} \widetilde{\hat{g}(\zeta)} d^4\zeta$$
$$= \frac{1}{(2\pi)^4} \int\limits_{\mathbb{R}^{(3,1)}} \hat{f}(\zeta)\, \mathbf{C}_\psi\, \widetilde{\hat{g}(\zeta)}\, d^4\zeta$$

where $\mathbf{C}_\psi$ is defined in [22].
From definition (2.3), follows:

$$\langle W_\psi f, W_\psi g\rangle_{L^2(G,Cl_{(3,1)})} = \int\limits_{\mathbb{R}^{(3,1)}} f(\mathbf{x})\, \mathbf{C}_\psi\, \widetilde{g(\mathbf{x})}\, d^4\mathbf{x} = \left\langle f\, \mathbf{C}_\psi, g\right\rangle_{L^2(\mathbb{R}^{(3,1)},Cl_{(3,1)})}.$$

Hence the proof.

Corollary 5.3. ***Norm Relation*** *Let* $\psi \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ *be an admissible Clifford-mother-wavelet, then for any* $f \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ *the scalar part of the inner product gives the* L^2*-norm*

$$\|W_\psi f\|^2{}_{L^2(\mathbb{R}^{(3,1)}Cl_{(3,1)})} = \langle \mathbf{C}_\psi\rangle\, \|f\|^2{}_{L^2(\mathbb{R}^{(3,1)}Cl_{(3,1)})} + \left\langle \left(f\,\langle \mathbf{C}_\psi\rangle_1, f\right)_{L^2(\mathbb{R}^{(3,1)}Cl_{(3,1)})}\right\rangle. \tag{5.6}$$

Proof. For $\psi \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$, $f \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ and from (5.3) can be given as:

$$\begin{aligned}\|W_\psi f\|^2{}_{L^2(\mathbb{R}^{(3,1)}Cl_{(3,1)})} &= \left\langle (W_\psi f, W_\psi f)_{L^2(\mathbb{R}^{(3,1)}Cl_{(3,1)})}\right\rangle \\ &= \left\langle (f\, \mathbf{C}_\psi, f)_{L^2(\mathbb{R}^{(3,1)}Cl_{(3,1)})}\right\rangle \\ &= \mathbf{C}_\psi\, (f\,, f)_{L^2(\mathbb{R}^{(3,1)}Cl_{(3,1)})} \\ &= \langle \mathbf{C}_\psi\rangle\, \|f\|^2{}_{L^2(\mathbb{R}^{(3,1)}Cl_{(3,1)})} + \left\langle \left(f\,\langle \mathbf{C}_\psi\rangle_1, f\right)_{L^2(\mathbb{R}^{(3,1)}Cl_{(3,1)})}\right\rangle.\end{aligned}$$

Hence the proof.

6. Inverse Clifford-wavelet transform in $Cl_{(3,1)}$

Theorem 6.1. *Let ψ be an admissible Clifford-mother-wavelet and f, g satisfy the admissibility conditions, then for any $f \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ can be decomposed as:*

$$f\boldsymbol{(x)} = \int_G W_\psi f\left(a, \Theta, \boldsymbol{b}\right) \psi\left(a, \Theta, \boldsymbol{b}\right) \boldsymbol{C}_\psi^{-1} d\boldsymbol{\mu} d^4\boldsymbol{b}. \tag{6.1}$$

Further (6.1) can also be represented from (3.16) as

$$f(\mathbf{x}) = \int_G \langle f, \psi\left(a, \Theta, \mathbf{b}\right)\rangle_{L^2(\mathbb{R}^{(3,1)}, Cl_{(3,1)})} \psi\left(a, \Theta, \mathbf{b}\right) \mathbf{C}_\psi^{-1} d\boldsymbol{\mu} d^4\mathbf{b}. \tag{6.2}$$

