South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, No. 2 (2024), pp. 247-268

DOI: 10.56827/SEAJMMS.2024.2002.19 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

PSEUDO CHEBYSHEV WAVELETS IN TWO DIMENSIONS AND THEIR APPLICATIONS IN THE THEORY OF APPROXIMATION OF FUNCTIONS BELONGING TO LIPSCHITZ CLASS

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(**Received:** *Nov. 07, 2023* **Accepted:** *Jul. 11, 2024* **Published:** *Aug. 30, 2024)*

Abstract: In 2022, the concept of one-dimensional pseudo Chebyshev wavelets was introduced by the authors. Building upon this research, the present article extends the study to two-dimensional pseudo Chebyshev wavelets. It defines and verifies the two-dimensional pseudo Chebyshev wavelet expansion for a functions of two variables. The paper proposes a novel algorithm utilizing the two-dimensional pseudo Chebyshev wavelet method to address computation problems in approximation theory. To demonstrate the validity and applicability of the results, the methods are illustrated through an example and compared with well-known Chebyshev wavelet methods. The research includes error analysis and convergence analysis for signals f belonging to the $\mathrm{Lip}_{\Omega^2}^{(\alpha,\beta)}(\mathbb{R})$, where Ω^2 is a finite connected domain in $\mathbb{R}^2$, classes using these wavelets. Furthermore, the paper estimates the error of approximation for a functions in the Lipschitz class using orthogonal projection operators of the two-dimensional pseudo Chebyshev wavelets. These findings represent significant advancements in wavelet analysis.

Keywords and Phrases: Pseudo Chebyshev functions, Pseudo Chebyshev wavelet (PCW), Two dimensional pseudo Chebyshev wavelet (2D-PCW).

2020 Mathematics Subject Classification: 40A30, 42C15, 42A16, 65T60, 65L10, 65L60, 65R20.

1. Introduction and Preliminaries

Wavelets have garnered significant interest from the mathematical community and researchers across a wide range of scientific and technological disciplines since their emergence in the early 1980s. As a result of this heightened interest, numerous researchers, including Daubechies [14], Chui [12], Morlet et al. [36], Meyer [34], Strang [46], Natanson [37], Chui [13], Daubechies and Lagarias [15], Walter [50, 51], Islam et al. [17], Mohammadi [35], Venkatesh [49], Keshavarz et al. [18], Lal et al. [19, 20, 21, 22, 23, 25, 26, 27], Bastin [1], Biazar et al. [4], Babolian and Fattahzadeh [2, 3] have made notable contributions to wavelet analysis as well as various areas of mathematics and mathematical sciences. Wavelets have experienced significant growth in conjunction with Fourier analysis and harmonic theory, owing to the influence of approximation theory and fractals. Researchers such as Strang [45], Lal et al. [28, 29, 30, 31, 32], Rehman and Siddiqi [40] among others, have actively worked in this direction and have made substantial contributions to the application of wavelets in various fields of science and technology. In recent years, the polynomials have emerged as key players in the realm of approximation theory and using it in the developing of new wavelets. Their increasing prominence is attributed to their versatility in representing and solving a wide array of problems across applied and theoretical mathematics (*see* [9, 10, 11, 16, 38]).

The natural inclination when working with wavelets is to seek complete orthonormal bases for the Hilbert space $L^2(\mathbb{R})$ that possess qualities reflecting the applications of translations and dilations. Considering these observations, orthogonal functions play a crucial role in the construction of new wavelets. The approach to utilizing wavelets involves transforming complex underlying problems into simpler approximations using truncated orthogonal functions. They are several sets of orthogonal functions in $L^2(\mathbb{R})$. Among the various sets of orthogonal functions, one notable example is the Chebyshev polynomials. The Chebyshev polynomials $T_m(t)$; $m \geq 0$, where $0 \leq t \leq 1$, is numerically more effective *see* [5, 7, 33, 42, 43, 44]. The pseudo Chebyshev functions of fractional degree is introduced by Ricci [41] and some of its important properties like orthogonality and more many studied by Cesarano and Ricci [8], Brandi and Ricci [6]. Lal et al. [24] introduced the pseudo Chebyshev wavelet for the first time in June 2022. These wavelets have a wide range of applications in Mathematics and Mathematical

Sciences, especially in the field of fractals, owing to their inherent characteristics.

Fractals, as described by Lal et al. [24], are mathematical objects that exhibit continuity throughout their structure but lack differentiability at any point. The fractional Brownian motion, complex Bernoulli spiral, Brownian trajectories, typical Feynman path, and turbulent fluid motion are all associated with irregular fractals. These phenomena exhibit complex and non-smooth structures, characteristic of fractal behaviour. Irregular fractals are characterized by a local Lipschitz condition at every point within any finite interval. This condition ensures that the fractal exhibits a certain degree of regularity and smoothness, albeit with variations and complexities that define its fractal nature. This fact is to motivate the inspiration for considering the approximation of functions belonging to Lipschitz class via the two-dimensional pseudo Chebyshev wavelet. But till now no work seems to have been done to obtain the error of a signals f belonging to Lipschitz class and its extension into the two dimensional pseudo Chebyshev wavelet expansion.

1.1. Functions of $\mathrm{Lip}_{\Omega^2}^{(\alpha,\beta)}(\mathbb{R})$

A signal $f:\Omega^2\to\mathbb{R}$ where $\Omega=[0,1)$, is said to be signal of $\mathrm{Lip}_{\Omega^2}^{(\alpha,\beta)}(\mathbb{R})$ class

$$i.e.\ f(x,y)\in Lip_{\Omega^2}^{(\alpha,\beta)}(\mathbb{R}),$$

if there exists a non negative real number κ such that

$$\begin{aligned}|f(x+t,y+u)-f(x,y)| &= \kappa\,(|t|^\alpha+|u|^\beta)\\ &= O(|t|^\alpha+|u|^\beta),\quad \text{for}\ \ 0<\alpha,\beta\le 1,(\text{see [48]})\,.\end{aligned}$$

Example 1.1. Define a signal $f:\Omega^2\to\mathbb{R}$ such that

$$f(x,y) \;=\; x^{1/2}+y^{1/2}+y^{3/2}+x^{5/2}\quad \forall\,(x,y)\ \in\Omega^2=(0,1]\times(0,1]\,.$$

Then $f\in\mathrm{Lip}_{\Omega^2}^{(1/2,1/2)}(\mathbb{R})$.

