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COMMON FIXED POINT THEOREMS FOR INTEGRAL TYPE MAPPINGS IN METRIC SPACES

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Abstract: In this paper, we prove a common fixed point theorem for a pair of R-weakly commutative of type (Ag) mappings satisfying a general contractive condition of integral type which generalizes the result of Branciari [1].

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1. Introduction

In 1976, Jungck [2] gave a common fixed point theorem for commuting maps, which generalizes the Banach's fixed point theorem. Later on Sessa [7] defined weak commutativity and proved common fixed point theorem for weakly commuting maps. Further, Jungck [3] introduced more generalized commutativity, so called compatibility as follows:

The pair of f and g is said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X, which is more general than that of weak commutativity.

However, the study of common fixed point of noncompatible mappings in metric space has been initiated by Pant [4,5]. In 1994, Pant [4] introduced the concept of R-weakly commuting mappings as follows:

Two self maps f and g of a metric space (X, d) are called R-weakly commuting at a point $x \in X$ if

$$d(fgx, gfx) \le Rd(fx, gx) \text{ for some } R > 0.$$

Two self maps f and g are called pointwise R-weakly commuting on X if given x in X, there exists $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$. Also we note that pointwise R-weak commutativity is equivalent to commutativity at coincidence points and hence a minimal commutativity condition for the existence of common fixed point of contractive type mappings.

In 1997, Pathak *et al.* [6] gave an analogue of R-weak commutativity known as R-weak commutativity of type (Ag) as follows:

Two self maps f and g of a metric space (X, d) are called R-weakly commutative of type (Ag) if there exists some positive real number R such that

$$d(ffx, gfx) \ \leq \ Rd(fx, gx) \ \text{ for all } x \in X.$$

Moreover, such mappings also commute at their coincidence points.

In 2002, Branciari [1] proved the following theorem:

Theorem 1.1. Let (X, d) be complete metric space, $c \in [0, 1)$, $f : X \to X$ a mapping such that, for each $x, y \in X$,

$$\int_0^{d(fx,fy)} \varphi(t)dt \ \leq \ c \int_0^{d(x,y)} \varphi(t)dt,$$

where $\varphi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable, non-negative and such that, for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that, for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.

2. Main Result

Theorem 2.1. Let f and g be noncompatible self mapping of a complete metric space (X, d) such that

(i) $\overline{f(X)} \subset g(x)$, where $\overline{f(X)}$ denotes the closure of the range of the mapping f;

(ii) $\displaystyle\int_0^{d(fx,fy)} \varphi(t)dt \leq c \int_0^{d(gx,gy)} \varphi(t)dt$, for each $x, y \in X$, $c \in [0, 1)$,

where $\varphi : R^+ \to R^+$ is a Lebesgue-integrable mapping which is summable, non-negative and such that, for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t)dt > 0$. If f and g are R-weakly commutative of type (Ag) then f and g have an unique common fixed point and the fixed point is a point of discontinuity.

Proof. Since f and g are noncompatible maps, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n \ = \ \lim_{n\to\infty} gx_n \ = \ t \tag{2.1}$$

for some t in X but either $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is nonzero or the limit does not exists. Since $t \in \overline{f(X)}$ and $\overline{f(X)} \subset g(X)$, so there exists some point u in X such that $t = gu$. Now, we prove $fu = gu$ if possible $fu \neq gu$, then the inequality (ii) becomes

$$\int_0^{d(fx_n,fu)} \varphi(t)dt \;\leq\; c\int_0^{d(gx_n,gu)} \varphi(t)dt, \quad \text{for } c \in [0,1).$$

Proceeding limit as $n \to \infty$, we obtain

$$\int_0^{d(gu,fu)} \varphi(t)dt \;\leq\; c\int_0^{d(gu,gu)} \varphi(t)dt,$$

hence we have $fu = gu$.