Proof. From (2.9) and (3.16), we get

$$\begin{aligned}
\langle W_\psi f, W_\psi g\rangle_{L^2(G, Cl_{(3,1)})} &= \int_G W_\psi f\left(a, \Theta, \mathbf{b}\right) W_\psi \widetilde{g\left(a, \Theta\right.}, \mathbf{b}) \, d\boldsymbol{\mu}\, d^4\mathbf{b} \\
&= \int_G \int_{\mathbb{R}^{(3,1)}} W_\psi f\left(a, \Theta, \mathbf{b}\right) \psi_{a,\Theta,\mathbf{b}}\left(\mathbf{x}\right) \widetilde{g(\mathbf{x})}\, d^4\mathbf{x}\, d\boldsymbol{\mu}\, d^4\mathbf{b} \\
&= \int_{\mathbb{R}^{(3,1)}} \left(\int_G W_\psi f\left(a, \Theta, \mathbf{b}\right) \psi_{a,\Theta,\mathbf{b}}\left(\mathbf{x}\right) \widetilde{g(\mathbf{x})} d^4\mathbf{b}\, d\boldsymbol{\mu} \right) d^4\mathbf{x}.
\end{aligned}$$

Using inner product relation (5.3), we get

$$\langle W_\psi f, W_\psi g\rangle_{L^2(G, Cl_{(3,1)})} = \left\langle f\, C_\psi, g\right\rangle_{L^2(\mathbb{R}^{(3,1)}, Cl_{(3,1)})}. \tag{6.3}$$

We use (3.16) in the left-hand-side of (6.3) and obtained as:

$$\langle f\mathbf{C}_\psi, g\rangle_{L^2(\mathbb{R}^{(3,1)}, Cl_{(3,1)})} = \left\langle \int_G W_\psi f\left(a, \Theta, \mathbf{b}\right) \psi_{a,\Theta,\mathbf{b}}\left(\mathbf{x}\right) d\boldsymbol{\mu} d^4\mathbf{b}, g(\mathbf{x}) \right\rangle_{L^2(\mathbb{R}^{(3,1)}, Cl_{(3,1)})}. \tag{6.4}$$

As the inner product identity holds for every $g \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ especially for all basis elements of the Clifford module [13], we can conclude that

$$f\left(\mathbf{x}\right) \mathbf{C}_\psi = \int_G W_\psi f\left(a, \Theta, \mathbf{b}\right) \psi_{a,\Theta,\mathbf{b}}\left(\mathbf{x}\right)\, d\boldsymbol{\mu}\, d^4\mathbf{b}.$$

Assuming the inevitability of $\mathbf{C}_\psi$, thus obtain (6.1) as follows:

$$f(\mathbf{x}) = \int_G W_\psi f(a,\Theta,\mathbf{b})\,\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})\;\mathbf{C}_\psi^{-1} d\boldsymbol{\mu}\, d^4\mathbf{b}.$$

And hence follows:

$$f(\mathbf{x}) = \int_G \langle f, \psi(a,\Theta,\mathbf{b})\rangle_{L^2(\mathbb{R}^{(3,1)},Cl_{(3,1)})}\psi(a,\Theta,\mathbf{b})\,\mathbf{C}_\psi^{-1} d\boldsymbol{\mu} d^4\mathbf{b}.$$

Hence the proof.

Remark 6.1. *Weak convergence of (6.1) explains that for all* $g \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ *holds*

$$\left\langle \int_G W_\psi f(a,\Theta,\boldsymbol{b})\,\psi_{a,\Theta,\boldsymbol{b}}(x)\;\boldsymbol{C}_\psi^{-1}\, d\boldsymbol{\mu}\, d^4\boldsymbol{b}, g(\boldsymbol{x})\right\rangle_{L^2(\mathbb{R}^{(3,1)},Cl_{(3,1)})}$$

converges to $\langle f, g\rangle_{L^2(\mathbb{R}^{(3,1)},Cl_{(3,1)})}$.