1.2. Two dimensional pseudo Chebyshev wavelets

In the recent research article Lal et al. [24] defined the notion of one dimensional pseudo Chebyshev wavelets with the help of the pseudo Chebyshev functions $T_{m+1/2}(x)$ of indices $m+1/2$. The one dimensional pseudo Chebyshev wavelets are given by

$$\begin{aligned}\psi_{n,m}(x) &:= \psi_{(k,n,m)}(x)\\ &= \begin{cases}\sqrt{\frac{2^{k+1}}{\pi}}T_{m+1/2}(2^kx-2n+1), & \text{for}\ \frac{n-1}{2^{k-1}}\le\ x\ \le\frac{n}{2^{k-1}},\\ 0\quad \text{otherwise, where}\ m\ge 0,\ n=1,2,\cdot\cdot 2^{k-1}\ \text{and}\ k\in\mathbb{N}, & \\ \qquad\qquad (\text{more detail see [24]})\,. & \end{cases}\end{aligned}$$

This definition of pseudo Chebyshev wavelets $\psi_{n,m}$ is generalized to introduce two dimensional pseudo Chebyshev wavelets $\psi_{(n,m;n',m')}$ as follows:

$$\begin{aligned} & \psi_{(n,m;n',m')}(x,y) := \psi_{(k,k':n,m;n',m')}(x,y) = \psi_{(k,:n,m)}(x) \times \psi_{(k':n',m')}(y) \\ = \; & \psi_{(n,m)}(x) \times \psi_{(n',m')}(y) \\ = \; & \begin{cases} \frac{2\sqrt{2^{k+k'}}}{\pi} T_{m+1/2}(2^k x - 2n + 1) T_{m'+1/2}(2^{k'} y - 2n' + 1), \\ \qquad\qquad \text{for } \frac{n-1}{2^{k-1}} \leq \; x \; \leq \frac{n}{2^{k-1}}, \;\; \& \; \frac{n'-1}{2^{k'-1}} \leq \; y \; \leq \frac{n'}{2^{k'-1}}, \\ 0, \qquad\qquad \text{otherwise}, \\ \text{where } m, m' \geq 0, \;\; n = 1, 2, \cdot\cdot 2^{k-1} \;\;\; n' = 1, 2, \cdot\cdot 2^{k'-1} \text{ and } k, k' \in \mathbb{N}, \end{cases} \end{aligned}$$

where,

$$T_{m+1/2}(x) = \cos\left((m + 1/2)\left(arc\cos x\right)\right) \quad m = 0, 1, 2, \cdots, \quad \text{and,}$$

$$T_{m'+1/2}(x) = 2xT_{(m'-1/2)}(x) - T_{(m'-3/2)}(x), \text{ with } T_{\pm 1/2}(x) = \sqrt{\frac{1+x}{2}}, \quad m' \in \mathbb{N}.$$

1.3. Two dimensional pseudo Chebyshev wavelet series

A signal $f \in L^2_\Omega(\mathbb{R})$ where $\Omega = [0, 1)$ is an expanded by one dimensional pseudo Chebyshev wavelet series as follows:

$$\begin{aligned} f \;\; &= \;\; \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \alpha_{(n,m)} \psi_{(n,m)} \text{ where } \alpha_{(n,m)} = \int f(t) \psi_{(n,m)}(t) \omega_{k,n}(t) dt \\ &= \;\; \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \langle f, \psi_{n,m} \rangle_{\omega_{k,n}} \psi_{n,m}, \quad (\text{see [39]}). \end{aligned}$$

If $f \in L^2_{\Omega^2}(\mathbb{R})$ be a signal, then the two dimensional pseudo Chebyshev wavelet series expansion is given by

$$f = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n'=1}^{\infty} \sum_{m'=0}^{\infty} \alpha_{(n,m;n',m')} \psi_{(n,m;n',m')}, \tag{1.1}$$

$$\text{where } \; \alpha_{(n,m;n',m')} = \int_{\Omega^2} f(x,y) \psi_{n,m}(x) \omega_{k,n}(x) \psi_{n',m'}(y) \omega_{k',n'}(y) dx dy.$$

1.4. Orthogonal Projection Operator

An orthogonal projection operator is a surjective map $P^f_n : L^2_\Omega \to V_n$ given by

$$P^f_n \;\; = \;\; \sum_{m=0}^{\infty} \alpha_{(n,m)} \psi_{(n,m)}, \quad \text{where } n = 1, 2, 3, \cdots 2^{k-1}, \quad k \in \mathbb{N},$$

$$= \sum_{m=0}^{\infty} \left\langle f, \psi_{(n,m)} \right\rangle_{\omega_{k,n}} \psi_{n,m}(t) \quad \text{where} \quad \alpha_{(n,m)} = \int\limits_{\Omega} f(t)\psi_{(n,m)}(t)\omega_{k,n}(t)dt, \text{(see[47])}$$

The two dimensional orthogonal projection operator $P^f_{(n,n')} : L^2_{\Omega^2} \to V_{(n,n')}$ is given by

$$\begin{aligned} P^f_{(n,n')} &= \sum_{m'=0}^{\infty} \sum_{m=0}^{\infty} \alpha_{(n,m;n',m')} \psi_{(n,m;n',m')}, \\ &= \sum_{m=0}^{\infty} \left\langle f, \psi_{(n,m;n',m')} \right\rangle_{\omega_{(k,n;k',n')}} \psi_{(n,m;n',m')}, \end{aligned}$$

where $\alpha_{n,m} = \int\limits_{\Omega^2} f(t,u)\psi_{n,m;n',m'}(t,u)\omega_{k,n;k',n'}(t)dtdu, \quad n = 1,2,3,\cdots 2^{k-1} \quad n' = 1,2,3,\cdots 2^{k'-1}$, and k, k' is fixed positive integers.

