South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, No. 3 (2024), pp. 397-412

DOI: 10.56827/SEAJMMS.2024.2003.30 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

FUZZY WEAK n-INNER PRODUCT SPACE

Bimalendu Kalita and Sinam Rajkishore Singha

Department of Mathematics,
The Assam Royal Global University,
Guwahati - 781035, Assam, INDIA

E-mail : bkalita84@gmail.com, rajkishoresingha15@gmail.com

(**Received:** *Oct. 10, 2024* **Accepted:** *Dec. 02, 2024* **Published:** *Dec. 30, 2024)*

Abstract: The paper is concerned with fuzzy real numbers and Felbin-type fuzzy inner product spaces. At first, we study fuzzy 2-inner product and discuss a few basic results of fuzzy inner product and fuzzy 2-inner product. The existence of fuzzy 2-inner product is proved with the help of an example. We introduce the notion of Felbin- type fuzzy weak n-inner product, which is a generalized concept of fuzzy n-inner product. Finally, we construct an n-iterated fuzzy 2-inner product and prove that it is a fuzzy weak n-inner product, also furnish an example of a 3-iterated fuzzy 2-inner product which is not a fuzzy 3-inner product.

Keywords and Phrases: Fuzzy real numbers, fuzzy inner product, weak n-inner product.

2020 Mathematics Subject Classification: Primary 54A40, 03E72; Secondary 46Cxx.

1. Introduction

A. Misiak [12], in 1989 generalized the idea of 2-inner product to n-inner product. Recently Minculete and Păltănea initiated the concept of weak n-inner product [9], with several applications. A classification of results related to the theory of 2-inner product and n-inner product can be found in [2], [3], [4], [5], [7], [12]. The notion of fuzzy norm on a vector space was first introduced by Katsaras, in 1984 [10]. In 1992 [6], Felbin introduced an alternative definition of fuzzy norm and discussed standard results of general normed linear spaces in Felbin-type fuzzy normed space. A. Hasankhani et al., [8] introduced the concept of Felbin-type fuzzy inner product space and studied various results of general inner product space in fuzzy inner product spaces. Misiak [12] demonstrated representation of n-inner product in terms of the basic inner product. In 2021, Minculete and Păltănea [11] developed the idea of the n-iterated 2-inner product, proved that it satisfies the properties of weak n-inner product, and showed its representation in terms of the standard n-inner product. They also explored several applications of the n-iterated 2-inner product. This motivates the investigation of the existence of a fuzzy n-iterated 2-inner product in the sense of a fuzzy weak n-inner product. Furthermore, it raises the problem of describing the relationship between the fuzzy n-iterated 2-inner product and the fuzzy n-inner product. The notion of fuzzy n-inner product is defined in [9] as follows:

Let n be a natural number greater than 1 and X be a vector space over $\mathbb{R}$ and $\dim(X) \geq n$. A fuzzy n-inner product on X is a mapping $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle : \underbrace{X \times X \times \ldots \times X}_{n+1} \to F(\mathbb{R})$ such that for all vectors $x, y, z, x_2, \ldots, x_n \in X$, $r \in \mathbb{R}$ and $\alpha \in (0, 1]$, we have:

A1) $\langle x, x \,|\, x_2, \ldots, x_n \rangle \succeq \tilde{0}$ and $\langle x, x \,|\, x_2, \ldots, x_n \rangle = \tilde{0}$ if and only if $x, x_2, \ldots, x_n$ are linearly dependent;

A2) $\langle x, y \,|\, x_2, \ldots, x_n \rangle = \langle y, x | x_2, \ldots, x_n \rangle$;

A3) $\langle x, y | x_2, \ldots, x_n \rangle$ is invariant under any permutation of $x_2, \ldots, x_n$;

A4) $\langle x, x \,|\, x_2, x_3, \ldots, x_n \rangle = \langle x_2, x_2 \,|\, x, x_3, \ldots, x_n \rangle$;

A5) $\langle rx, y \,|\, x_2, \ldots, x_n \rangle = \tilde{r} \otimes \langle x, y \,|\, x_2, \ldots, x_n \rangle$ for all $r \in \mathbb{R}$;

A6) $\langle x + y, z \,|\, x_2, \ldots, x_n \rangle = \langle x, z \,|\, x_2, \ldots, x_n \rangle \oplus \langle y, z \,|\, x_2, \ldots, x_n \rangle$;

A7) $\inf_{\alpha \in (0,1]} \langle x, x \,|\, x_2, \ldots, x_n \rangle_\alpha^- > 0$, if $x, x_2, \ldots, x_n$ are linearly independent.