From [22] $\mathbf{C}_\psi^{-1} = \widetilde{\left[\mathbf{C}_\psi^{-1}\right]}$; thus [13] analogously can be represented as:

$$f(\mathbf{x}) = \mathbf{C}_\psi^{-1} \int_G \widetilde{[\psi_{a,\Theta,\mathbf{b}}]}\left\langle \psi_{a,\Theta,\mathbf{b}}, \tilde{f}\right\rangle_{L^2(\mathbb{R}^{(3,1)},Cl_{(3,1)})} d\boldsymbol{\mu}\, d^4\mathbf{b}.$$

Theorem 6.2. ***Reproducing kernel:*** *For an admissible Clifford-mother-wavelet* $\psi \in L^2\left(\mathbb{R}^{(3,1)}, Cl_{(3,1)}\right)$ *reproducing kernel in* $L^2(G; d\boldsymbol{\lambda})$ *can be defined as:*

$$K_\psi(a,\Theta,\mathbf{b}; a', \Theta', \mathbf{b}') = \left\langle \psi_{a,\Theta,\mathbf{b}}\,\mathbf{C}_\psi^{-1}, \psi_{a',\Theta',\mathbf{b}'}\right\rangle_{L^2(\mathbb{R}^{(3,1)},Cl_{(3,1)})} \tag{6.5}$$

$$W_\psi(a', \Theta', \mathbf{b}') = \int_G W_\psi f(a,\Theta,\mathbf{b}) K_\psi(a,\Theta,\mathbf{b}; a', \Theta', \mathbf{b}') d\boldsymbol{\lambda}. \tag{6.6}$$

Proof. Substituting (6.1) in (3.16), we get

$$\begin{aligned} W_\psi f(a', \Theta', \mathbf{b}') &= \int_{\mathbb{R}^{(3,1)}} \left\{\int_G W_\psi f(a,\Theta,\mathbf{b})\psi_{a,\Theta,b}(\mathbf{x})\mathbf{C}_\psi^{-1} d\boldsymbol{\lambda}\right\} \left[\widetilde{\psi_{a',\Theta',\mathbf{b}'}(\mathbf{x})}\right] d^4\mathbf{x} \\ &= \int_G W_\psi f(a,\Theta,\mathbf{b}) \left\{\int_{\mathbb{R}^{(3,1)}} \psi_{a,\Theta,\mathbf{b}}(\mathbf{x})\mathbf{C}_\psi^{-1}\left[\widetilde{\psi_{a',\Theta',\mathbf{b}'}(\mathbf{x})}\right] d^4\mathbf{x}\right\} d\boldsymbol{\lambda} \\ &= \int_G W_\psi f(a,\Theta,\mathbf{b}) K_\psi(a,\Theta,\mathbf{b}; a', \Theta', \mathbf{b}') d\boldsymbol{\lambda}. \end{aligned}$$

Hence completes the proof.

7. Example

Example 7.1. Consider a Clifford-wavelet in $Cl_{(3,1)}$ defined by

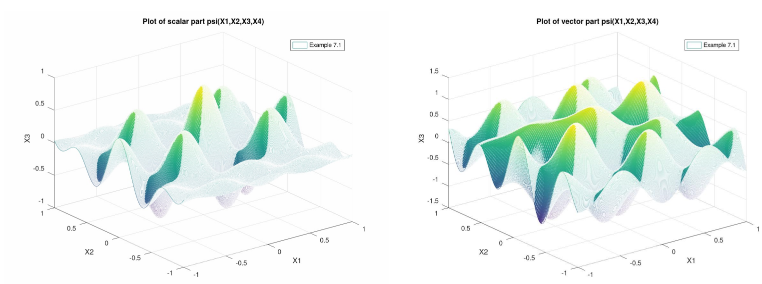
$$\psi_{a,\Theta,\mathbf{b}}(\mathbf{x}) = \begin{cases} e^{-e_1\omega_1 x_1 - e_2\omega_2 x_2 - e_3\omega_3 x_3 - e_4\omega_4 x_4}; -1 \le \mathbf{x}_p \le 1; p = 1,2,3,4 \\ = 0; \text{otherwise.} \end{cases} \tag{7.1}$$