1.5. Function Approximation

A signal $f \in L^2_{\Omega^2}(\mathbb{R})$, may be expanded in terms of the two dimensional pseudo Chebyshev wavelet series expansion by equation (1.1).

If there exist a signal $f_0 \in L^2_{\Omega^2}(\mathbb{R})$, such that

$$\begin{aligned} f \approx f_0 &= \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} \sum_{n'=1}^{2^{k'-1}} \sum_{m'=0}^{M'-1} \alpha_{n,m;n',m'} \psi_{n,m;n',m'} = \langle A, \Psi \rangle \\ &= A^{\tau}\Psi \text{ where } A^{\tau} \text{ indicates transpose of a matrix } A, \end{aligned}$$

where A and Ψ are $2^{k-1}M2^{k'-1}M' \times 1$ matrices and $\langle A, \Psi \rangle$ is an inner product of column vectors A and Ψ, then this f_0 is called approximation of signal f.

1.6. Error of Wavelet Approximation

The error of wavelet approximation $E^f_{\left(2^{k-1},M\right)}$ of a signal $f \in L^2_{\Omega}(\mathbb{R})$ using the operators $P^f_{\left(2^{k-1},M\right)}$ is

$$E^f_{\left(2^{k-1},M\right)} = \inf_{P^f_{\left(2^{k-1},M\right)}} \|P^f_{\left(2^{k-1},M\right)} - f\|_2.$$

If $E^f_{\left(2^{k-1},M\right)} \to 0$ as $k \to \infty$ or $M \to \infty$ then $P^f_{\left(2^{k-1},M\right)}$ is called the best wavelet approximation of a function $f \in L^2_{\Omega^2}(\mathbb{R})$ (see[52]).

The error of two dimensional pseudo Chebyshev wavelet approximation $E^{f}_{\left(2^{k-1},M;2^{k'-1},M'\right)}$ of a function $f \in L^2_{\Omega^2}(\mathbb{R})$ using the orthogonal projection operators $P^{f}_{\left(2^{k-1},M;2^{k'-1},M'\right)}$ is

$$E^{f}_{\left(2^{k-1},M;2^{k'-1},M'\right)} = \inf_{P^{f}_{\left(2^{k-1},M;2^{k'-1},M'\right)}} \|P^{f}_{\left(2^{k-1},M;2^{k'-1},M'\right)} - f\|_2$$
$$\text{where} M, M' \text{ and } k, k' \in \mathbb{N}.$$

If $E^{f}_{\left(2^{k-1},M;2^{k'-1},M'\right)} \to 0$ as $k, k' \to \infty$ or $M, M' \to \infty$ then $P^{f}_{\left(2^{k-1},M;2^{k'-1},M'\right)}$ is called the best wavelet approximation for the signal $f \in L^2_{\Omega^2}(\mathbb{R})$ of an order $\left(2^{k-1}, M; 2^{k'-1}, M'\right)$.

1.7. Lemmas

The following Lemmas are required hereafter.

Lemma 1.1. (Cauchy Integral Test) *Let N be an integer and a $f : [N, \infty) \to \mathbb{R}$ be a real valued monotonic decreasing signal. Then*

$$\int_N^\infty f(t)dt \leq \sum_N^\infty f(n) \leq f(N) + \int_N^\infty f(t)dt.$$

Lemma 1.2. *If f be a bounded real valued measurable signal on the non negative countably additive finite measurable space $(X, \Im, \mu)$ and Y be a measurable subset of X. Then there exist $\kappa_0 > 0$ such that*

$$|f(t_0, u_0)| \leq \kappa_0 \mu(X \times X')\mu(Y \times Y') \text{ a.e.}, \quad \text{where } (t_0, u_0) \in Y \times Y'.$$

In particular, if

$$(X \times X') = ([0,1) \times [0,1)) \text{ and } (Y \times Y') = \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right] \times \left[\frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}\right],$$

where $n = 1, 2, 3, \cdots, 2^{k-1}$, $n' = 1, 2, 3, \cdots, 2^{k'-1}$. Then

$$f\left(\frac{2n-1}{2^k}, \frac{2n'-1}{2^{k'}}\right) \leq \frac{4\kappa_0}{2^k 2^{k'}}.$$

For the proof of Lemma 1.2, (*see* [24]) .

2. Main Results

In this section, we develop some important theorem ascertaining that two dimensional pseudo Chebyshev wavelets series expansions for the Lipschitz class of signals.

Theorem 2.1. *Let $f \in L^{(\alpha,\beta)}_{\Omega^2}(\mathbb{R})$ and its two dimensional pseudo Chebyshev wavelet series can be expanded as*

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty}\left\langle f,\psi_{(n,m;n',m')}\right\rangle_{\omega_{(k,n;k',n')}}\psi_{(n,m;n',m')}.$$

Then the order of wavelet approximation
$P^{f}_{2^k,M;2^{k'},M'}=\sum\sum\sum\sum\alpha_{n,m;n'm'}\psi_{(n,m;n',m')}$ *coefficient is*

$$\left|\alpha^{f}_{2^k,m;2^{k'},m'}\right| = O\left(\left(\frac{1}{2^{k(\alpha+1)}}+\frac{1}{2^{k'(\beta+1)}}\right)\left(\frac{1}{\left(m+\frac{1}{2}\right)\left(m'+\frac{1}{2}\right)}\right)\right).$$

Proof. Consider a signal $f(x,y)\in L^2_{\Omega^2}(\mathbb{R})$, and its two dimensional pseudo Chebyshev wavelet series

