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NEW TYPE OF DEGENERATE CHANGHEE POLYNOMIALS OF THE SECOND KIND

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Abstract: In this paper, we consider the new type of degenerate Changhee numbers and polynomials of the second kind which are different from the previously introduced degenerate Changhee numbers and polynomials of the second kind by Kim-Kim. We investigate some properties of these numbers and polynomials. In addition, we give some new relations between the new type of degenerate Changhee polynomials of the second kind and the Carlitz's degenerate Euler polynomials.

Keywords and Phrases: Degenerate Changhee polynomials and numbers of the second, new type degenerate Changhee polynomials of the second kind, Stirling numbers.

2020 Mathematics Subject Classification: 11B83, 11B73, 11S80.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the filed of p-adic rational numbers and the completion of an algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\ .\ |_p$ is normalized by $|\ p\ |_p = \frac{1}{p}$. Let $C(\mathbb{Z}_p)$ be the space of continuous function on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows

$$I(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x + p^N\mathbb{Z}_p)$$

$$= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{(see [3, 4, 5, 30])} \tag{1.1}$$

From (1.1), we note that

$$I(f_n)+(-1)^{n-1}I(f) = 2\sum_{a=0}^{n-1}(-1)^{n-1-a}f(a), \text{(see [5, 6, 20-24, 33, 28, 9, 33, 30, 31])}, \tag{1.2}$$

where $f_n(x) = f(x+n), (n \in \mathbb{N})$.

Let the Changhee polynomials are defined by the generating function as follows (see [20, 21])

$$\frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}. \tag{1.3}$$

Letting $x = 0$, $Ch_n = Ch_n(0), (n \geq 0)$ are called the Changhee numbers. From (1.3), we note that

$$\int_{\mathbb{Z}_p} (1+t)^{x+y} d\mu_{-1}(y) = \frac{2}{2+t}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}. \tag{1.4}$$

Thus, by (1.4), we have

$$\int_{\mathbb{Z}_p} (x+y)_n d\mu_{-1}(y) = Ch_n(x), (n \geq 0), \text{(see [34, 38])}, \tag{1.5}$$

where $(x)_0 = 1, (x)_n = x(x-1)\cdots(x-n+1), (n \geq 1)$,

As well known, the Euler polynomials are defined by

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.6}$$

In the case $x = 0$, $E_n = E_n(0)$ are called the Euler numbers.

By using (1.1) and (1.6), we note that

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \text{(see [7-14, 25-28])}. \tag{1.7}$$

By (1.7), we get

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x), (n \geq 0), \text{(see [20, 21])}, \tag{1.8}$$

For $n \geq 0$, the Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1 n, l x^l, \text{(see [1-7, 20-26, 30])}. \tag{1.9}$$

The Stirling numbers of the second kind are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, (n \geq 0), \text{(see [14-19, 29-38])}. \tag{1.10}$$

From (1.3), (1.6), (1.9) and (1.10), we get

$$Ch_n(x) = \sum_{l=0}^{n} E_l(x) S_1(n,l), \tag{1.11}$$

and

$$E_n(x) = \sum_{l=0}^{n} Ch_n(x) S_2(n,l), \text{(see [20, 21])} \tag{1.12}$$

For any $\lambda \in \mathbb{R}$, degenerate version of the exponential function $e_\lambda^x(t)$ is defined as follows (see [18-27])

$$e_\lambda^x(t) := (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \text{(see, [7, 22-65, 12-19, 29-34])} \tag{1.13}$$

It follows from (1.13) is $\lim_{\lambda \to 0} e_\lambda^x(t) = e^{xt}$. Note that $e_\lambda^1(t) := e_\lambda(t)$.

For $n \geq 0$, the degenerate Stirling numbers of the first kind (see [33]) are defined by

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0). \tag{1.14}$$

Note that $\lim_{\lambda \to 0} S_{1,\lambda}(n,k) = S_1(n,k)$, where $S_1(n,k)$ are called the Stirling numbers of the first kind.

Kim introduced the degenerate Stirling numbers of the second kind (see [32]) are given by

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k) \frac{t^n}{n!}, \quad (k \geq 0). \tag{1.15}$$

It is clear that $\lim_{\lambda \to 0} S_{2,\lambda}(n,k) = S_2(n,k)$, where $S_2(n,k)$ are called the Stirling numbers of the second.