Obtain Clifford-wavelet transform for $f(\mathbf{x}) = e^{x_1+x_2+x_3+x_4}; \quad -\infty < \mathbf{x}_p < 0.$
Solution. By applying Clifford-wavelet transform (3.16) and [3], we get
$W_\psi f\,(a, \Theta, \mathbf{b})$

$$= \frac{1}{a^2}\int_{-1+b_1}^{m_1}\int_{-1+b_2}^{m_2}\int_{-1+b_3}^{m_3}\int_{-1+b_4}^{m_4} e^{x_1+x_2+x_3+x_4} \;\; e^{-e_1\omega_1 x_1 - e_2\omega_2 x_2 - e_3\omega_3 x_3 - e_4\omega_4 x_4} d^4\mathbf{x}$$

for $m_p = \min(0, -1 + b_p)$.
Further simplifying we get
$W_\psi f\,(a, \Theta, \mathbf{b})$

$$= \frac{1}{a^2}\int_{-1+b_1}^{m_1}\int_{-1+b_2}^{m_2}\int_{-1+b_3}^{m_3}\int_{-1+b_4}^{m_4} e^{x_1(1-e_1\omega_1)}\, e^{x_2(1-e_2\omega_2)} e^{x_3(1-e_3\omega_3)} e^{x_4(1-e_4\omega_4)}\; dx_1\, dx_2 dx_3 dx_4$$
$$= \frac{1}{a^2}\left\{\int_{-1+b_1}^{m_1} e^{x_1(1-e_1\omega_1)} dx_1 \int_{-1+b_2}^{m_2} e^{x_2(1-e_2\omega_2)}\, dx_2 \int_{-1+b_3}^{m_3} e^{x_3(1-e_3\omega_3)} dx_3 \int_{-1+b_4}^{m_4} e^{x_4(1-e_4\omega_4)}\; dx_4\right\}$$
$$= \frac{1}{a^2}\left\{\left[\frac{e^{x_1(1-e_1\omega_1)}}{(1-e_1\omega_1)}\right]_{-1+b_1}^{m_1} \left[\frac{e^{x_2(1-e_2\omega_2)}}{(1-e_2\omega_2)}\right]_{-1+b_2}^{m_2} \left[\frac{e^{x_3(1-e_3\omega_3)}}{(1-e_3\omega_3)}\right]_{-1+b_3}^{m_3} \left[\frac{e^{x_4(1-e_4\omega_4)}}{(1-e_4\omega_4)}\right]_{-1+b_4}^{m_4}\right\}$$
$$W_\psi f\,(a, \Theta, \mathbf{b}) = \frac{1}{a^2}\prod_{p=1}^{p=4}\left\{\frac{\left(e^{m_p(1-e_p\omega_p)} - e^{(b_p-1)(1-e_p\omega_p)}\right)}{(1-e_p\omega_p)}\right\}.$$



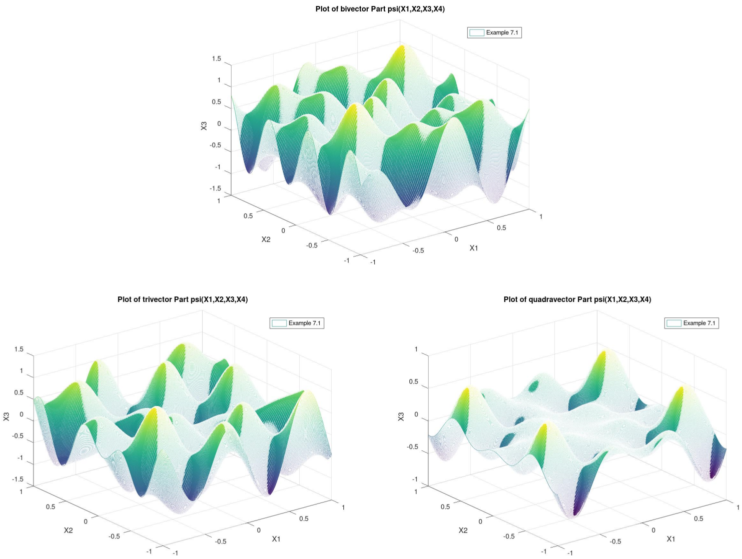


Figure 1. Plot of $\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})$ considering equation (7.1)