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty}\alpha_{n,m;n',m'}\psi_{n,m}(x)\psi_{n',m'}(y),$$

$$\text{where}\quad \alpha_{n,m;n',m'}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)\psi_{n,m}(x)\omega_{k,n}(x)\psi_{n',m'}(y)\omega_{k',n'}(y)dxdy,$$

and the sequence of partial sums

$$S_{(N,M;N',M')}f(x,y) = \sum_{n=1}^{N}\sum_{m=0}^{M-1}\sum_{n'=1}^{N'}\sum_{m'=0}^{M'-1}\alpha_{n,m;n',m'}\psi_{n,m;n'm'}(x,y),$$

$$\text{where}\quad \psi_{n,m;n'm'}(x,y)=\psi_{n,m}(x)\psi_{n',m'}(y).$$

Next

$$\begin{aligned} f(x,y)-S_{(N,M;N',M')}f(x,y) &= \left(\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty}-\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{M'-1}\right) \\ &\qquad \alpha_{n,m;n',m'}\psi_{n,m;n'm'}(x,y), \\ &= \left(\sum_{n=1}^{N}\left(\sum_{m=0}^{M-1}+\sum_{m=M}^{\infty}\right)\sum_{n'=1}^{N'}\left(\sum_{m=0}^{M'-1}+\sum_{m=M'}^{\infty}\right)\right) \end{aligned}$$

$$+ \left(\sum_{n=N+1}^{\infty} \left(\sum_{m=0}^{M-1} + \sum_{m=M}^{\infty} \right) \sum_{n'=N'+1}^{\infty} \left(\sum_{m=0}^{M'-1} + \sum_{m=M'}^{\infty} \right) \right)$$
$$- \left(\sum_{n=1}^{N} \sum_{m=0}^{M-1} \sum_{n'=1}^{N'} \sum_{m'=0}^{M'-1} \right) \alpha_{n,m;n',m'} \psi_{n,m;n'm'}(x,y).$$

Now by the orthonormal property of the $\{\psi_{n,m;n',m'}\}$ in the disjoint intervals $\left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right] \times \left[\frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}\right]$ and take $N = 2^{k-1}, N' = 2^{k'-1}\ \ k, k' \in \mathbb{N}$, we have

$$\| f(x,y) - S_{2^k,M;2^{k'},M'} f(x,y) \|_2^2 = \sum_{n=1}^{2^{k-1}} \sum_{m=M}^{\infty} \sum_{n'=1}^{2^{k'-1}} \sum_{m=M'}^{\infty} |\alpha_{n,m;n',m'}|^2$$

Since

$$\alpha_{n,m;n',m'} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \psi_{n,m;n',m'}(x,y) \omega_{k,n;k',n'}(x,y) dxdy,$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) - f\left(\frac{2n-1}{2^k}, \frac{2n'-1}{2^{k'}}\right) \psi_{n,m;n',m'}(x,y) \omega_{k,n;k',n'}(x,y) dxdy$$
$$+ f\left(\frac{2n-1}{2^k}, \frac{2n'-1}{2^{k'}}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_{n,m;n',m'}(x,y) \omega_{k,n;k',n'}(x,y) dxdy.$$

Now, $f(x,y) \in \mathrm{Lip}_{\Omega^2}^{(\alpha,\beta)}$ and $\sup\left(2^k t - 2n + 1\right) = 1\ \forall\ \ t\ \ \in \left(\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$ and using Lemma 1.2, we have,

$$\left|\alpha_{n,m;n',m'}\right| \le \left(\kappa \left(\frac{1}{2^{k\alpha}} + \frac{1}{2^{k'\beta}} \right) + \frac{4\kappa_0}{2^k 2^{k'}} \right) \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \int_{\frac{n'-1}{2^{k'-1}}}^{\frac{n'}{2^{k'-1}}} \left|\psi_{n,m;n',m'}(x,y) \omega_{k,n;k',n'}(x,y)\right| dxdy.$$

If $k \neq k'$ or $n \neq n'$ then $\psi_{n,m;n',m'} = 0$

$$\left|\alpha_{n,m;n,m'}\right| \le \left(\frac{\kappa}{2^{k\alpha}} + \frac{\kappa}{2^{k\beta}} + \frac{4\kappa_0}{2^k 2^k} \right) \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left|\psi_{n,m;n,m'}(x,y) \omega_{k,n;k,n}(x,y)\right| dxdy,$$
$$\le \max\{\kappa, 2\kappa_0\} \left(\frac{1}{2^{k\alpha}} + \frac{1}{2^{k\beta}} + \frac{2}{2^{k\alpha} 2^{k\beta}} \right) \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \left|\psi_{n,m;n,m'}(x,y) \omega_{k,n;k,n}(x,y)\right| dxdy,$$

$$\leq \quad 2\max\{\kappa, 2\kappa_0\}\left(\frac{1}{2^{k\alpha}}+\frac{1}{2^{k\beta}}\right)\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left|\psi_{n,m;n,m'}(x,y)\omega_{k,n;k,n}(x,y)\right|dxdy.$$

Next, $\displaystyle\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\psi_{n,j}(t)\omega_{k,n}(t)dt = \sqrt{\frac{2^{k+1}}{\pi}}\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}T_{j+1/2}(2^k t-2n+1)\omega(2^k t-2n+1)dt,$

$$= \frac{1}{2^k}\sqrt{\frac{2^{k+1}}{\pi}}\int_0^{\pi}T_{j+1/2}(cos\theta)d\theta = \frac{(-1)^j}{2^k}\sqrt{\frac{2^{k+1}}{\pi}}\frac{1}{j+1/2}.$$

