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SOME NORMED LINEAR SPACE AND INTEGRAL INEQUALITIES OF COMPOSITE CONVEX FUNCTIONS

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Abstract: Convex functions play an important role in finding the inequalities, those help in finding the solutions of different types of equations and equations involving functions. In this article, we have considered convex functions in a normed linear space. We have established some results on composite convex sets and composite convex functions. We have considered quasi-arithmetic mean, that unifies efficiently all types of power means. On applying the principles of composite convex functions, we have established a Hermite-Hadamard like inequality. The functions considered are composite convex functions with respect to a strictly monotonic continuous composite function. The composite convex functions serve as a comprehensive generalization of composite convex functions. As an application, we

have established some inequalities on integrable composite convex functions. The results are deferred mean type inequalities. The results on inequalities can be applied for further investigations as well as for application in finding the solutions in different areas of research.

Keywords and Phrases: Integral inequality; Convex function; Composite function.

2020 Mathematics Subject Classification: 26A51, 26B25, 26D15, 90C26.

1. Introduction

A set $K \sqsubseteq \mathbb{R}$ is termed convex if, for all $x_1, y_1 \in K, \alpha \in [0,1]$, it satisfies the condition $(1-\alpha)x_1 + \alpha y_1 \in K$ A function g : K $\to \mathbb{R}$ is considered to be convex in the classical sense for a normed linear space if, for all $x_1, y_1 \in K, \alpha \in [0,1]$, the following holds:

$$\|g((1-\alpha)x_1 + \alpha y_1\| \leq |(1-\alpha)|\|g(x_1)\| + |\alpha|\|g(y_1)\|$$

The significance of convexity theory extends across various domains in both pure and applied sciences. As a result, the traditional principles surrounding convex sets and convex functions have undergone broad generalizations. A compelling factor that draws numerous researchers to the field is the intimate connection between convexity theory and the theory of inequalities. Numerous well-known inequalities can be derived by employing the framework of convex functions. One extensively studied outcome is the Hermite-Hadamard inequality, serving as a pivotal condition for a function to be considered convex. The formulation of this result by Charles Hermite and Jacques Hadamard is articulated as follows:

Theorm 1.1. *Let $g : [c,d] \sqsubseteq \mathbb{R} \to \mathbb{R}$ be an integrable normed linear space convex function. Then*

$$\left\|g\left(\frac{c+d}{2}\right)\right\| \leq \frac{1}{|d-c|}\int_c^d \|g(x_1)\|dx_1 \leq \frac{\|g(c)\| + \|g(d)\|}{2}$$

The primary motivation behind this paper is to introduce the concepts of composite convex sets and composite convex functions. The quasi-arithmetic mean or generalised f-maen or Kolmogorov-Nogumo-de Finotti men is one, which is the generalisation of different means such as the arithmetic mean, the geometric mean etc. The results are established through the utilization of quasi-arithmetic means, which effectively unify all the power means $\mu_{goh}(x_1, y_1)$

$$\mu_{goh(x_1,y_1)} = h^{-1}og^{-1}[(1-\alpha)goh(x_1) + \alpha\, goh(y_1)]$$

These concepts are particularly tied to strictly monotonic continuous composite functions. Applying the principles of composite convex functions, we proceed to derive novel Hermite Hadamard-like inequalities. Simultaneously, we delve into detailed discussions on various significant special cases.

2. Composite-convex functions

In this section we will now introduce novel categories encompassing composite convex sets and composite convex functions.