Then the vector space X equipped with this fuzzy n-inner product $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$ is called a fuzzy n-inner product space.

Here we have mentioned below few basic results related to the theory of fuzzy n-inner product proved in [9]:

1. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be a fuzzy n-inner product space. Then, for all $\alpha \in (0, 1]$, $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle_\alpha^-$ and $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle_\alpha^+$ satisfy all the properties of n-inner product except homogeneity.

2. In a fuzzy n-inner product space X, for all $x, x_2, \ldots, x_n \in X$ and real β, $\langle x, x \,|\, \beta\, x_2, x_3, \ldots, x_n\rangle = \widetilde{\beta^2} \otimes \langle x, x \,|\, x_2, x_3, \ldots, x_n\rangle$.

3. In a fuzzy n-inner product space, if the vectors $x, x_2, \ldots, x_n$ are linearly dependent, then $\langle x, y \,|\, x_2, \ldots, x_n\rangle = \tilde{0}$.

4. For any $x, y, x_2, \ldots, x_n$ in a fuzzy n-inner product space X, we have $\langle x_2, y \,|\, x_2, \ldots, x_n\rangle = \langle x, x_2 \,|\, x_2, \ldots, x_n\rangle = \tilde{0}$. In particular, $\langle \vec{0}, y \,|\, x_n, \ldots, x_2\rangle = \langle x, \vec{0} \,|\, x_n, \ldots, x_2\rangle = \langle x, y \,|\, \vec{0}, \ldots, x_2\rangle = \tilde{0}$.

This article pertains to the construction of an n-iterated fuzzy 2-inner product that satisfies all the conditions of a fuzzy weak n-inner product.

2. Preliminaries

In this section, basic definitions and notations are given.

Definition 2.1. [8] *A mapping $\eta : \mathbb{R} \to [0, 1]$ is called a fuzzy real number with α-level set $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies the following conditions:*

1. *there exist $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.*

2. *for each $\alpha \in (0, 1]$, there exist real numbers $-\infty < \eta^-_\alpha \leq \eta^+_\alpha < +\infty$ such that the α-level set $[\eta]_\alpha$ is equal to the closed interval $[\eta^-_\alpha, \eta^+_\alpha]$.*

The set of all fuzzy real numbers (fuzzy intervals) is denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^+(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. The real number $\eta^-_\alpha \geq 0$ for all $\eta \in F^+(\mathbb{R})$ and $\alpha \in (0, 1]$.
Since each $r \in \mathbb{R}$ can be considered as the fuzzy real number $\tilde{r} \in F(\mathbb{R})$ defined by

$$\tilde{r}(t) = \begin{cases} 1, \; \text{if } t = r \\ 0, \; \text{if } t \neq r \end{cases} \tag{2.1}$$

it follows that $\mathbb{R}$ can be embedded in $F(\mathbb{R})$. Also α-level set of $\tilde{r}$ is given by $[\tilde{r}]_\alpha = [r, r]$, $0 < \alpha \leq 1$.

Lemma 2.2. [8] *Let $\eta, \gamma \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha]$, $[\gamma]_\alpha = [\gamma^-_\alpha, \gamma^+_\alpha]$. Then for all*

$\alpha \in (0,1]$,

$$[\eta \oplus \gamma]_\alpha = [\eta_\alpha^- + \gamma_\alpha^-, \eta_\alpha^+ + \gamma_\alpha^+],$$
$$[\eta \ominus \gamma]_\alpha = [\eta_\alpha^- - \gamma_\alpha^+, \eta_\alpha^+ - \gamma_\alpha^-],$$
$$[\eta \otimes \gamma]_\alpha = [\eta_\alpha^- \gamma_\alpha^-, \eta_\alpha^+ \gamma_\alpha^+], \forall \eta, \gamma \in F^+(\mathbb{R}),$$
$$[\tilde{1} \oslash \eta]_\alpha = [\frac{1}{\eta_\alpha^+}, \frac{1}{\eta_\alpha^-}], \forall \eta_\alpha^- > 0,$$
$$[|\eta|]_\alpha = [max(0, \eta_\alpha^-, -\eta_\alpha^+), max(|\eta_\alpha^-|, |\eta_\alpha^+|)].$$