Therefore,

$$\begin{aligned}
|\alpha_{n,m;n,m'}| &\leq 2\max\{\kappa, 2\kappa_0\}\left(\frac{1}{2^{k\alpha}}+\frac{1}{2^{k\beta}}\right)\\
&\qquad \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left|\psi_{n,m}(x)\omega_{k,n}(x)\right|dx\int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}}\left|\psi_{n,m'}(y)\omega_{k,n}(y)\right|dy,\\
&\leq 2\max\{\kappa, 2\kappa_0\}\left(\frac{1}{2^{k\alpha}}+\frac{1}{2^{k\beta}}\right)\frac{1}{2^k}\sqrt{\frac{2^{k+1}}{\pi}}\frac{1}{(m+1/2)}\frac{1}{2^k}\sqrt{\frac{2^{k+1}}{\pi}}\frac{1}{(m'+1/2)},\\
&= \frac{4}{\pi}\max\{\kappa, 2\kappa_0\}\frac{1}{2^k}\left(\frac{1}{2^{k\alpha}}+\frac{1}{2^{k\beta}}\right)\frac{1}{(m+1/2)(m'+1/2)},\\
&= \frac{4}{\pi}\max\{\kappa, 2\kappa_0\}\left(\frac{1}{2^{k(\alpha+1)}}+\frac{1}{2^{k(\beta+1)}}\right)\frac{1}{(m+1/2)(m'+1/2)}.
\end{aligned}$$

Hence,

$$\left|\alpha_{2^k,m;2^{k'},m'}\right| \;=\; O\left(\left(\frac{1}{2^{k(\alpha+1)}}+\frac{1}{2^{k'(\beta+1)}}\right)\frac{1}{(m+1/2)(m'+1/2)}\right), \quad \text{where } \; 0<\alpha,\beta\leq 1.$$

Thus the Theorem 2.1 is completely established.

Theorem 2.2. *Let a function* $f:\Omega^2\to\mathbb{R}$ *be a real valued function belongs to*

Lipschitz class and its two dimensional pseudo-Chebyshev wavele series

$$\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\langle f,\psi_{n,m;n',m'}\rangle_{\omega_{k,n;k',n'}}\psi_{n,m;n',m'}(x,y).$$

Then the error $E_{2^{k-1},M;2^{k'-1},M'}f$ *of function* $f(x,y)$ *converges uniformly to* 0. *More explicitly,*

$$\left|E_{2^{k-1},M;2^{k'-1},M'}f\right| = O\left(\left(\frac{1}{2^{k\alpha}}+\frac{1}{2^{k'\beta}}\right)\frac{1}{\sqrt{\left(M+\frac{1}{2}\right)\left(M'+\frac{1}{2}\right)}}\right),$$
$$\text{for } 0<\alpha,\beta\leq 1.$$

Proof. Following the proof of theorem 2.1 we have

$$0\leq\parallel E_{2^{k-1},M;2^{k'-1},M'}f\parallel^2\leq\frac{1}{\pi^2}\left(\frac{1}{2^{k\alpha}}+\frac{1}{2^{k\beta}}\right)^2\sum_{m=M}^{\infty}\sum_{m'=M'}^{\infty}\frac{1}{(m+1/2)^2(m'+1/2)^2}$$
$$\leq\frac{1}{\pi^2}\left(\frac{1}{2^{k\alpha}}+\frac{1}{2^{k\beta}}\right)^2\frac{1}{(M+1/2)^2(M'+1/2)^2}\ \text{by Lemma 1.1}$$
$$\to 0 \text{ as } M \text{ or } M' \to\infty.$$

So error function uniformly converges to 0, and more over,

$$\left|E_{2^{k-1},M;2^{k'-1},M'}f\right| = O\left(\left(\frac{1}{2^{k\alpha}}+\frac{1}{2^{k\beta}}\right)\frac{1}{\sqrt{(M+1/2)(M'+1/2)}}\right)$$

Thus the Theorem 2.2 is completely established.

3. Corollaries

In this section, very important corollaries related to Theorem 2.1 and Theorem 2.2, have been established in the following forms:

Corollary 3.1. *If* $f\in Lip^{(\alpha,\beta)}_{((0,1]\times(0,1])}(\mathbb{R})$ *and it can be expanded as an infinite series of the pseudo Chebyshev wavelets for* $m=0$ *and* $m'=0$ *is given by*

$$f(x,y)=\sum_{n=1}^{\infty}\langle f,\psi_{n,0;n',0}\rangle_{\omega_{k,n;k',n'}}\psi_{n,0;n',0},$$

then the series converges uniformly to f. *More explicitly, the order of wavelet coefficients* $a_{n,m:n',m'}$ *in the series expansion satisfy*

$$|a_{n,m'n',m}| = O\left(\frac{1}{N^{\alpha}}+\frac{1}{N'^{\beta}}\right)\ \text{for } 0<\alpha,\beta\leq 1.$$

Corollary 3.2. *If $f \in Lip^{(\alpha,\beta)}_{((0,1]\times(0,1])}(\mathbb{R})$ and it can be expanded as an infinite series of the pseudo Chebyshev wavelets for $k = k' = 1$ is given by*

$$f(x,y) = \sum_{n=1}^{\infty} \langle f, \psi_{1,m;1,m'} \rangle_{\omega_{1,1;1,1}} \psi_{1,m;1,m'},$$

then the series converges uniformly to f. More explicitly, the order of wavelet coefficients $a_{n,m:n',m'}$ in the series expansion satisfy

$$|a_{n,m;n',m'}| = O\left(\frac{1}{2^{\alpha}} + \frac{1}{2^{\beta}}\right)\left(\frac{1}{m+1/2} + \frac{1}{m'+1/2}\right) \; 0 < \alpha, \beta \leq 1.$$

Corollary 3.3. *If f is single variable real valued function in the class $Lip^{\alpha}_{(0,1]}(\mathbb{R})$ and it can be expanded as an infinite series of the pseudo Chebyshev wavelets for $k = n = 1$ is given by*

$$f(x) = \sum_{m=1}^{\infty} \langle f, \psi_{1,m} \rangle_{\omega_{1,1}} \psi_{1,m},$$

then the series converges uniformly to f. More explicitly, the order of wavelet coefficients $a_{1,m}$ in the series expansion satisfy

$$|a_{n,m}| = O\left(\frac{1}{2^{\alpha}(m+1/2)}\right) \; 0 < \alpha \leq 1.$$

Corollary 3.4. *If $f \in Lip^{(\alpha,\beta)}_{((0,1]\times(0,1])}(\mathbb{R})$, then the pseudo Chebyshev wavelet error $E_{2^{k-1},0;2^{k'-1},0}(f)$ of a function f by $P_{2^{k-1},0;2^{k'-1},0}f$ satisfy*

$$\begin{aligned} \left|E_{2^{k-1},0;2^{k'-1},0}(f)\right| &= \left(\min_{P_{2^{k-1},0;2^{k'-1},0}f} ||f - P_{2^{k-1},0;2^{k-1},0}f||\right) \\ &= O\left(\frac{1}{2^{k\alpha+3}} + \frac{1}{2^{k\beta+3}}\right) \text{ for } 0 < \alpha, \beta \leq 1. \end{aligned}$$