The degenerate Euler polynomials are defined by (see [22, 23])

$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+y}{\lambda}}d\mu_{-1}(y)=\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}$$

$$=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.16}$$

From (1.16), we note that

$$\int_{\mathbb{Z}_p}(x+y)_{n,\lambda}d\mu_{-1}(y)=\mathcal{E}_{n,\lambda}(x),(n\geq 0).$$

Recently, Kim [31] introduced the degenerate Changhee polynomials of the second kind are defined by

$$\int_{\mathbb{Z}_p}(1+\lambda\log(1+t))^{\frac{x+y}{\lambda}}d\mu_{-1}(y)=\frac{2}{1+(1+\lambda\log(1+t))}(1+\lambda\log(1+t))^{\frac{x}{\lambda}}$$

$$=\sum_{n=0}^{\infty}Ch_{n,\lambda}(x)\frac{t^n}{n!}, \tag{1.17}$$

where $\lambda\in\mathbb{C}_p$ with $|\lambda|_p\leq 1$.

When $x=0$, $Ch_{n,\lambda}=Ch_{n,\lambda}(0)$ are called the degenerate Changhee numbers of the second kind.

This paper is organized as follows. In sect 2, we study new type of degenerate Changhee numbers and polynomials of the second kind and investigate some properties of these numbers and polynomials. In sect 3, we introduce higher-order new type of degenerate Changhee polynomials and numbers of the second kind and we derive their explicit expressions and some other polynomials. Moreover, we obtain identities involving those polynomials and some other special numbers and polynomials.

2. New type of degenerate Changhee polynomials of the second kind

In this section, we introduce new type of degenerate Changhee polynomials of the second and investigate some properties of these polynomials which are derived from the fermionic p-adic integral on $\mathbb{Z}_p$.

For $\lambda\in\mathbb{R}$, the degenerate logarithm function $\log_\lambda(1+t)$, which is the compositional inverse of the degenerate exponential function $e_\lambda(t)$ and the motivation for the definition of degenerate polylogarithm function, as follows (see [36])

$$\log_\lambda(1+t)=\sum_{n=1}^{\infty}\lambda^{n-1}(1)_{n,1/\lambda}\frac{t^n}{n!}=\frac{1}{\lambda}\sum_{n=1}^{\infty}(\lambda)_n\frac{t^n}{n!}=\frac{1}{\lambda}\left((1+t)^\lambda-1\right). \tag{2.1}$$

Note that

$$\lim_{\lambda\to 0}\log_\lambda(1+t)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^n}{n!}=\log(1+t).$$

We start following definition as follows.

For $\lambda\in\mathbb{C}_p$ with $\mid\lambda\mid_p\leq 1$. Now, we define the new type of degenerate Changhee polynomials of the second kind by

$$\int_{\mathbb{Z}_p}(1+\lambda\log_\lambda(1+t))^{\frac{x+y}{\lambda}}d\mu_{-1}(y)=\frac{2}{1+(1+\lambda\log_\lambda(1+t))}(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}$$
$$=\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.2}$$

Note that, $\lim_{\lambda\to 0}\tilde{Ch}_{n,\lambda}(x)=Ch_n(x),\quad(n\geq 0)$, (see [20, 21]). We note that $x=0,\tilde{Ch}_{n,\lambda}=\tilde{Ch}_{n,\lambda}(0)$ are called the new type of degenerate Changhee numbers of the second kind.

Theorem 2.1. *For $n\geq 0$, we have*

$$\tilde{Ch}_{n,\lambda}(x)=\sum_{l=0}^{n}S_{1,\lambda}(n,l)\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l}l!d\mu_{-1}(y)\lambda^l$$
$$=\sum_{l=0}^{n}\int_{\mathbb{Z}_p}(x+y)_{l,\lambda}d\mu_{-1}(y)\lambda^l S_{1,\lambda}(n,l), \tag{2.3}$$

where $(x)_{0,\lambda}=1$ and $(x)_{n,\lambda}=x(x-\lambda)\cdots(x-(n-1)\lambda)$ for $n\geq 1$.