Since f and g are R-weakly commutative of type (Ag), we get

$$d(ffu, gfu) \;\leq\; Rd(fu, gu) \;=\; 0 \;\text{ i.e., } ffu = gfu.$$

Now, we prove the existence of fixed point, if possible $fu \neq ffu$, then using (ii), we have

$$\int_0^{d(fu,ffu)} \varphi(t)dt \;\leq\; c\int_0^{d(gu,gfu)} \varphi(t)dt \;=\; c\int_0^{d(fu,ffu)} \varphi(t)dt,$$

a contradiction, hence $fu = ffu = gfu$ and therefore, fu is a common fixed point of f and g.

Uniqueness. Let fu and fv be two distinct fixed point of f and g, then using (ii), we have

$$\int_0^{d(fu,fv)} \varphi(t)dt = \int_0^{d(ffu,ffv)} \varphi(t)dt \leq c\int_0^{d(gfu,gfv)} \varphi(t)dt = c\int_0^{d(fu,fv)} \varphi(t)dt,$$

which is a contradiction, hence fu is a unique common fixed point of f and g.

Now, we show that f and g are discontinuous at the common fixed point $t = fu = gu$. If possible, suppose f is continuous, then considering the sequence $\{x_n\}$ of (1.1) we get $\lim_{n\to\infty} ffx_n = ft = t$ and $\lim_{n\to\infty} fgx_n = ft$. R-weakly commutative of type (Ag) implies that

$$d(ffx_n, gfx_n) \;\leq\; Rd(fx_n, gx_n).$$

On letting as $n \to \infty$, $\lim\limits_{n\to\infty} gfx_n = ft = t$. This in turn, yields $\lim\limits_{n\to\infty} d(fgx_n, gfx_n)$ $= f(ft, ft) = 0$. This contradicts the fact that $\lim\limits_{n\to\infty} d(fgx_n, gfx_n)$ is either nonzero or non existent for the sequence $\{x_n\}$ of (2.1). Hence f is discontinuous at the fixed point.

Next, suppose that g is continuous. Then for the sequence $\{x_n\}$ of (2.1), we get

$$\lim_{n\to\infty} gfx_n = gt = t \text{ and } \lim_{n\to\infty} ggx_n = gt = t.$$

Thus

$$\int_0^{d(ft,fgx_n)} \varphi(t)dt \leq c\int_0^{d(gt,ggx_n)} \varphi(t)dt.$$

Proceeding the limit as $n \to \infty$, we have $\lim\limits_{n\to\infty} fgx_n = ft = t$. But $\lim\limits_{n\to\infty} gfx_n = gt = t$. Therefore, it contradicts the fact that $\lim\limits_{n\to\infty} d(fgx_n, gfx_n)$ is either nonzero or nonexistent. Thus f and g are discontinuous at their common fixed point. Hence the theorem.

Now, we give an example to illustrate the above theorem.

Example 2.1. Let $X = [3, 22]$ and d be usual metric on X. Define $f, g : X \to X$ by

$$fx = 3 \text{ if } x = 3 \text{ or } x > 7; \quad fx = 8 \text{ if } 3 < x \leq 7$$

$$g3 = 3; \quad gx = 10 \text{ if } 3 < x \leq 7; \quad gx = (x+2)/3 \text{ if } x > 7$$

$\psi, \varphi : [0,\infty) \to [0,\infty)$, where $\psi(t) = (t+1)^{t+1} - 1$ and $\varphi(t) = \psi'(t)$ which gives $f(X) = \{3\} \cup \{8\}$, $g(X) = [3, 8] \cup \{10\}$ and $\overline{f(X)} \subset g(X)$. Clearly f and g are noncompatible since for $\{x_n = 7+1/n,\ n \geq 1\}$ in X, $\lim_{n\to\infty} fx_n = 3$, $\lim_{n\to\infty} gx_n = 3$, $\lim_{n\to\infty} fgx_n = 8$ and $\lim_{n\to\infty} gfx_n = 3$. Hence f and g are noncompatible. Moreover, f and g are discontinuous at the common fixed point $x = 3$. Thus, f and g satisfy all the conditions of the theorem and have a unique common fixed point at $x = 3$.

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