Corollary 3.5. *If $f \in Lip^{(\alpha,\beta)}_{((0,1]\times(0,1])}(\mathbb{R})$, then the pseudo Chebyshev wavelet error $E_{1,M;1,M'}(f)$ of a function f by $P_{1,M;1,M'}f$ satisfy*

$$\begin{aligned} |E_{1,M;1,M'}(f)| &= \left(\min_{P_{1,M;1,M'}f} ||f - P_{1,M;1,M'}f||\right) \\ &= O\left(\frac{1}{2^{\alpha+1}} + \frac{1}{2^{\beta+1}}\right)\left(\frac{1}{\sqrt{(M+1/2)(M'+1/2)}}\right) \text{for } 0 < \alpha, \beta \leq 1. \end{aligned}$$

Corollary 3.6. *If f is a single real valued function in the class $Lip^{\alpha}_{(0,1]}(\mathbb{R})$, then the Pseudo Chebyshev wavelet error $E_{1,M}(f)$ of a function f by $P_{1,M}f$ satisfy*

$$\begin{aligned} E_{1,M}(f) &= \left(\min_{P_{1,M}f} ||f - P_{1,M}f||\right) \\ &= O\left(\frac{1}{2^{\alpha+1}(M+1/2)}\right) \text{ for } 0 < \alpha \leq 1. \end{aligned}$$

4. Effectiveness of the pseudo Chebyshev wavelet

In this section, we calculate the approximation of a function and it effectiveness show by an example, more over this results are compared with pseudo Chebyshev wavelet and Chebyshev wavelet approximation method.

4.1. Illustrative Examples

$$f(x) = \begin{cases} 2x^{1/2} - 3x^{3/2} - 5x^{5/2} + 7x^{7/2} & \text{for } \quad 0 \leq x \leq 1, \\ 0 \quad \text{otherwise.} \end{cases}$$

by the pseudo-Chebyshev wavelet approximation method.

In the Corollary 3.3, if $k = 1$, then $n = 1$ and

$$\begin{aligned} f_0^{1,M}(x) &= \sum_{m=0}^{M-1} \langle f, \psi_{1,m} \rangle_{\omega_{1,1}} \psi_{1,m}(x) = \sum_{m=0}^{M-1} a_{1,m}\psi_{1,m}(x) \\ \& \qquad a_{1,m} &= \frac{\int_0^1 f(x)\psi_{1,m}(x)\omega(x)dx}{\int_0^1 \psi_{1,m}(x)\psi_{1,m}(x)\omega(x)dx}. \end{aligned}$$

Next, we evaluate $f_0^{1,1}(x)$, $f_0^{1,2}(x)$, $f_0^{1,3}(x)$, $f_0^{1,4}(x)$, $E_{1,1}(f)(x)$, $E_{1,2}(f)(x)$, $E_{1,3}(f)(x)$, $E_{1,4}(f)(x)$ and $f_0^{1,M}(x)$ & $E_{1,M}(f)(x)$. If,

$A_1^M = (a_{1,0}, a_{1,1}, a_{1,2}, \cdots, a_{1,M-1})^{\tau}$ and $\Psi_1^M = (\psi_{1,0}, \psi_{1,1}, \psi_{1,2}, \cdots, \psi_{1,M-1})^{\tau}$

then

$$\begin{aligned} f_0 &= \sum_{m=0}^{\infty} a_{1,m}\psi_{1,m} = \sum_{m=0}^{\infty} \langle f, \psi_{1,m} \rangle_{\omega_{1,1}} \psi_{1,m} = \lim_{M\to\infty} \sum_{m=0}^{M-1} a_{1,m}\psi_{1,m}, \\ &= \lim_{M\to\infty} \langle A_1^M, \Psi_1^M \rangle = \lim_{M\to\infty} \left(\left(A_1^M\right)^{\tau} \Psi_1^M \right) = \lim_{M\to\infty} f_0^{1,M}, \end{aligned}$$

where $\quad a_{1,m} = \langle f, \psi_{1,m} \rangle_{\omega_{1,1}} = \int\limits_0^1 f(t)\psi_{1,m}(t)\omega_{1,1}(t)dt.$

Now,

$$a_{1,0} = \int\limits_0^1 f(t)\psi_{1,0}(t)\omega_{1,1}(t)dt \approx 0.4016,$$

$$a_{1,1} = \int_0^1 f(t)\psi_{1,1}(t)\omega_{1,1}(t)dt \approx -0.0138,$$

$$a_{1,2} = \int_0^1 f(t)\psi_{1,2}(t)\omega_{1,1}(t)dt \approx 0.4016,$$

$$a_{1,3} = \int_0^1 f(t)\psi_{1,3}(t)\omega_{1,1}(t)dt \approx 0.0969,$$

$$a_{1,4} = \int_0^1 f(t)\psi_{1,4}(t)\omega_{1,1}(t)dt = 0,$$

$$a_{1,5} = \cdots = a_{1,M-1} = 0, \text{for} M \geq 5,$$

$$\text{then } A_1^M = (0.4016, -0.0138, 0.4016, 0.0969, 0, 0, \cdots, 0)^\tau .$$