Definition 2.1. *A set $\tilde{A} \sqsubseteq \mathbb{R}$ is defined as a composite convex set concerning a strictly monotonic continuous composite function if*

$$\mu_{goh(x_1,y_1)} = h^{-1}og^{-1}[(1-\alpha)goh(x_1) + \alpha\, goh(y_1)] \in \tilde{\mathrm{A}}, x_1, y_1 \in \tilde{\mathrm{A}}, \alpha \in [0,1]$$

Definition 2.2. *A function $g : \tilde{A} \to \mathbb{R}$ is regarded as a composite convex function with respect to a strictly monotonic continuous composite function if*

$$g\left(\mu_{goh(x_1,y_1)}\right) \leq (1-\alpha)g(x_1) + \alpha g(y_1), \forall x_1, y_1 \in \tilde{\mathrm{A}}, \alpha \in [0,1] \tag{2.1}$$

Note that the function g is referred to strictly composite convex on $\tilde{A}$ if the aforementioned. Inequality holds strictly, as an inequality for each distinct x_1 and $y_1 \in \tilde{A}$ and for each α in the interval $[0,1]$.
The function $g : \tilde{A} \to \mathbb{R}$ is termed composite -concave(strictly composite-concave) on $\tilde{A}$ if its negation, $-g$ is composite-convex(strictly composite-convex) on $\tilde{A}$.
If we take $\alpha = 1/2$ in (2.1), then we obtain

$$g\left(h^{-1}og^{-1}\left(\frac{goh(x_1) + goh(y_1)}{2}\right)\right) \leq \frac{g(x_1) + g(y_1)}{2}, \text{ for all } x_1, y_1 \in \tilde{\mathrm{A}} \tag{2.2}$$

The function g is then referred to as a mid-convex function.
Now, let's explore some special cases of Definition 2.2 :
Case-I. If we set $\mathrm{goh}(\mathrm{x}_1) = \ln \mathrm{x}_1$, then condition(2.1) becomes

$$g\left(x_1^{1-\alpha} y_1^{\alpha}\right) \leq (1-\alpha)g(x_1) + \alpha g(y_1), \text{ for all } x_1, y_1 \in [a_1, b_1] \sqsubseteq (0,\infty), \alpha \in [0,1],$$

which is the idea of geometric convexity referred to in [1].
Case-2. If we set $g \circ h(x_1) = 1/x_1$, then condition (2.1) becomes

$$g\left(\frac{x_1 y_1}{\alpha x_1 + (1-\alpha) y_1}\right) \leq (1-\alpha)g(x_1) + \alpha g(y_1), \text{ for all } x_1, y_1 \in [a_1, b_1] \sqsubseteq (0,\infty), \alpha \in [0,1].$$

Which is the idea of harmonic convexity as referred to in [15].

Case-3. If we set $g \circ h(x_1) = x_1^p (p > 0)$, then condition (2.1) becomes

$$g\left(((1-\alpha)x_1^p + \alpha y_1^p)^{\frac{1}{p}}\right) \leq (1-\alpha)g(x_1)+\alpha g(y_1), \text{ for all } x_1, y_1 \in [a_1, b_1] \sqsubseteq (0,\infty), \alpha \in [0,1],$$

which is the idea of p-convexity as referred to in [16].
Case-4. If we take goh(x_1) $= e^{x_1}$, then the condition (2.1) becomes

$$g\left(\ln\left((1-\alpha)e^{x_1} + \alpha e^{x_1}\right)\right) \leq (1-\alpha)g(x_1)+\alpha g(y_1), \text{ for all } x_1, y_1 \in [a_1, b_1] \sqsubseteq (0,\infty), \alpha \in [0,1],$$

which is the concept of log-exponential convex functions on $[a_1, b_1]$.

3. Application of composite-convex functions to integral inequalities

In this section, we illustrate a noteworthy application of composite-convex functions by introducing fresh integral inequalities of the Hermite-Hadamard type. Unless stated otherwise, the interval I $= [a_1, b_1]$ is applied, with goh representing a continuously differentiable and strictly monotonic function in its domain. The set $\mathbb{R}^+$ represents the collection of positive real numbers.