Proof. Using (2.2), we note that

$$\int_{\mathbb{Z}_p}(1+\lambda\log_\lambda(1+t))^{\frac{x+y}{\lambda}}d\mu_{-1}(y)=\sum_{l=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l}d\mu_{-1}(y)\lambda^l(\log_{\lambda(1+t)})^l$$
$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}S_{1,\lambda}(n,l)\int_{\mathbb{Z}_p}\binom{\frac{x+y}{\lambda}}{l}l!d\mu_{-1}(y)\lambda^l\right)\frac{t^n}{n!}. \tag{2.4}$$

Comparing the coefficients of on both sides of (2.2) and (2.4), we obtain the result (2.3).

Theorem 2.2. *For $n \geq 0$, we have*

$$E_{n,\lambda}(x) = \sum_{m=0}^{n} \widehat{Ch}_{m,\lambda}(x) S_{2,\lambda}(n,m). \tag{2.5}$$

Proof. By replacing t by $e_\lambda(t) - 1$ in (2.2), we get

$$\sum_{m=0}^{\infty} \widehat{Ch}_{m,\lambda}(x) \frac{(e_\lambda(t)-1)^m}{m!} = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}} + 1}(1+\lambda t)^{\frac{x}{\lambda}}$$

$$= \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!}. \tag{2.6}$$

On the other hand,

$$\sum_{m=0}^{\infty} \widehat{Ch}_{m,\lambda}(x) \frac{(e_\lambda(t)-1)^m}{m!} = \sum_{m=0}^{\infty} \widehat{Ch}_{m,\lambda}(x) \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \widehat{Ch}_{m,\lambda}(x) S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}. \tag{2.7}$$

By (2.6) and (2.7), we get the result.

Theorem 2.3. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\lambda}(x) = \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} (x)_{m,\lambda} S_{1,\lambda}(k,m) \widehat{Ch}_{n-k,\lambda}. \tag{2.8}$$

Proof. From (2.2), we note that

$$\sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda}(x) \frac{t^n}{n!} = \frac{2}{1 + (1 + \lambda \log_\lambda(1+t))}(1 + \lambda \log_\lambda(1+t))^{\frac{x}{\lambda}}$$

$$= \left(\sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda} \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} \binom{\frac{x}{\lambda}}{m} (\log_\lambda(1+t))^m \right)$$

$$= \left(\sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda} \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} (x)_{m,\lambda} \sum_{k=m}^{\infty} S_{1,\lambda}(k,m) \frac{t^k}{k!} \right)$$

$$= \left(\sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda} \frac{t^n}{n!} \right) \left(\sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} (x)_{m,\lambda} S_{1,\lambda}(k,m) \right) \frac{t^k}{k!} \right)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} (x)_{m,\lambda} S_{1,\lambda}(k,m) \widehat{Ch}_{n-k,\lambda} \right) \frac{t^n}{n!}. \tag{2.9}$$

Therefore, by (2.2) and (2.9), we obtain at the required result.

Theorem 2.4. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\lambda}(1) + \widehat{Ch}_{n,\lambda} = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \geq 1, \end{cases} \tag{2.10}$$

Proof. By (1.2), we easily get

$$\int_{\mathbb{Z}_p} f(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2f(0). \tag{2.11}$$

Now, equation (2.11) can be written as

$$\int_{\mathbb{Z}_p} (1 + \lambda \log_\lambda(1+t))^{\frac{x+1}{\lambda}} d\mu_{-1}(x) + \int_{\mathbb{Z}_p} (1 + \lambda \log_\lambda(1+t))^{\frac{x}{\lambda}} d\mu_{-1}(x) = 2. \tag{2.12}$$

From (2.2) and (2.12), we have

$$\frac{2}{1 + (1 + \lambda \log_\lambda(1+t))} (1 + \lambda \log_\lambda(1+t))^{\frac{x}{\lambda}} + \frac{2}{1 + (1 + \lambda \log_\lambda(1+t))} = 2. \tag{2.13}$$

From (2.2) and (2.13), we have

$$\sum_{n=0}^{\infty} \left(\widehat{Ch}_{n,\lambda}(1) + \widehat{Ch}_{n,\lambda} \right) \frac{t^n}{n!} = 2. \tag{2.14}$$

In view of (2.14), we complete the proof.