Since $f_0^{1,M} = \sum_{m=0}^{M-1} a_{1,m}\psi_{1,m} = \left(A_1^M\right)^\tau \Psi_1^M$ and $E_{1,M}(f) = \sum_{m=M}^{\infty} a_{1,m}\psi_{1,m}$. Therefore

$$\begin{aligned} f_0^{1,M}(x) &\approx 0.4015715755\ \psi_{1,0}(x) - 0.01384729574\ \psi_{1,1}(x) + 0.4015715755\ \psi_{1,2}(x) \\ &+ 0.09693106944\ \psi_{1,3}(x) + 0 + 0 + \cdots + 0, = f_0^{1,4}(x) = f_0^1(x) \approx f(x), \end{aligned}$$

and the energy of function f & $f_0^{1,M}$ are given by

$$\begin{aligned} \| f \|_2^2 &= \langle f, f\rangle_{\omega_{1,1}} = \int_0^1 |f(t)|^2 \omega(2t-1)\, dt \approx 0.3321068405772413 \\ &= \lim_{M\to\infty} \sum_{m=0}^{M} |a_{1,m}|^2 = \| f_0^1 \|_2^2, \text{ and } E^{1,M}(x) \approx 0, \ for\ M \geq 4. \end{aligned}$$

x	f(x)	$f_0^{1,1}(x)$	$E^{1,1}(f)(x)$	$f_0^{1,2}(x)$	$E^{1,2}(f)(x)$	$f_0^{1,3}(x)$	$E^{1,3}(f)(x)$	$f_0^{1,4}(f)(x)$	$E^{1,4}(f)(x)$
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1000	0.5240	0.1433	0.3807	0.1561	1.3679	0.6089	0.0849	0.5240	0.0000
0.2000	0.5617	0.2026	0.3591	0.2180	1.3437	0.5504	0.0113	0.5617	0.0000
0.3000	0.4995	0.2482	0.2114	0.2636	0.1959	0.3728	0.0867	0.4995	0.0000
0.4000	0.2833	0.2866	0.0032	0.3004	0.0171	0.1743	0.1090	0.2833	0.0000
0.5000	0.0884	0.3204	0.2320	0.3315	0.2431	0.0110	0.0773	0.0884	0.0000
0.6000	-0.0682	0.3510	0.4192	0.3583	0.4264	0.0770	0.0088	-0.0682	0.0000
0.7000	-0.1247	0.3791	0.5038	0.3817	0.5064	0.0580	0.0666	-0.1247	0.0000
0.8000	-0.0143	0.4053	0.4196	0.4052	0.4168	0.0945	0.1088	-0.0143	0.0000
0.9000	0.3349	0.4299	0.0950	0.4210	0.0861	0.4038	0.0689	0.3349	0.0000
1.0000	1.0000	0.4531	0.5469	0.4375	0.5225	0.8906	0.1094	1.0000	0.0000

Table 1: Comparison between truncated $f_0^{1,M}$ and exact f

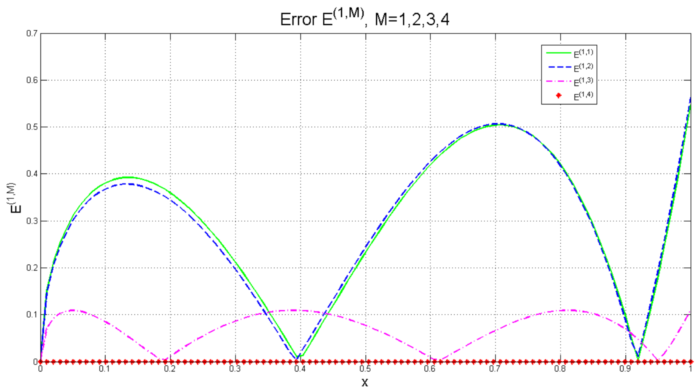

Figure 1: Graph of $\left(f, f_0^{1,1}\right), \left(f, f_0^{1,2}\right), \left(f, f_0^{1,3}\right), \left(f, f_0^{1,4}\right)$.

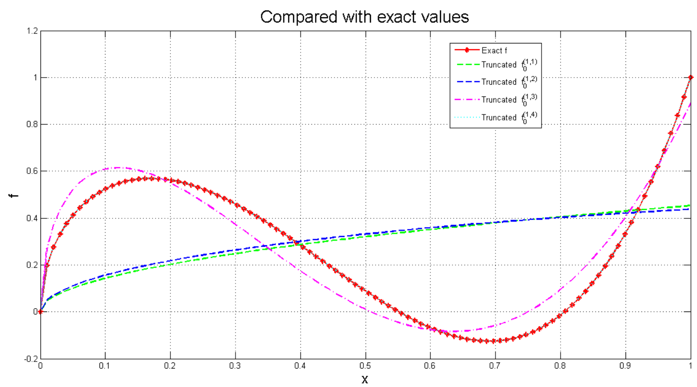

Figure 2: Graph of f and truncated $f_0^{1,M}$ for $M = 1, 2, 3, 4,$ & $k = 1$.

x	f(x)	*CW* $f_0^{1,3}(x)$	*CW* $E^{1,3}(f)(x)$	*PCW* $f_0^{1,3}(x)$	*PCW* $E^{1,3}(f)(x)$	*CW* $f_0^{1,4}(x)$	*CW* $E^{1,4}(f)(x)$	*PCW* $f_0^{1,4}(f)(x)$	*PCW* $E^{1,4}(f)(x)$
0.0000	0.0000	-0.0721	0.0721	0.0000	0.0000	-0.0817	0.0817	0.0000	0.0000
0.1000	0.5240	0.0839	0.4401	0.6089	0.0849	0.0920	0.4320	0.5240	0.0000
0.2000	0.5617	0.1227	0.4390	0.5504	0.0113	0.1308	0.4309	0.5617	0.0000
0.3000	0.4995	0.0786	0.3810	0.3728	0.0867	0.0793	0.3802	0.4995	0.0000
0.4000	0.2833	-0.0144	0.2977	0.1743	0.1090	-0.0210	0.3044	0.2833	0.0000
0.5000	0.0884	-0.1220	0.2104	0.0110	0.0773	-0.1316	0.2200	0.0884	0.0000
0.6000	-0.0682	-0.2101	0.1419	0.0770	0.0088	-0.2167	0.1486	-0.0682	0.0000
0.7000	-0.1247	-0.2445	0.1198	0.0580	0.0666	-0.2438	0.1191	-0.1247	0.0000
0.8000	-0.0143	-0.1910	0.1767	0.0945	0.1088	-0.1829	0.1686	-0.0143	0.0000
0.9000	0.3349	-0.0155	0.3504	0.4038	0.0689	-0.0074	0.3423	0.3349	0.0000
1.0000	1.0000	0.3161	0.6839	0.8906	0.1094	0.3065	0.6935	1.0000	0.0000