Theorem 3.1. *Considering* g :I $\to \mathbb{R}^+$ *is an integrable composite-convex function, then we have following inequalities*

$$g\left(h^{-1}og^{-1}\left(\frac{goh(x_1)+goh(y_1)}{2}\right)\right) \leq \frac{1}{goh(b_1)-goh(a_1)}\int_{a_1}^{b_1} g(x_1)(g\circ h)'(x_1)dx_1 \leq \frac{g(a_1)+g(b_1)}{2} \tag{3.1}$$

Proof. Since g is a composite-convex function, we have

$$g\left(h^{-1}og^{-1}\left(\frac{goh(x_1)+goh(y_1)}{2}\right)\right) \leq \frac{g(x_1)+g(y_1)}{2}.$$

Putting $x_1 = h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))$ and
$y_1 = h^{-1}og^{-1}(\alpha goh(a_1)+(1-\alpha)goh(b_1))$ in the above inequality, we have

$$\begin{aligned} &g\left(h^{-1}og^{-1}\left(\frac{goh(a_1)+goh(b_1)}{2}\right)\right) \\ &\leq \frac{g\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)}{2} \\ &\qquad + \frac{g\left(h^{-1}og^{-1}(\alpha goh(a_1)+(1-\alpha)goh(b_1))\right)}{2}. \end{aligned} \tag{3.2}$$

Integrating both sides with respect to α on [0,1] of (3.2), we get

$$g\left(g^{-1}\circ h^{-1}\left(\frac{g\circ h(a_1)+goh(b_1)}{2}\right)\right) \leq \frac{1}{g\circ h(b_1)-goh(a_1)}\int_a^b g(x_1)(g\circ h)'(x_1)dx_1. \tag{3.3}$$

Similarly, in light of the assumption in Theorem 3.1 that g is the composite-convex function, we have

$$g\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)\leq(1-\alpha)goh(a_1)+\alpha goh(b_1).$$

Integrating both sides with respect to α on $[0,1]$ of the above inequality, we get

$$\frac{1}{g\circ h(b_1)-goh(a_1)}\int_{a_1}^{b_1} g(x_1)(g\circ h)'(x_1)dx_1\leq\frac{g(a_1)+g(b_1)}{2}. \tag{3.4}$$

From equation (3.3) and equation (3.4) we get the proof of Theorem.

Theorem 3.2. *Let* g : I $\to\mathbb{R}^+$ *be an integrable composite-convex function, then we have the following inequalities*

$$\begin{aligned}
&\frac{2g(a_1)}{goh(b_1)-goh(a_1)}\int_{a_1}^{b_1}\left(\frac{g\circ h(b_1)-g\circ h(x_1)}{goh(b_1)-goh(a_1)}\right)g(x_1)(g\circ h)'(x_1)dx_1\\
&\qquad+\frac{2g(b_1)}{goh(b_1)-goh(a_1)}\int_{a_1}^{b_1}\left(\frac{g\circ h(x_1)-goh(a_1)}{goh(b_1)-goh(a_1)}\right)g(x_1)(g\circ h)'(x_1)dx_1\\
&\leq\frac{1}{g\circ h(b_1)-goh(a_1)}\int_{a_1}^{b_1} g^2(x_1)(g\circ h)'(x_1)dx_1+\frac{g^2(a_1)+g(a_1)g(b_1)+g^2(b_1)}{3}\\
&\leq\frac{2\left[g^2(a_1)+g(a_1)g(b_1)+g^2(b_1)\right]}{3}
\end{aligned}\tag{3.5}$$

Proof. Using the arithmetic-geometric means inequality gives

$$\begin{aligned}
&2g\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)\{(1-\alpha)g(a_1)+\alpha g(b_1)\}\\
&\leq\left\{g\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)\right\}^2+\{(1-\alpha)g(a_1)+\alpha g(b_1)\}^2\\
&=\left\{g\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)\right\}^2\\
&\quad+(1-\alpha)^2g^2(a_1)+\alpha^2g^2(b_1)+2\alpha(1-\alpha)g(a_1)g(b_1).
\end{aligned}$$

