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NEW GENERALIZATION OF CHEBYSHEV-LIKE POLYNOMIALS AND THEIR APPLICATIONS

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Abstract: This study is focused on the development of a new generalized version of four known types of Chebyshev polynomials. We come up with four different kind of generalized Chebyshev polynomials using a modified recurrence relationship with different starting points. We also get Binet's formula for generalized Chebyshev's polynomials. The Binet formula is obtained by mathematical induction. The matrix representation and the characteristic equation are presented using matrix algebra properties for these polynomials. We also explore about the sum, products, and subtraction of the roots of the characteristic equation of the generalized Chebyshev polynomials. Finally, we have shown how Chebyshev-like polynomials can be used in practice with examples.

Keywords and Phrases: Generalized Chebyshev polynomials, Recurrences Relation, Binet-Like Formula, Matrix Representation, Characteristic Equations.

2020 Mathematics Subject Classification: 11C08, 11B37, 11B39.

1. Introduction

Chebyshev polynomials occupy prominent attention because of their substantial use in mathematics. This study is very useful in the theoretical as well as practical aspects of mathematics like in approximation theory. The authors Gultekin and Betul Sakiroglu, conducted a study on the analysis of Chebyshev generalized polynomials forms using matrixes and combination forms [14]. Akmak and Uslu, developed a generalized version of all four Chebyshev polynomials. Additionally, they demonstrated a Binet-style formula and demonstrated the relationship between the four known types of the Chebyshev polynomials and the generalized version of the Chebyshev polynomials [6]. Abd-Elhameed and Al-Harbi, primarily concerned with the generalization of Chebyshev's third-kind polynomials, with contributions from different perspectives. Additionally, some new formulas were discussed [2]. In order to gain new insights into the properties of Lucas-polynomials, Abd-Elhameed and Napoli explored different approaches to obtaining results. Matrix representation was also discussed in order to identify certain properties of the polynomials [1]. Semaa et al. studied the second type of Chebyshev polynomials for generalization was used to solve differential equation with their higher order. The authors presented very promising results with the examples of this generalization. The authors reduced the actual differential equation to the solution of algebraic equations with computing programs [9]. Cesarano introduced generalizations for the first kind of Chebyshev polynomials using Hermite polynomials, which served as integral representations for the generalized Chebyshev polynomials [12]. Wituła and Słota, presented a variation of the Chebyshev polynomials with novel outcomes and applications related to the Morgan-Voyce polynomials [47]. Bilgici research on generalizing new sequences was based on the relationship between the recurrence relation with the basic conditions. They were able to obtain Binet's formula and the generating function for these sequences [10]. The authors Verma and Priyanka, used first-order derivatives to derive a generalized version of Fibonacci polynomials. Additionally, they discussed Fibonacci polynomials with dual variable [46].

Uygun et al. proposed a generalized version of some of the polynomials names; Pell, Pell Lucas, and Pell-Vieta. They identified properties such as sum formula, generating function, Binet-like formula, and differentiation, as well as generating a matrix whose values were extracted from a generalized version of the Vieta- Pell -Lucas's polynomials [44]. Djordjevi´c, conducted a series of studies on the various categories of polynomials associated with Chebyshev's polynomials and the derived results associated with them [13]. Alqudah used the Bernstein base to came up with a new way to look at Chebyshev's second-kind polynomials and used obtained results in the approximation of functions [5]. Kizilates et al. instigates (p, q) first and second kind of Chebyshev polynomials in to Fibonacci, Luca's polynomials. They also talked about n^{th} generalizations and properties of derivatives, which are represented by determinants of the polynomials [32]. Abd-Elhameed et al. objective of the paper was to evolve the connection between generalized types of Lucas and Fibonacci polynomials. Hypergeometric functions were employed by authors to link the well-known polynomials, such as Lucas, Pell, Fermat, and Fermat Lucas [3]. Abed modified the first-type Chebyshev polynomials and employed the variable separation approach to solve the partial differential equations for diffusion [4]. Marchi et al. familiarize themselves with the generalizations of the first-kind Chebyshev polynomials and identify a number of properties associated with orthogonal polynomials [33]. Sarhan et al. came up with a new way to solve Pell's polynomials problem using two variables. A significant observation was made in two variables to resolve PDE. These techniques are employed to resolve the underlying issue with error-free calculations. This paper also provides examples demonstrating the rationale of the strategy [40]. Soykan, delved into a generalized form of the Fibonacci numbers and presented Simson's generating function formulas derived from matrix results, as well as calculating the infinitesimal sums of these polynomials [41]. Nemaniy et al., in their study, established the sophisticated properties of the Fibonacci sequence. Their findings concerned the divisibility of Fibonacci sequences and the representation of matrices by determinants including sequence terms [35]. Bychkov and Shabat, their research focused on the generalizations of the Catalan numbers and the generalizations of Chebyshev's polynomials [11]. The main focus of Anna Tatarczak study was to generalize the Chebyshev polynomials of two distinct types and to presented some prominent results demonstrating the relationship between these two types [43] (see also: [29] Karaatli and Kesk). Ali1 et al., explained the partial diff. of space fractional using the 5th kind of shifted Chebyshev polynomials [7]. Adem and Sahin, utilized a generalized form of the Fibonacci numbers to yield remarkable results with a recurrence relationship for square pyramids numbers [38]. Salih and Shihab, primary objective of the research was to identify a variant of Chebyshev polynomials. Furthermore, the authors discussed their integration, derivative operational matrix, and estimation techniques to address the issue of optimal control [39]. Using the Dilcher-Stolarsky approach for their study, Kim modified the second Chebyshev polynomials in various ways to determine the properties such as irreducibility and zero distribution [30]. Various modifications have been made to the Fibonacci sequence and the Lucas sequence, in some cases by maintaining the original conditions and in other cases by maintaining the recurrence relationship by Musraini et al. [34].