Theorem 2.5. *For $n \geq 0$, $d \in \mathbb{N}$ with $d \equiv 1$(mod 2) we have*

$$\widehat{Ch}_{n,\lambda} = \sum_{m=0}^{n} d^m \sum_{a=0}^{d-1} (-1)^a E_{m,\frac{d}{\lambda}} S_{1,\lambda}(n,m). \tag{2.15}$$

Proof. For $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$, by (1.2), we have

$$\int_{\mathbb{Z}_p} f(x+d) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2\sum_{a=0}^{d-1} (-1)^a f(a). \tag{2.16}$$

Let us take $f(x) = (1+\lambda \log_\lambda(1+t))^{\frac{x}{\lambda}}$. Then by (2.16), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+\lambda \log_\lambda(1+t))^{\frac{x}{\lambda}} d\mu_{-1}(x) &= \frac{2}{(1+\lambda\log_\lambda(1+t))^{\frac{d}{\lambda}}+1} \sum_{a=0}^{d-1} (-1)^a (1+\lambda\log_\lambda(1+t))^{\frac{a}{\lambda}}. \\ &= \sum_{a=0}^{d-1} (-1)^a \frac{2}{1+(1+\frac{\lambda}{d} d \log_\lambda(1+t))^{\frac{d}{\lambda}}} (1+\frac{\lambda}{d} d\log_\lambda(1+t))^{\frac{d}{\lambda}\frac{a}{d}}. \end{aligned} \tag{2.17}$$

By (1.16), we easily get

$$\begin{aligned} \frac{2}{1+(1+\frac{\lambda}{d} d\log_\lambda(1+t))^{\frac{d}{\lambda}}} (1+\frac{\lambda}{d} d\log_\lambda(1+t))^{\frac{d}{\lambda}\frac{a}{d}} &= \sum_{m=0}^{\infty} E_{m,\frac{d}{\lambda}} \left(\frac{a}{d}\right) \frac{d^m}{m!} (\log_\lambda(1+t))^m \\ &= \sum_{m=0}^{\infty} E_{m,\frac{d}{\lambda}} d^m \sum_{n=m}^{\infty} S_{1,\lambda}(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} d^m E_{m,\frac{d}{\lambda}} S_{1,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.18}$$

From (2.17) and (2.18), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda} \frac{t^n}{n!} &= \int_{\mathbb{Z}_p} (1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}} d\mu_{-1}(x) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} d^m \sum_{a=0}^{d-1} (-1)^a E_{m,\frac{d}{\lambda}} S_{1,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.19}$$

Thus, by (2.19), we complete the proof.

Theorem 2.6. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\lambda}(x+1) + \widehat{Ch}_{n,\lambda}(x) = 2\sum_{m=0}^{n} (x)_{m,\lambda} S_{1,\lambda}(n,m). \tag{2.20}$$

Proof. Suppose that

$$\frac{2}{1+(1+\lambda\log_\lambda(1+t))^{\frac{1}{\lambda}}}(1+\lambda\log_\lambda(1+t))^{\frac{x+1}{\lambda}}+\frac{2(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}}{1+(1+\lambda\log_\lambda(1+t))^{\frac{1}{\lambda}}}$$
$$=2(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}. \tag{2.21}$$

Thus, by (2.1) and (2.21), we get

$$\sum_{n=0}^{\infty}\left(\widehat{Ch}_{n,\lambda}(x+1)+\widehat{Ch}_{n,\lambda}(x)\right)\frac{t^n}{n!}$$
$$=2\sum_{m=0}^{\infty}(x)_{m,\lambda}\frac{1}{m!}(\log_\lambda(1+t))^m$$
$$=\sum_{n=0}^{\infty}\left(2\sum_{m=0}^{n}(x)_{m,\lambda}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}. \tag{2.22}$$

By comparing the coefficients of t, we get (2.20).

Theorem 2.7. *For $n\geq 0$, $d\in\mathbb{N}$ with $d\equiv 1$(mod 2), we have*

$$\widehat{Ch}_{n,\lambda}(d)+\widehat{Ch}_{n,\lambda}=2\sum_{a=0}^{d-1}(-1)^a\sum_{m=0}^{n}(a)_{m,\lambda}S_{1,\lambda}(n,m). \tag{2.23}$$

Proof. From (2.2), we have

$$\frac{2}{1+(1+\lambda\log_\lambda(1+t))^{\frac{1}{\lambda}}}(1+\lambda\log_\lambda(1+t))^{\frac{d}{\lambda}}+\frac{2}{1+(1+\lambda\log_\lambda(1+t))^{\frac{1}{\lambda}}}$$
$$=2\sum_{a=0}^{d-1}(-1)^a(1+\lambda\log_\lambda(1+t))^{\frac{a}{\lambda}}, \tag{2.24}$$

where $d\in\mathbb{N}$ with $d\equiv 1(\text{mod } 2)$.