Example 7.2. Considering a Clifford-wavelet in $Cl_{(3,1)}$ defined by

$$\psi_{a,\Theta,\mathbf{b}}(\mathbf{x}) = \begin{cases} e^{-e_1\omega_1 x_1 - e_2\omega_2 x_2 - e_3\omega_3 x_3 - e_4\omega_4 x_4}; 0 \leq \mathbf{x}_p \leq \frac{1}{2}; p = 1,2,3,4 \\ -e^{-e_1\omega_1 x_1 - e_2\omega_2 x_2 - e_3\omega_3 x_3 - e_4\omega_4 x_4}; \frac{1}{2} \leq \mathbf{x}_p \leq 1 \\ 0; \text{otherwise.} \end{cases} \tag{7.2}$$

Obtain Clifford-wavelet transform for Gaussian function $f(\mathbf{x}) = e^{x_1^2+x_2^2+x_3^2+x_4^2}$.
Solution. By applying Clifford-wavelet transform (3.15) and [3], we get

$$\begin{aligned} &W_\psi f\,(a,\Theta,b) \\ &= \frac{1}{a^2}\int_{b_1}^{1/2+b_1} e^{x_1^2} e^{-i_1\omega_1 x_1} dx_1 \int_{b_2}^{1/2+b_2} e^{x_2^2} e^{-i_2\omega_2 x_2} dx_2 \\ &\quad \int_{b_3}^{1/2+b_3} e^{x_3^2} e^{-i_3\omega_3 x_3} dx_3 \int_{b_4}^{1/2+b_4} e^{x_4^2} e^{-i_4\omega_4 x_4} dx_4 \\ &\quad -\frac{1}{a^2}\int_{1/2+b_1}^{1+b_1} e^{x_1^2} e^{-i_1\omega_1 x_1} dx_1 \int_{1/2+b_2}^{1+b_2} e^{x_2^2} e^{-i_2\omega_2 x_2} dx_2 \\ &\quad \int_{1/2+b_3}^{1+b_3} e^{x_3^2} e^{-i_3\omega_3 x_3} dx_3 \int_{1/2+b_4}^{1+b_4} e^{x_4^2} e^{-i_4\omega_4 x_4} dx_4 \end{aligned}$$

Thus
$W_\psi f\,(a,\Theta,\mathbf{b})$

$$= \frac{1}{a^2}\int_{b_1}^{1/2+b_1} e^{-(x_1+e_1\omega_1/2)^2-(\omega_1/2)^2}dx_1 \int_{b_2}^{1/2+b_2} e^{-(x_2+e_2\omega_2/2)^2-(\omega_2/2)^2}dx_2$$
$$\times\int_{b_3}^{1/2+b_3} e^{-(x_3+e_3\omega_3/2)^2-(\omega_3/2)^2}dx_3 \int_{b_4}^{1/2+b_4} e^{-(x_4+e_4\omega_4/2)^2-(\omega_4/2)^2}dx_4$$
$$-\frac{1}{a^2}\int_{b_1}^{1/2+b_1} e^{-(x_1+e_1\omega_1/2)^2-(\omega_1/2)^2}dx_1 \int_{b_2}^{1/2+b_2} e^{-(x_2+e_2\omega_2/2)^2-(\omega_2/2)^2}dx_2$$
$$\times\int_{b_3}^{1/2+b_3} e^{-(x_3+e_3\omega_3/2)^2-(\omega_3/2)^2}dx_3 \int_{b_4}^{1/2+b_4} e^{-(x_4+e_4\omega_4/2)^2-(\omega_4/2)^2}dx_4.$$