Definition 2.3. [8] *Let $\eta, \gamma \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$, for all $\alpha \in (0,1]$. Define a partial ordering by $\eta \preceq \gamma$ in $F(\mathbb{R})$ if and only if $\eta_\alpha^- \leq \gamma_\alpha^-$ and $\eta_\alpha^+ \leq \gamma_\alpha^+$, for all $\alpha \in (0,1]$.*

Remark 2.4. [8] *Let $\eta, \gamma \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^-, \eta_\alpha^+]$, $[\gamma]_\alpha = [\gamma_\alpha^-, \gamma_\alpha^+]$, for all $\alpha \in (0,1]$. By above definition, if $\eta_\alpha^- = \gamma_\alpha^-$ and $\eta_\alpha^+ = \gamma_\alpha^+$, then $\eta = \gamma$ and vice versa.*

Definition 2.5. [8] *For a non-negative fuzzy real number η we define $\sqrt{\eta} = \gamma$ where $[\gamma]_\alpha = [\sqrt{\eta_\alpha^-}, \sqrt{\eta_\alpha^+}]$, $\alpha \in (0,1]$.*

Lemma 2.6. [8] *Let $\eta \in F^+(\mathbb{R})$ and $\gamma \in F(\mathbb{R})$. Then*

1. $(\sqrt{\eta})^2 = \eta$,
2. $\gamma \preceq |\gamma|$.

Lemma 2.7. *For any real number $r \in \mathbb{R}$, $\widetilde{|r|} = |\tilde{r}| = \begin{cases} \tilde{r}, & \text{if } r \geq 0; \\ \ominus\tilde{r}, & \text{if } r < 0. \end{cases}$*

Proof. For $r \geq 0$, $[\widetilde{|r|}]_\alpha = [|r|, |r|] = [r, r]$ and $[|\tilde{r}|]_\alpha = [\max(0, r, -r), \max(|r|, |r|)] = [r, r]$. For $r < 0$, let $r = -p$, where $p > 0$, $[\widetilde{|r|}]_\alpha = [\widetilde{|-p|}]_\alpha = [\widetilde{p}]_\alpha = [p, p] = [-r, -r] = [\ominus\tilde{r}]_\alpha$ and $[|\tilde{r}|]_\alpha = [|\widetilde{-p}|]_\alpha = [\max(0, -p, -(-p)), \max(|-p|, |-p|)] = [p, p] = [-r, -r] = [\ominus\tilde{r}]_\alpha$.

3. Fuzzy Inner Product

Definition 3.1. [8] *Let X be a vector space over $\mathbb{R}$. A real-valued fuzzy inner product on X is a mapping $\langle \cdot, \cdot \rangle : X \times X \to F(\mathbb{R})$ such that for all vectors $x, y, z \in X$ and $r \in \mathbb{R}$, we have*

B1) $\langle x + y, z \rangle = \langle x, z \rangle \oplus \langle y, z \rangle$,

B2) $\langle rx, y \rangle = \tilde{r} \otimes \langle x, y \rangle$, *B3)* $\langle x, y\rangle = \langle y, x\rangle$,

B4) $\langle x, x\rangle \succeq \tilde{0}$,

B5) $\inf_{0<\alpha\leq 1}\langle x, x\rangle_\alpha^- > 0$, *if* $x \neq 0$,

B6) $\langle x, x\rangle = \tilde{0}$ *if and only if* $x = 0$.

The vector space X with a real-valued fuzzy inner product is called a real fuzzy inner product space. We write $[\langle\cdot,\cdot\rangle]_\alpha = [\langle\cdot,\cdot\rangle_\alpha^-, \langle\cdot,\cdot\rangle_\alpha^+]$.