By (2.2) and (2.24), we get

$$\sum_{n=0}^{\infty}\left(\widehat{Ch}_{n,\lambda}(d)+\widehat{Ch}_{n,\lambda}\right)\frac{t^n}{n!}$$
$$=2\sum_{a=0}^{d-1}(-1)^a\sum_{m=0}^{\infty}(a)_{m,\lambda}\frac{1}{m!}(\log_\lambda(1+t))^m$$

$$= 2\sum_{a=0}^{d-1}(-1)^a\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(a)_{m,\lambda}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(2\sum_{a=0}^{d-1}(-1)^a\sum_{m=0}^{n}(a)_{m,\lambda}S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}. \qquad (2.25)$$

Therefore, by (2.25), we obtain the result.

3. New type of higher-order degenerate Changhee polynomials of the second kind

In this section, we introduce new type of higher-order degenerate Changhee polynomials of the second kind which are derived from the multivariate fermionic p-adic integral on $\mathbb{Z}_p$.

For $r \in \mathbb{N}$, we define the new type of higher-order degenerate Changhee polynomials of the second kind which are given multivariate fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log_\lambda(1+t))^{\frac{x+x_1+\cdots+x_r}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$

$$\left(\frac{2}{1+(1+\lambda\log_\lambda(1+t))}\right)^r(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \qquad (3.1)$$

When $x=0, \widehat{Ch}_{n,\lambda}^{(r)} = \widehat{Ch}_{n,\lambda}^{(r)}(0)$ are called the new type of higher-order degenerate Changhee numbers of the second kind.

Theorem 3.1. *For $n \geq 0$ and $r \in \mathbb{N}$, we have*

$$\widehat{Ch}_{n,\lambda}^{(r)}(x) = \sum_{m=0}^{n}E_{m,\lambda}^{(r)}S_{1,\lambda}(n,m).$$

Proof. From (3.1), we note that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda\log_\lambda(1+t))^{\frac{x+x_1+\cdots+x_r}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$

$$= \sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\binom{\frac{x_1+\cdots+x_r+x}{\lambda}}{m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\lambda^m(\log_\lambda(1+t)))^m$$

$$= \sum_{m=0}^{\infty}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)_{\lambda,m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\frac{1}{m!}(\log_\lambda(1+t)))^m$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)_{\lambda,m}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)S_{1,\lambda}(n,m)\right)\frac{t^n}{n!}. \tag{3.2}$$

It is easy to show that

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1+\cdots+x_r}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$

$$= \left(\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}\right)^r(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}E_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}, \tag{3.3}$$

where $E_{n,\lambda}^{(r)}(x)$ are the Carlitz's degenerate Euler polynomials of order r.

Thus, by (3.3), we get

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+\cdots+x_r+x)_{\lambda,m}d\mu_{-1}(x_1)\cdots d\mu_{-1} = E_{n,\lambda}^{(r)}(x), (m\geq 0). \tag{3.4}$$

Therefore, by (3.2), (3.3) and (3.4), we obtain the result.

Theorem 3.2. *For $n\geq 0$, we have*

$$E_{n,\lambda}^{(r)}(x) = \sum_{m=0}^{n}\widehat{Ch}_{m,\lambda}^{(r)}(x)S_{2,\lambda}(n,m).$$

Proof. By changing t by $e_\lambda(t)-1$ in (3.1), we have

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+x_1+\cdots+x_r}{\lambda}}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$

$$= \sum_{m=0}^{\infty}\widehat{Ch}_{m,\lambda}^{(r)}(x)\frac{(e_\lambda(t)-1)^m}{m!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\widehat{Ch}_{m,\lambda}^{(r)}(x)S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}. \tag{3.5}$$

Therefore, by (3.3) and (3.5), we get the result.