Integrating both sides with respect to α on $[0,1]$ of the above inequality, we get

$$\begin{aligned}
&2g(a_1)\int_0^1 (1-\alpha)g\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)d\alpha \\
&\qquad + 2g(b_1)\int_0^1 \alpha g\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)d\alpha \\
&\leq \int_0^1 g^2\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)d\alpha \\
&\qquad + g^2(a_1)\int_0^1 (1-\alpha)^2 d\alpha + g^2(b_1)\int_0^1 \alpha^2 d\alpha \\
&\qquad\qquad + 2g(a_1)g(b_1)\int_0^1 \alpha(1-\alpha)d\alpha. \qquad (3.6)
\end{aligned}$$

By making the change of variables, inequality (3.6) can be written as

$$\begin{aligned}
&2g(a_1)\cdot\frac{1}{g\circ h(b_1)-goh(a_1)}\int_a^b \left(\frac{goh(b_1)-goh(x_1)}{goh(b_1)-goh(a_1)}\right)g(x_1)(g\circ h)'(x_1)dx_1 \\
&+ 2g(b_1)\cdot\frac{1}{g\circ h(b_1)-goh(a_1)}\int_a^b \left(\frac{goh(x_1)-goh(a_1)}{goh(b_1)-goh(a_1)}\right)g(x_1)(g\circ h)'(x_1)dx_1 \\
&\leq \frac{1}{g\circ h(b_1)-goh(a_1)}\int_a^b g^2(x_1)(g\circ h)'(x_1)dx_1 + \frac{g^2(a_1)+g(a_1)g(b_1)+g^2(b_1)}{3} \qquad (3.7)
\end{aligned}$$

On the other hand, since g is a composite-convex function, we have
$g\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right) \leq (1-\alpha)g(a_1)+\alpha g(b_1), \forall\alpha\in[0,1]$
therefore, we have

$$\begin{aligned}
&\frac{1}{goh(b_1)-goh(a_1)}\int_a^b g(x_1)(g\circ h)'(x_1)dx_1 \\
&\quad = \int_0^1 g^2\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)d\alpha \\
&\quad \leq \int_0^1 [(1-\alpha)g(a_1)+\alpha g(b_1)]^2 d\alpha = \frac{g^2(a_1)+g(a_1)g(b_1)+g^2(b_1)}{3}. \qquad (3.8)
\end{aligned}$$

Combining (3.7) and (3.8) leads to the inequalities described in Theorem 3.2.

Theorem 3.3. *Let* g : $l \to \mathbb{R}^+$ *be integrable composite- convex function, then we* *have the following inequalities*

$$\begin{aligned}
&\frac{1}{goh(b_1) - goh(a_1)} \int_a^b g(x_1)(g \circ h)'(x_1)dx_1 \\
&\leq \frac{1}{2} g\left(h^{-1}og^{-1}\left(\frac{goh(a_1)+goh(b_1)}{2}\right)\right) \\
&\quad + \frac{1}{4(goh(b_1) - goh(a_1))g\left(h^{-1}og^{-1}\left(\frac{goh(a_1)+goh(b_1)}{2}\right)\right)} \int_a^b g^2(x_1)(g \circ h)'(x_1)dx_1 \\
&\quad\quad + \frac{1}{24g\left(h^{-1}og^{-1}\left(\frac{goh(a_1)+goh(b_1)}{2}\right)\right)} \left(g^2(a_1) + g^2(b_1) + 4g(a_1)g(b_1)\right).
\end{aligned}$$