Kim and Lee [31] explored the analytical and algebraic characteristics of difference quotients associated with Chebyshev polynomials of the first kind. Verma et al. [45] studied about some identities involving Chebyshev polynomial of third kind, Lucas numbers, fourth kind, and Fibonacci numbers. Hong et al. [15] extend Edgar's identity relating Fibonacci and Lucas numbers, building upon previous work by Benjamin–Quinn and Marques. Dafnis's has also presented a similar identity recently. Here, they contribute further generalizations to both Edgar's and Dafnis's identities. The variety of studies have delved into the intrinsic characteristics of Chebyshev polynomials, particularly in connection with Fibonacci and Lucas numbers. Specifically, the references highlight significant research contributions by Kim and his team (see [16], [17], [18], [19], [20], [21], [22]) as well as notable work by Dolgy et al. (see [23]). Additionally, other relevant papers on these polynomials and sequences are listed in references (see [24], [25], [26]). Aruldoss and Devi [8] studied operational matrix for fractional integration based on Chebyshev wavelets. This matrix proves to be a valuable tool in addressing multi-order fractional differential equations, offering the significant advantage of transforming any fractional differential equation into a set of algebraic equations. The effectiveness, simplicity, and suitability of this approach are demonstrated through various numerical examples. Panwar, Mansuri and Bhandari [36] explored the summation of $(s+1)$ consecutive terms from bivariate Fibonacci polynomials and bivariate Lucas's polynomials, unveiling associated identities for both even and odd terms. The focus is on presenting two by two matrices and uncovering noteworthy properties, particularly those related to the n^{th} power of the matrix. Binet's formula is utilized to derive these identities. Rathore, Sisodiya and Panwar [37] generalized the matrix sequence termed as the $(s,t)-$ Pell matrix sequence and extending the concepts from both the $(s,t)-$ Pell matrix sequence and the $(s,t)-$ Pell-Lucas's matrix sequence. Various properties of the generalized $(s,t)-$Pell matrix sequence is outlined, and connections between the $(s,t)-$ Pell matrix and $(s,t)-$ Pell-Lucas's matrix sequences are established. In [42] Swamy, Nirmala, and Sailaja introduced specific families of holomorphic and Al-Oboudi type bi-univalent functions that are associated with $(m,n)-$ Lucas's polynomials.

The main goals of the article are outlined below:
In this paper, we have proposed a new generalized version of the Chebyshev kind of polynomials. We have addressed the determinant representation of this generalized version with its characteristic equation, as well as the Binet-like formulas and the practical application of generalized polynomials in the approximation of the functions.