Substituting $y_p = x_p + (i_p\omega_p)/2$ we get:
$W_\psi f\,(a,\Theta,\mathbf{b}) =$

$$\frac{e^{\omega_1^2/4+\omega_2^2/4+\omega_3^2/4+\omega_4^2/4}}{a^2}\left(\int_0^{b_1+(e_1\omega_1)/2}\left(-e^{-y_1^2}\right)dx_1 + \int_0^{1/2+b_1+(e_1\omega_1)/2} e^{-y_1^2}dx_1\right)$$
$$\times\left(\int_0^{b_2+(e_2\omega_2)/2}\left(-e^{-y_2^2}\right)dx_2 + \int_0^{1/2+b_2+(e_2\omega_2)/2} e^{-y_2^2}dx_2\right)$$
$$\times\left(\int_0^{b_3+(e_3\omega_3)/2}(-e^{-y_3^2})dx_3 + \int_0^{1/2+b_3+(e_3\omega_3)/2} e^{-y_3^2}dx_3\right)$$
$$\times\left(\int_0^{b_4+(e_4\omega_4)/2}(-e^{-y_4^2})dx_4 + \int_0^{1/2+b_4+(e_4\omega_4)/2} e^{-y_4^2}dx_4\right)$$
$$-\frac{e^{\omega_1^2/4+\omega_2^2/4+\omega_3^2/4+\omega_4^2/4}}{a^2}\left(\int_0^{1/2+b_1+(e_1\omega_1)/2}\left(-e^{-y_1^2}\right)dx_1 + \int_0^{1+b_1+(e_1\omega_1)/2} e^{-y_1^2}dx_1\right)$$
$$\times\left(\int_0^{1/2+b_2+e_2\omega_2/2}\left(-e^{-y_2^2}\right)dx_2 + \int_0^{1+b_2+e_2\omega_2/2} e^{-y_2^2}dx_2\right)$$
$$\times\left(\int_0^{1/2+b_3+e_3\omega_3/2}(-e^{-y_3^2})dx_3 + \int_0^{1+b_3+e_3\omega_3/2} e^{-y_3^2}dx_3\right)$$

$$\times\left(\int_0^{1/2+b_4+e_4\omega_4/2}(-e^{-y_4^2})dx_4+\int_0^{1+b_4+e_4\omega_4/2}e^{-y_4^2}dx_4\right).$$

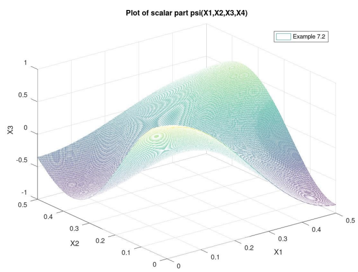
Thus gives values in error function:
$W_\psi f\left(a,\Theta,\mathbf{b}\right)$