Proof. Using the arithmetic-geometric means inequality and α the composite convexity of g, it follows that

$$\begin{aligned}
&g\left(h^{-1}og^{-1}\left(\frac{goh(a_1)+goh(b_1)}{2}\right)\right)\left[g\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)\right. \\
&\qquad\qquad \left.+ g\left(h^{-1}og^{-1}(\alpha goh(a_1)+(1-\alpha)goh(b_1))\right)\right] \\
&\leq g^2\left(h^{-1}og^{-1}\left(\frac{goh(a_1)+goh(b_1)}{2}\right)\right) \\
&\qquad\qquad + \frac{1}{4}\left[g\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)\right. \\
&\qquad\qquad\qquad + g(h^{-1}og^{-1}(\alpha goh(a_1)+(1-\alpha)goh(b_1)))]^2 \\
&= g^2\left(h^{-1}og^{-1}\left(\frac{goh(a_1)+goh(b_1)}{2}\right)\right) \\
&\quad + \frac{1}{4}\left[g^2\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right) + g^2(h^{-1}og^{-1}(\alpha goh(a_1)\right. \\
&\qquad + (1-\alpha)goh(b_1))) + 2g\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right) \\
&\qquad\qquad g\left(h^{-1}og^{-1}(\alpha goh(a_1)+ (1-\alpha)goh(b_1))\right)] \\
&\leq g^2\left(h^{-1}og^{-1}\left(\frac{goh(a_1)+goh(b_1)}{2}\right)\right) \\
&+ \frac{1}{4}\left[g^2\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right) + g^2\left(h^{-1}og^{-1}(\alpha goh(a_1)+\right.\right. \\
&(1-\alpha)goh(b_1))) + 2((1-\alpha)goh(a_1)+\alpha goh(b_1)))(\alpha goh(a_1)+(1-\alpha)goh(b_1)))].
\end{aligned}$$

Integrating both sides of the above inequality with respect to α on $[0,1]$, we obtain

$$
\begin{aligned}
& g\left(h^{-1}og^{-1}\left(\frac{goh(a_1)+goh(b_1)}{2}\right)\right)\left[\int_0^1 g\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)d\alpha\right. \\
& \quad \left. +\int_0^1 g\left(h^{-1}og^{-1}(\alpha goh(a_1)+(1-\alpha)goh(b_1))\right)d\alpha\right] \\
& \leq g^2\left(h^{-1}og^{-1}\left(\frac{goh(a_1)+goh(b_1)}{2}\right)\right)\int_0^1 d\alpha \\
& \quad +\frac{1}{4}\left[\int_0^1 g^2\left(h^{-1}og^{-1}((1-\alpha)goh(a_1)+\alpha goh(b_1))\right)d\alpha\right. \\
& \qquad +\int_0^1 g^2\left((h^{-1}og^{-1}(\alpha goh(a_1)+(1-\alpha)goh(b_1))\right)d\alpha+2(g(a_1)+g(b_1)) \\
& \qquad\qquad \left.\int_0^1 \alpha(1-\alpha)d\alpha+2g(a_1)g(b_1)\int_0^1\left(\alpha^2+(1-\alpha)^2\right)d\alpha\right]
\end{aligned}
$$

Performing the change of variable, we get

$$
\begin{aligned}
& g\left(h^{-1}og^{-1}\left(\frac{g\circ h(a_1)+goh(b_1)}{2}\right)\right)\frac{2}{g\circ h(b_1)-goh(a_1)}\int_a^b g(x_1)(g\circ h)'(x_1)dx_1 \\
& \leq g^2\left(h^{-1}og^{-1}\left(\frac{g\circ h(a_1)+goh(b_1)}{2}\right)\right) \\
& \quad +\frac{1}{2(g\circ h(b_1)-goh(a_1))}\int_a^b g^2(x_1)(g\circ h)'(x_1)dx_1+\frac{g^2(a_1)+g^2(b_1)+4g(a_1)g(b_1)}{12}
\end{aligned}
$$

Upon performing straightforward computations, the aforementioned inequality can be transformed into the desired form of Theorem 3.3.

4. Conclusion

We've presented the concept of composite-convex functions and demonstrated that this class encompasses several traditional convexity classes. Indeed, composite convex functions serve as a comprehensive generalization of convex functions associated with various power means. Additionally, by employing the notion of composite convexity, we've derived novel integral inequalities akin to the Hermite-Hadamard type. We anticipate that the ideas and methodologies presented in this paper may inspire further exploration and research in this particular field.

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