2. Generalized Chebyshev polynomials

In this section, we are going to introduce the new generalizations of Chebyshev-like polynomials and also derive a characteristic equation for generalized Chebyshev polynomials. We will also talk about sum, product, subtraction, and sum of squares of roots. The Chebyshev generalized polynomials is defined by the following recurrence relation:

$$R_n(x) = uxR_{n-1}(x) + vxR_{n-2}(x), n \geq 2. \tag{2.1}$$

With initial condition,

$$R_0(x) = 1, R_1(x) = rx - s. \tag{2.2}$$

Where u, v, r, s are integers.
The following are the few terms of $R_n(x)$ for $n = 2, 3, 4, 5$.

$$R_2(x) = ux(rx - s) + vx = urx^2 - uxs + vx. \tag{2.3}$$

$$R_3(x) = ux[ux(rx-s)+vx]+vx[rx-s] = u^2rx^3-u^2sx^2+uvx^2+vrx^2-vsx. \tag{2.4}$$

$$\begin{aligned} R_4(x) &= ux[u^2rx^3 - u^2sx^2 + uvx^2 + vrx^2 - vsx] + vx[urx^2 - uxs + vx] \\ &= u^3rx^4 - u^3sx^3 + u^2vx^3 + uvrx^3 - uvsx^2 + uvrx^3 - uvsx^2 + v^2x^2. \end{aligned} \tag{2.5}$$

$$\begin{aligned} R_5(x) &= ux[u^3rx^4 - u^3sx^3 + u^2vx^3 + uvrx^3 - uvsx^2 + uvrx^3 - uvsx^2 + v^2x^2] \\ &\quad + vx[u^2rx^3 - u^2sx^2 + uvx^2 + vrx^2 - vsx] \\ &\quad = u^4rx^5 - u^4sx^4 + u^3vx^4 + u^2vrx^4 - u^2vsx^3 + u^2vrx^4 - u^2vsx^3 + uv^2x^3 \\ &\quad + u^2vrx^4 - u^2vsx^3 + uv^2x^3 + v^2rx^3 - v^2sx^2. \end{aligned}$$

The characteristic equation of the generalized Chebyshev polynomials is:

$$E^n = uxE^{n-1} + vxE^{n-2},$$

$$E^2 - uxE - vx = 0. \tag{2.6}$$

$I_1(x) and I_2(x)$ denote the roots of the above equation and defined by;

$$I_1(x) = \frac{ux + \sqrt{u^2x^2 + 4vx}}{2}, \qquad I_2(x) = \frac{ux - \sqrt{u^2x^2 + 4vx}}{2}. \tag{2.7}$$

The sum and the product of the roots are obtained, respectively

$$I_1(x) + I_2(x) = ux, \tag{2.8}$$

and

$$I_1(x)I_2(x) == -vx. \tag{2.9}$$

Subtraction of the roots is given by:

$$I_1(x) - I_2(x) = \sqrt{u^2 + x^2 + 4vx}, \tag{2.10}$$

and sum of the square of the roots provides the relation:

$$I_1^2(x) + I_2^2(x) = u^2x^2 + 2vx. \tag{2.11}$$

2.1. Binet Formula for Generalized Chebyshev polynomials

In this subsection we obtain the Binet formula for generalized Chebyshev polynomials.
Let the general solution of above equation is;

$$R_n(x) = Z_1 {I_I}^{n+1} + Z_2 {I_2}^{n+1}. \tag{2.12}$$

To find

$$\begin{aligned} &Z_1, Z_2; \\ &\quad 1 = I_1(x). \\ &\quad rx - s = I_2(x). \end{aligned} \tag{2.13}$$

$$1 = Z_1 I_1(x) + Z_2 I_2(x). \tag{2.14}$$

$$rx - s = Z_1 I_1^2(x) + Z_2 I_2^2(x). \tag{2.15}$$

Multiply (2.14) by $I_1(x)$,

$$I_1(x) = Z_1 I_1(x) I_1(x) + Z_2 I_1(x) I_2(x). \tag{2.16}$$

Now subtract (2.16) and (2.15);

$$I_1(x) - (rx - s) = Z_2[I_1(x) I_2(x) - I_2^2(x)]. \tag{2.17}$$

$$Z_2 = \frac{I_1(x) - (rx - s)}{[I_1(x) I_2(x) - I_2^2(x)]},$$

$$Z_2 = \frac{I_1(x) - (rx - s)}{I_2(x)[I_1(x) - I_2(x)]}. \tag{2.18}$$

Now use the value of Z_2 in (2.14),

$$1 = Z_1 I_1(x) + \frac{I_1(x) - (rx - s)}{I_2(x)[I_1(x) - I_2(x)]} I_2(x),$$

$$1 = Z_1 I_1(x) + \frac{I_1(x) - (rx - s)}{[I_1(x) - I_2(x)]},$$

$$Z_1 I_1(x) = 1 - \frac{(I_1(x) - (rx - s)}{[I_1(x) - I_2(x)]}),$$

$$Z_1 = \frac{(rx - s) - I_2(x)}{I_1(x)[I_1(x) - I_2(x)]}, \tag{2.19}$$

Now use the values of Z_1, Z_2 in (2.12)

$$R_n(x) = \frac{(rx-s) - I_2(x)}{I_1(x)[I_1(x) - I_2(x)]} I_1^{(n+1)}(x) + \frac{I_1(x) - (rx-s)}{I_2(x)[I_1(x) - I_2(x)]} I_2^{(n+1)}(x). \tag{2.20}$$