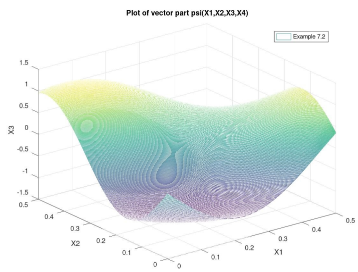
$$\begin{aligned}
&=\frac{16\,e^{\omega_1^2/4+\omega_2^2/4+\omega_3^2/4+\omega_4^2/4}}{\left(\pi a\right)^2}\left[-\operatorname{erf}\left(b_1+\left(e_1\omega_1\right)/2\right)+\operatorname{erf}\left(1/2+b_1+\left(e_1\omega_1\right)/2\right)\right]\\
&\quad\times\left[-\operatorname{erf}\left(b_2+\left(e_2\omega_2\right)/2\right)+\operatorname{erf}\left(1/2+b_2+\left(e_2\omega_2\right)/2\right)\right]\\
&\quad\times\left[-\operatorname{erf}\left(b_3+\left(e_3\omega_3\right)/2\right)+\operatorname{erf}\left(1/2+b_3+\left(e_3\omega_3\right)/2\right)\right]\\
&\quad\times\left[-\operatorname{erf}\left(b_4+\left(e_4\omega_4\right)/2\right)+\operatorname{erf}\left(1/2+b_4+\left(e_4\omega_4\right)/2\right)\right]\\
&-\frac{16\,e^{\omega_1^2/4+\omega_2^2/4+\omega_3^2/4+\omega_4^2/4}}{\left(\pi a\right)^2}\left[-\operatorname{erf}\left(1/2+b_1+\left(e_1\omega_1\right)/2\right)+\operatorname{erf}\left(1+b_1+\left(e_1\omega_1\right)/2\right)\right]\\
&\quad\times\left[-\operatorname{erf}\left(1/2+b_2+\left(e_2\omega_2\right)/2\right)+\operatorname{erf}\left(1+b_2+\left(e_2\omega_2\right)/2\right)\right]\\
&\quad\times\left[-\operatorname{erf}\left(1/2+b_3+\left(e_3\omega_3\right)/2\right)+\operatorname{erf}\left(1+b_3+\left(e_3\omega_3\right)/2\right)\right]\\
&\quad\times\left[-\operatorname{erf}\left(1/2+b_4+\left(e_4\omega_4\right)/2\right)+\operatorname{erf}\left(1+b_4+\left(e_4\omega_4\right)/2\right)\right].
\end{aligned}$$

Finally the results can be represented as:

$$\begin{aligned}
W_\psi f\left(a,\Theta,\mathbf{b}\right)&=\frac{16\,e^{\sum_{p=1}^{p=4}\omega_p^2/4}}{\left(\pi a\right)^2}\prod_{p=1}^{p=4}\left[-\operatorname{erf}\left(b_p+\left(e_p\omega_p\right)/2\right)+\operatorname{erf}\left(1/2+b_p+\left(e_p\omega_p\right)/2\right)\right]\\
&-\frac{16\,e^{\sum_{p=1}^{p=4}\omega_p^2/4}}{\left(\pi a\right)^2}\prod_{p=1}^{p=4}\left[-\operatorname{erf}\left(1/2+b_p+\left(e_p\omega_p\right)/2\right)+\operatorname{erf}\left(1+b_p+\left(e_p\omega_p\right)/2\right)\right].
\end{aligned}$$

Hence the results.





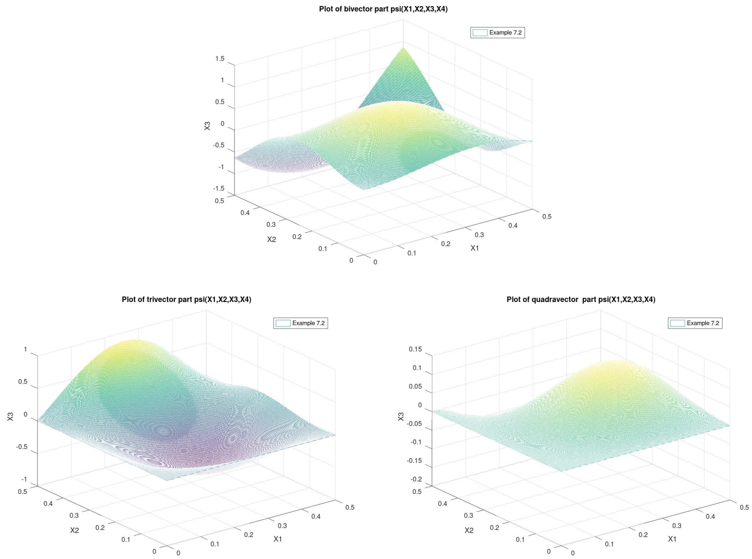


Figure 2. Plot of $\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})$ considering equation (7.2)