Theorem 3.3. *For $n\geq 0$, we have*

$$\widehat{Ch}_{n,\lambda}^{(r)}(x) = \sum_{l=0}^{n}\binom{n}{l}\widehat{Ch}_{n-l,\lambda}^{(k)}\widehat{Ch}_{l,\lambda}^{(r-k)}(x).$$

Proof. From (3.1), we have

$$\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}=\left(\frac{2}{1+(1+\lambda\log_{\lambda}(1+t))^{\frac{1}{\lambda}}}\right)^{r}(1+\lambda\log_{\lambda}(1+t))^{\frac{x}{\lambda}}$$

$$=\left(\frac{2}{1+(1+\lambda\log_{\lambda}(1+t))^{\frac{1}{\lambda}}}\right)^{k}\left(\frac{2}{1+(1+\lambda\log_{\lambda}(1+t))^{\frac{1}{\lambda}}}\right)^{r-k}(1+\lambda\log_{\lambda}(1+t))^{\frac{x}{\lambda}}$$

$$=\left(\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda}^{(k)}\frac{t^n}{n!}\right)\left(\sum_{l=0}^{\infty}\widehat{Ch}_{l,\lambda}^{(r-k)}(x)\frac{t^l}{l!}\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\widehat{Ch}_{n-l,\lambda}^{(k)}\widehat{Ch}_{l,\lambda}^{(r-k)}(x)\right)\frac{t^n}{n!}. \tag{3.6}$$

In view of (3.5), we complete the proof.

Theorem 3.4. *For $n \geq 0$, we have*

$$\widehat{Ch}_{m,\lambda}^{(r)}=\sum_{k=0}^{n}\sum_{m=0}^{k}(-1)^m m!\binom{r+m-1}{m}2^{-m}S_{2,\lambda}(k,m)S_{1,\lambda}(n,k).$$

Proof. By (3.1), we have

$$\left(\frac{2}{1+(1+\lambda\log_{\lambda}(1+t))^{\frac{1}{\lambda}}}\right)^{r}=\left(\frac{(1+\lambda\log_{\lambda}(1+t))^{\frac{1}{\lambda}}-1}{2}+1\right)^{-r}$$

$$=\sum_{m=0}^{\infty}\binom{-r}{m}2^{-m}((1+\lambda\log_{\lambda}(1+t))^{\frac{1}{\lambda}}-1)^m$$

$$=\sum_{m=0}^{\infty}(-1)^m2^{-m}\binom{r+m-1}{m}m!\sum_{k=m}^{\infty}S_{2,\lambda}(k,m)\frac{1}{k!}(\log_{\lambda}(1+t))^k$$

$$=\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}(-1)^m m!\binom{r+m-1}{m}2^{-m}S_{2,\lambda}(k,m)\right)\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}(-1)^m m!\binom{r+m-1}{m}2^{-m}S_{2,\lambda}(k,m)S_{1,\lambda}(n,k)\right)\frac{t^n}{n!}. \tag{3.7}$$

In view of (3.1) and (3.7), we get the result.

Theorem 3.5. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\lambda}^{(r)}(x) = \sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} (x)_{m,\lambda} S_{1,\lambda}(k,m) \widehat{Ch}_{n-k,\lambda}^{(r)}.$$

Proof. From (3.1), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda}^{(r)}(x) \frac{t^n}{n!} &= \left(\frac{2}{1 + (1 + \lambda \log_\lambda(1+t))} \right)^r (1 + \lambda \log_\lambda(1+t))^{\frac{x}{\lambda}} \\
&= \left(\sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda}^{(r)} \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} \binom{\frac{x}{\lambda}}{m} (\log_\lambda(1+t))^m \right) \\
&= \left(\sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda}^{(r)} \frac{t^n}{n!} \right) \left(\sum_{m=0}^{\infty} (x)_{m,\lambda} \sum_{k=m}^{\infty} S_{1,\lambda}(k,m) \frac{t^k}{k!} \right) \\
&= \left(\sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda}^{(r)} \frac{t^n}{n!} \right) \left(\sum_{k=0}^{\infty} \left(\sum_{m=0}^{k} (x)_{m,\lambda} S_{1,\lambda}(k,m) \right) \frac{t^k}{k!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \sum_{m=0}^{k} \binom{n}{k} (x)_{m,\lambda} S_{1,\lambda}(k,m) \widehat{Ch}_{n-k,\lambda}^{(r)} \right) \frac{t^n}{n!}. \qquad (3.8)
\end{aligned}$$

Therefore, by (3.1) and (3.8), we obtain at the required result.