Now further its solutions can be modified;

$$R_n(x) = \frac{(rx-s) - I_2(x)}{[I_1(x) - I_2(x)]} I_1^{n}(x) + \frac{I_1(x) - (rx-s)}{[I_1(x) - I_2(x)]} I_2^{n}(x), \tag{2.21}$$

$$(rx-s) - I_2(x) = \frac{2rx - 2s - ux + \sqrt{u^2x^2 + 4vx}}{2}, \tag{2.22}$$

$$I_1(x) - I_2(x) = \sqrt{u^2x^2 + 4vx}, \tag{2.23}$$

$$I_1(x) - (rx-s) = \frac{ux - 2rx + 2s + \sqrt{u^2x^2 + 4vx}}{2}, \tag{2.24}$$

Hence,

$$\begin{aligned} R_n(x) = \frac{1}{2\sqrt{u^2x^2 + 4vx}} &[(2rx - 2s - ux + \sqrt{u^2x^2 + 4vx}) I_1^n(x) \\ &+ (ux - 2rx + 2s + \sqrt{u^2x^2 + 4vx}) I_2^n(x)]. \end{aligned} \tag{2.25}$$

If we put

$$H = \sqrt{u^2x^2 + 4vx}, \tag{2.26}$$

we get the Binet Formula for generalized Chebyshev polynomials as

$$R_n(x) = \frac{1}{2H}[(2rx - 2s - ux + H) I_1^n(x) + (ux - 2rx + 2s + H) I_2^n(x)]. \tag{2.27}$$

3. Generalized Chebyshev's Polynomials Through Matrix Representation

In this section, we explore a matrix-based approach to generalized the representation of Chebyshev polynomials (GCPs). The generalized Chebyshev polynomials are defined by the recurrence relation

$$\begin{aligned} &R_0(x) = 1, \qquad R_1(x) = rx - s, \\ &R_n(x) = uxR_{n-1}(x) + vxR_{n-2}(x), n \geq 2. \end{aligned} \tag{3.1}$$

$[O_{b,c}]$ define the tri-diagonal matrix succession for the generalized Chebyshev polynomials as follows:

$$[O_{b,c}] = \begin{cases} O_{b,c} = rx - s, & if \quad b = c = 1 \\ O_{b,c} = ux, & if \quad b = c \geq 2 \\ O_{b,c} = -vx, & if \quad b = c - 1 \\ O_{b,c} = 1, & if \quad b = c + 1 \\ O_{b,c} = 0, & otherwise \end{cases} \tag{3.2}$$

In general, the determinant representation is given by:

$$|k(n)| = \begin{vmatrix} rx - s & 1 & 0 & \dots & \dots & 0 \\ -vx & ux & 1 & \dots & \dots & 0 \\ 0 & -vx & ux & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & ux & 1 \\ 0 & 0 & \dots & \dots & -vx & ux \end{vmatrix}. \tag{3.3}$$

The $|k(n)|$ denotes the determinant of Chebyshev matrices $R_n(x)$ and given by:

$$|k(1)| = O_{1,1} = rx - s = R_1(x). \tag{3.4}$$

$$\begin{aligned} |k(2)| &= O_{1,1}O_{2,2} - O_{2,1}O_{1,2} \\ &= Det \begin{pmatrix} rx - s & 1 \\ -vx & ux \end{pmatrix} \\ &= urx^2 - uxs + vx \\ &= R_2(x). \end{aligned} \tag{3.5}$$