8. Applications

8.1. Fermionic field in $Cl_{(3,1)}$

The spin $(-1/2)$ of fermionic field is the Dirac field and considered as $\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})$ -fermionic wave-function in $Cl_{(3,1)}$. The equation of motion for a free spin $(1/2)$ particle is the Dirac equation given by:

$$(i_4 R_\alpha \partial_{\mathbf{x}} - m)\, \psi_{a,\Theta,\mathbf{b}}(\mathbf{x}) = 0. \tag{8.1}$$

where m-fermionic mass, $\partial_{\mathbf{x}}$-derivative in $4D$. The solution of (8.1) are plane wave solutions given by

$$\psi_{a,\Theta,\mathbf{b}}(\mathbf{x}) = \begin{bmatrix} c_1 e^{-m} \\ c_2 e^{m} \\ c_3 e^{-m/k_1} \\ c_4 \widetilde{e^{m/k_1}} \end{bmatrix} \tag{8.2}$$

where $k_1 = -e_{1234}\cos(\alpha) + e_{34}\sin(\alpha)$ for $\mathbf{u} = \begin{bmatrix} c_1 e^{-m} \\ c_3 e^{-m/k_1} \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} c_2 e^{m} \\ c_4 \widetilde{e^{m/k_1}} \end{bmatrix}$ are spinors, labelled by spin, s and spinor indices $\alpha \in \{1, 2, 3, 4\}$.
$\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})$ and $\psi_{a,\Theta,\mathbf{b}}(\mathbf{y})$ obey the anticommutation relation:

$$\{\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})), \psi_{a,\Theta,\mathbf{b}}(\mathbf{y})\} = \delta^{(4)}(\mathbf{x} - \mathbf{y})\delta_{\alpha\beta}. \tag{8.3}$$

The Feynman propagator for the fermion field considering Clifford-wavelet transform of $\psi_{a,\Theta,\mathbf{b}}(\mathbf{x})$ is represented as:

$$\Delta_{W_\psi}(\mathbf{x}\text{-}\mathbf{y}) = \langle 0 \,|W_\psi((\mathbf{x})(\mathbf{y}))|\, 0\rangle \,. \tag{8.4}$$

8.2. Klein—Gordon equation in $Cl_{(3,1)}$

Klein-Gordon equation in natural units using [23]

$$(\Delta + m^2 i_4 \boldsymbol{\omega}.\mathbf{x})\psi_{a,\Theta,\mathbf{b}}(\mathbf{x}) = 0 \tag{8.5}$$

with the metric signature diag$(-1,+1,+1,+1)$.
The solution

$$\psi_{a,\Theta,\mathbf{b}}(\mathbf{x}) = \begin{bmatrix} c_5 e^{m^2 e_1 \omega_1 . x_1} \\ c_6 e^{-m^2 e_2 \omega_2 . x_2} \\ c_7 e^{-m^2 e_3 \omega_3 . x_3} \\ c_8 e^{-m^2 e_4 \omega_4 . x_4} \end{bmatrix} . \tag{8.6}$$

And hence the general solution of wavelet function using Klein-Gorden equation is given by

$$\psi_{a,\Theta,\mathbf{b}}(\mathbf{x}) = k_2(\boldsymbol{\omega}) \int_{\mathbb{R}^{(3,1)}} \frac{\mathrm{d}^4\boldsymbol{\omega}}{(2\pi)^4} e^{m^2 i_4 \boldsymbol{\omega}\cdot\,\mathbf{x}} \psi_{a,\Theta,\mathbf{b}}(\boldsymbol{\omega}) \tag{8.7}$$

where $k_2(\boldsymbol{\omega}) = c_5 c_6 c_7 c_8$.
This is the general solution to the Klein-Gordon equation in $Cl_{(3,1)}$.

9. Conclusion

Authors have developed Clifford-wavelet transform in space-time algebra $Cl_{(3,1)}$. The properties of Clifford-wavelet transform are studied. Plancherel's and Inversion formula have been established. The study is supported with examples and applications from Mathematical Physics.

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