Theorem 3.6. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} \widehat{Ch}_{m,\lambda}^{(r)}(x) S_{2,\lambda}(n,m) = \sum_{m=0}^{n} S_{2,\lambda}(n,m) Ch_m^{(r)}(x).$$

Proof. Now, we observe that

$$\begin{aligned}
(1+\lambda t)^{\frac{x_1 + \cdots + x_r + x}{\lambda}} &= \left((1+\lambda t)^{\frac{1}{\lambda}} - 1 + 1 \right)^{x_1 + \cdots + x_r + x} \\
&= \sum_{m=0}^{\infty} \binom{x_1 + \cdots + x_r + x}{m} ((1+\lambda t)^{\frac{1}{\lambda}} - 1)^m \\
&= \sum_{m=0}^{\infty} (x_1 + \cdots + x_r + x)_m \sum_{n=m}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_{2,\lambda}(n,m)(x_1 + \cdots + x_r + x)_m \right) \frac{t^n}{n!}. \tag{3.9}$$

Thus, by (3.5) and (3.9), we get

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+x_1+\cdots+x_r}{\lambda}} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_{2,\lambda}(n,m) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_m d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_{2,\lambda}(n,m) Ch_m^{(r)}(x) \right) \frac{t^n}{n!}. \tag{3.10}$$

Therefore, by (3.5) and (3.10), we obtain the result.

Theorem 3.7. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\lambda}^{(r)}(x) = \sum_{m=0}^{n} E_{m,\lambda}^{(r)}(x) S_{1,\lambda}(n,m).$$

Proof. By replacing t by $\log_\lambda(1+t)$ in (3.3), we have

$$\left(\frac{2}{1 + (1 + \lambda \log_\lambda(1+t))^{\frac{1}{\lambda}}} \right)^r (1 + \lambda \log_\lambda(1+t))^{\frac{x}{\lambda}} = \sum_{m=0}^{\infty} E_{m,\lambda}^{(r)}(x) \frac{(\log_\lambda(1+t))^m}{m!}$$

$$= \sum_{m=0}^{\infty} E_{m,\lambda}^{(r)}(x) \sum_{n=m}^{\infty} S_{1,\lambda}(n,m) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} E_{m,\lambda}^{(r)}(x) S_{1,\lambda}(n,m) \right) \frac{t^n}{n!}. \tag{3.11}$$

On the other hand, we have

$$\left(\frac{2}{1 + (1 + \lambda \log_\lambda(1+t))^{\frac{1}{\lambda}}} \right)^r (1 + \lambda \log_\lambda(1+t))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \widehat{Ch}_{n,\lambda}^{(r)}(x) \frac{t^n}{n!}. \tag{3.12}$$

In view of (3.11) and (3.12), we obtain the result.

Theorem 3.8. *For $n \geq 0$, we have*

$$\widehat{Ch}_{n,\lambda}^{(r)}(x+y) = \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(y)_{m,\lambda}S_{1,\lambda}(k,m)\widehat{Ch}_{n-k,\lambda}^{(r)}(x).$$

Proof. From (3.1), we have

$$\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda}^{(r)}(x+y)\frac{t^n}{n!} = \left(\frac{2}{1+(1+\lambda\log_\lambda(1+t))^{\frac{1}{\lambda}}}\right)^r (1+\lambda\log_\lambda(1+t))^{\frac{x+y}{\lambda}}$$

$$= \left(\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\binom{\frac{y}{\lambda}}{m}(\log_\lambda(1+t))^m\right)$$

$$= \left(\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(y)_{m,\lambda}\sum_{k=m}^{\infty}S_{1,\lambda}(k,m)\frac{t^k}{k!}\right)$$

$$= \left(\sum_{n=0}^{\infty}\widehat{Ch}_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}(y)_{m,\lambda}S_{1,\lambda}(k,m)\right)\frac{t^k}{k!}\right)$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(y)_{m,\lambda}S_{1,\lambda}(k,m)\widehat{Ch}_{n-k,\lambda}^{(r)}(x)\right)\frac{t^n}{n!}. \tag{3.13}$$

Thus, by (3.13), we obtain at the required result.

4. Conclusion

Motivated by the works of Kim-Kim [23], we defined new type of degenerate Changhee polynomials and numbers of the second kind. We derived their explicit expressions and some identities involving them. Further, we introduced the higher-order new type of degenerate Changhee numbers and polynomials of the second kind and deduced their explicit expressions and some identities related to them.

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