$$\begin{aligned} |k(3)| &= O_{3,3}|K(2)| - O_{3,2}O_{2,3}|K(1)| \\ &= ru^2x^3 + rvx^2 + uvx^2 - u^2x^2s - svx \\ &= Det \begin{pmatrix} rx - s & 1 & 0 \\ -vx & ux & 1 \\ 0 & -vx & ux \end{pmatrix} \\ &= R_3(x). \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
|k(4)| &= O_{4,4}|k(3)| - O_{4,3}O_{3,4}|k(2)| \\
&= Det\begin{pmatrix} rx-s & 1 & 0 & 0 \\ -vx & ux & 1 & 0 \\ 0 & -vx & ux & 1 \\ 0 & 0 & -vx & ux \end{pmatrix} \\
&= u^3rx^4 - u^3sx^3 + u^2vx^3 + uvrx^3 - uvsx^2 \\
&+ uvrx^3 - uvsx^2 + v^2x^2 \\
&= R_4(x).
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
|k(5)| &= O_{5,5}|k(4)| - O_{5,4}O_{4,5}|k(3)| \\
&= Det\begin{pmatrix} rx-s & 1 & 0 & 0 & 0 \\ -vx & ux & 1 & 0 & 0 \\ 0 & -vx & ux & 1 & 0 \\ 0 & 0 & -vx & ux & 1 \\ 0 & 0 & 0 & -vx & ux \end{pmatrix} \\
&= u^4rx^5 - u^4sx^4 + u^3vx^4 + u^2vrx^4 - u^2vsx^3 \\
&+ u^2vrx^4 - u^2vsx^3 + uv^2x^3 + u^2vrx^4 \\
&- u^2vsx^3 + uv^2x^3 + v^2rx^3 - v^2sx^2 \\
&= R_5(x).
\end{aligned} \tag{3.8}
$$

In general, we have

$$
\begin{aligned}
|k(n)| &= O_{n,n}|k(n-1)| - O_{n,n-1}O_{n-1,n}|k(n-2)| \\
&= Det\begin{pmatrix} rx-s & 1 & 0 & ... & ... & 0 \\ -vx & ux & 1 & ... & ... & 0 \\ 0 & -vx & ux & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ux & 1 \\ 0 & 0 & ... & ... & -vx & ux \end{pmatrix} \\
&= R_n(x).
\end{aligned} \tag{3.9}
$$

3.1. Characteristic Equations of the Generalized Chebyshev Polynomials

Here we obtain the characteristic equations for the generalized Chebyshev polynomials up to fifth degree.

1. $\lambda - R_1 = 0$.
2. $\lambda^2 - (rx + ux)\lambda + R_2 = 0$.
3. $\lambda^3 - (rx - s + 2ux)\lambda^2 + (rux^2 + 2sux - u^2x^2 - 2vx)\lambda - R_3 = 0$.

$$\begin{aligned}
&4.\ \lambda^4+\lambda^3(-rx+s-3ux)+\lambda^2(3ux^2r-3sux+2ux^2+u^2x^2+2vx+ux)+\\
&\lambda(-3u^2x^3r+3u^2x^3s+vxs+uxs-u^3x^3-4uvx^2-rvx^2-ux^2r)+R_4=0.\\
&5.\ \lambda^5+\lambda^4(4ux+rx-s)-\lambda^3(-4usx^2+4vxs-6x^2u^2-vs-2vx-ux)\\
&-\lambda^2((6u^2x^3r+2vx^2r-6u^2x^2s-2vxs+4u^3x^3+8uvx^2+u^2x^2+ux^2r-uxs)\\
&-\lambda(-4u^3x^4r-4uvx^3r-u^2x^3r+3u^3x^3s+5uvx^2s+u^2x^2s\\
&-u^4x^4-4vu^2x^3-3v^2x^2-uvx^4+u^3x^3-u^2x^2v)-R_5=0.
\end{aligned} \tag{3.10}$$

4. Practical Applications

Express: x^3-3x^2+2x+3 in to generalized Chebyshev polynomials.
Solution. Generalized Chebyshev polynomials are defined by:

$$R_n(x)=uxR_{n-1}(x)+vxR_{n-2}(x), n\geq 2;$$

with initial condition;

$$R_0(x)=1, \qquad R_1(x)=rx-s.$$

Now from above generalized recurrence relation, we get

$$\begin{aligned}
&R_0(x)=1, \quad R_1(x)=rx-s, \quad R_2(x)=urx^2-uxs+vx,\\
&R_3(x)=u^2rx^3-u^2sx^2+uvx^2+vrx^2-vsx.
\end{aligned} \tag{4.1}$$

From above series we have to find out the values of x, x^2, x^3.