Table 2: Comparison between truncated $f_0^{1,M}\&f$ using PCW & CW

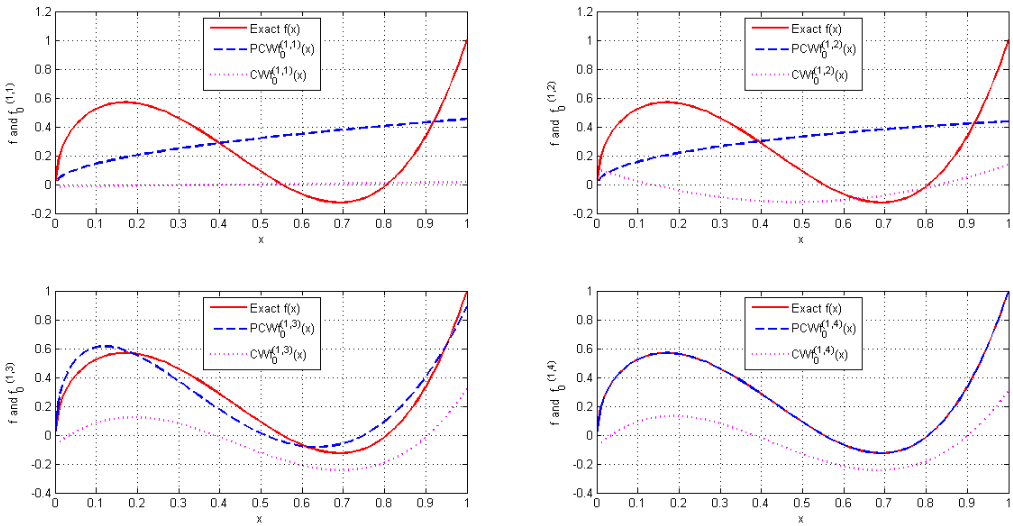

Figure 3: Graph of exact f and $f_0^{1,M}$ using PCW & CW

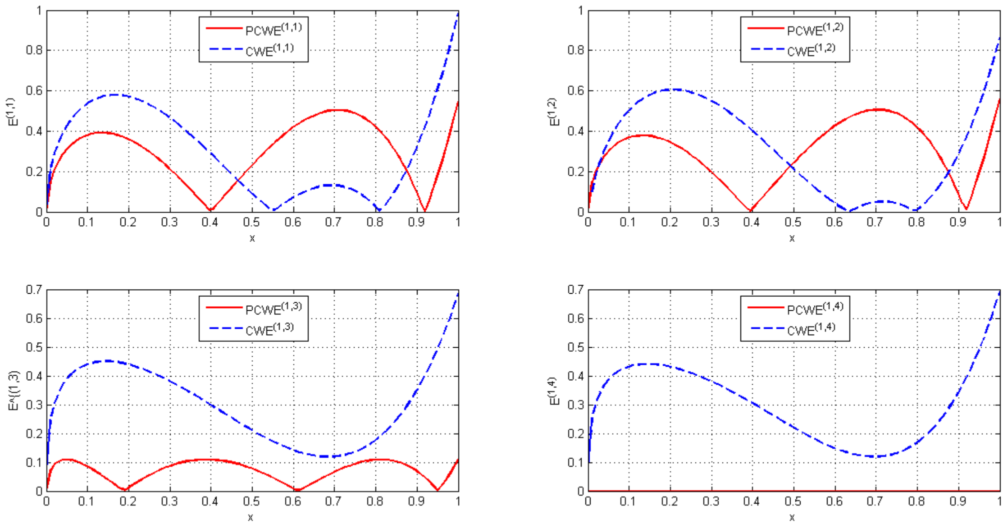

Figure 4: Graph of $E^{1,1}(x), E^{1,2}(x), E^{1,3}(x), E^{1,4}(x),$ by PCW and CW .

5. Conclusions

(i) Since $E_{2^{k-1},M;2^{k'-1},M'}f \to 0$ as $k, k' \to \infty$ or $M, M' \to \infty$ in above results. Therefore the wavelet approximations determined in this results are best possible in the wavelet analysis [52].

(ii) Some most important Corollaries 3.1, 3.2, 3.3 and 3.4, 3.5, 3.6 have been derived from our main Theorems 2.1 and 2.2 respectively.

(iii) Independent proofs of these Corollaries 3.1, 3.2, 3.3, 3.4, 3.5 and 3.6 can be developed for specific contributions of these estimates in wavelet analysis.

(iv) Figures (1, 2, 3 & 4) and Tables (1&2) are shows that the pseudo Chebyshev wavelet method is more effective rather than Chebyshev wavelet method in the case of fractional degree.

(v) Figures $(1, 2, 3\&4)$ are shows that how to error functions are rapidly converges to zero functions by the pseudo Chebyshev wavelet method rather than Chebyshev wavelet method in this case.

Acknowledgments

Authors are grateful to anonymous learned referees and the editor, for their exemplary guidance, valuable feedback and constant encouragement which improve the quality and presentation of this paper. The authors are also thankful to all the editorial board members and reviewers of this reputed journal. Shyam Lal, one of the authors, is thankful to DST - CIMS for encouragement to this work. Susheel Kumar, one of the authors, is grateful to dear esteemed friend Dr. Harish Chandra Yadav, which provides technical support for the numerical computation of this research work.

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