$$\begin{aligned}
x&=\frac{1}{r}[R_1(x)+s].\\
x^2&=\frac{1}{ur^2}[R_2(x)r+suR_1(x)+su-vR_1(x)+vs].\\
x^3&=\frac{1}{u^2r^2}[R_3(x)r^3+usR_2(x)r^2+u^2s^2R_1(x)r+u^2s^2r-vR_1(x)sr\\
&+uvs^2r-vR_2(x)r^2-vsuR_1(x)r-uvsr+v^2R_1(x)r-vs^2r^2\\
&-2vsR_1(x)r^2-v^2sr-vsr^2+v^2R_1(x)u-v^2su-vR_2(x)ru].
\end{aligned} \tag{4.2}$$

Put the above values of x, x^2, x^3 in the given polynomial, we get

$$\begin{aligned}
&\frac{1}{u^2r^2}[R_3(x)r^3+usR_2(x)r^2+u^2s^2R_1(x)r+u^2s^2r-vR_1(x)sr\\
&+uvs^2r-vR_2(x)r^2-vsuR_1(x)r-uvsr+v^2R_1(x)r-vs^2r^2\\
&-2vsR_1(x)r^2-v^2sr-vsr^2+v^2R_1(x)u-v^2su-vR_2(x)ru]\\
&-\frac{3}{ur^2}[R_2(x)r+suR_1(x)+su-vR_1(x)+vs]+\frac{2}{r}[R_1(x)+s]\\
&+3R_0(x).
\end{aligned} \tag{4.3}$$

Express: $7x^3 - 2x^2$ in to generalized Chebyshev polynomials.
Solution. We know that,
Now from above generalized recurrence relation, we get

$$\begin{aligned} R_0(x) &= 1. \\ R_1(x) &= rx - s. \\ R_2(x) &= urx^2 - uxs + vx. \\ R_3(x) &= u^2rx^3 - u^2sx^2 + uvx^2 + vrx^2 - vsx. \end{aligned} \tag{4.4}$$

From above, we get the values of the x, x^2, x^3.

$$\begin{aligned} x &= \frac{1}{r}[R_1(x) + s]. \\ x^2 &= \frac{1}{ur^2}[R_2(x)r + suR_1(x) + su - vR_1(x) + vs]. \\ x^3 &= \frac{1}{u^2r^2}[R_3(x)r^3 + usR_2(x)r^2 + u^2s^2R_1(x)r + u^2s^2r \\ &- vR_1(x)sr + uvs^2r - vR_2(x)r^2 - vsuR_1(x)r - uvsr \\ &+ v^2R_1(x)r - vs^2r^2 - 2vsR_1(x)r^2 - v^2sr - vsr^2 \\ &+ v^2R_1(x)u - v^2su - vR_2(x)ru]. \end{aligned} \tag{4.5}$$

Put the above values of x, x^2, x^3 in the given polynomial, we get

$$\begin{aligned} & R_3(x)\frac{7r^3}{u^2r^2} + R_2(x)[\frac{7usr^2}{u^2r^2} - \frac{7vr^2}{u^2r^2} - \frac{7uvr}{u^2r^2} - \frac{2r}{ur^2}] \\ & + R_1(x)[\frac{7r}{u^2r^2} - \frac{7vsr}{u^2r^2} - \frac{7vsur}{u^2r^2} + \frac{7v^2r}{u^2r^2} - \frac{14vsr^2}{u^2r^2} \\ & + \frac{7v^2u}{u^2r^2} - \frac{2su}{ur^2} - \frac{-2v}{ur^2}]. \end{aligned} \tag{4.6}$$

5. Conclusion

We have come up with a new way of looking at Chebyshev-like polynomials that have three term recurrence relations. We have already looked at the generalized version using matrix algebra. In the future, we will be looking at different types of modified and generalized Chebyshev types. We will be looking at them from a different angle, and we will be using matrix algebra for some basic properties.

6. Significance of the work

Following are some key points that summarize the importance of the current research:

• To obtain the generalized version of Chebyshev polynomials helps to know more about the hidden factors of Chebyshev polynomials.
• The Binet formula for the generalized version of Chebyshev polynomials presented the explicit form the current version.
• To discuss the general nature of generalized Chebyshev polynomials with characteristic equation and its roots; like sum, products, subtraction, and sum of square of roots.
• Via matrix representation of generalized Chebyshev polynomials we can apply matrix algebra to obtain more properties.
• Eigen values and Eigen vectors can be obtained with the aid of the characteristic equations of generalized Chebyshev polynomials.
• In approximation theory, generalized Chebyshev polynomials are helpful for approximating other polynomials.
• These polynomials can be used to calculate lower order approximations.

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