Lemma 3.2. [1] *In a fuzzy inner product space* $(X, \langle\cdot,\cdot\rangle)$, *for vectors* x, y *and for each* $\alpha \in (0, 1]$, *we have*

$$|\langle x, y\rangle|_\alpha^+ \leq \sqrt{\langle x, x\rangle_\alpha^-}\sqrt{\langle y, y\rangle_\alpha^-}. \tag{3.1}$$

Hence, it holds that

$$|\langle x, y\rangle| \preceq \sqrt{\langle x, x\rangle} \otimes \sqrt{\langle y, y\rangle}. \tag{3.2}$$

Corollary 3.3. *Let* $(X, \langle\cdot,\cdot\rangle)$ *be a fuzzy inner product space. Then for any positive fuzzy number* $\langle\cdot,\cdot\rangle$ *and vectors* x, y, *for each* $\alpha \in (0, 1]$, *we have*

$$\langle x, y\rangle_\alpha^+ \leq \sqrt{\langle x, x\rangle_\alpha^-}\sqrt{\langle y, y\rangle_\alpha^-}. \tag{3.3}$$

Hence, it holds that

$$\langle x, y\rangle \preceq \sqrt{\langle x, x\rangle} \otimes \sqrt{\langle y, y\rangle} \text{ or, } \langle x, y\rangle^2 \preceq \langle x, x\rangle \otimes \langle y, y\rangle. \tag{3.4}$$

Also, $\langle x, y\rangle^2 = \langle x, x\rangle \otimes \langle y, y\rangle$ *if the vectors* x *and* y *are linearly dependent.*

Proof. By Lemma 2.2 and Lemma 3.2, we have

$$\begin{aligned}\langle x, y\rangle_\alpha^+ \leq |\langle x, y\rangle|_\alpha^+ &= \max(|\langle x, y\rangle_\alpha^-|, |\langle x, y\rangle_\alpha^+|)\\ &= \max(\langle x, y\rangle_\alpha^-, \langle x, y\rangle_\alpha^+) \leq \sqrt{\langle x, x\rangle_\alpha^-}\sqrt{\langle y, y\rangle_\alpha^-}.\end{aligned}$$

Again if x, y are linearly dependent $\Rightarrow y = kx$ (for some $k \in \mathbb{R}$) $\Rightarrow \langle x, y\rangle = \sqrt{\langle x, x\rangle} \otimes \sqrt{\langle y, y\rangle}$.

Corollary 3.4. *For any positive fuzzy number in a fuzzy inner product space, if* $\langle x, y\rangle_\alpha^+ = \sqrt{\langle x, x\rangle_\alpha^-}\sqrt{\langle y, y\rangle_\alpha^-}$, *then vectors* x *and* y *are linearly dependent.*

Proof. As discussed in Theorem 2, [1], for $y \neq 0$,

$(\|x\|_\alpha^-)^2 - \frac{(|\langle x,y\rangle|_\alpha^+)^2}{(\|y\|_\alpha^-)^2} = (\|x\|_\alpha^-)^2 - \frac{(|v|_\alpha^+)^2}{(\|y\|_\alpha^-)^2} = \langle x + ay, x + ay\rangle_\alpha^-$,

where $a = \begin{cases} \frac{|v|_\alpha^+}{(\|y\|_\alpha^-)^2}, & \text{if } |v|_\alpha^+ = |v_\alpha^-| \\ \frac{-|v|_\alpha^+}{(\|y\|_\alpha^-)^2}, & \text{if } |v|_\alpha^+ = |v_\alpha^+| \end{cases}$
Again, $|\langle x,y\rangle|_\alpha^+ = \max\left(|\langle x,y\rangle_\alpha^-|, |\langle x,y\rangle_\alpha^+|\right) = \langle x,y\rangle_\alpha^+$.
Thus

$$\begin{aligned} &\langle x,y\rangle_\alpha^+ = \sqrt{\langle x,x\rangle_\alpha^-}\sqrt{\langle y,y\rangle_\alpha^-} \\ &\Rightarrow (\|x\|_\alpha^-)^2 - \frac{(|\langle x,y\rangle|_\alpha^+)^2}{(\|y\|_\alpha^-)^2} = 0 \\ &\Rightarrow \langle x+ay, x+ay\rangle_\alpha^- = 0 \\ &\Rightarrow x+ay = 0. \end{aligned}$$

Corollary 3.5. *If $\langle\cdot,\cdot\rangle$ is a fuzzy inner product space, then $\|x+y\| = \|x\| \oplus \|y\|$ if and only if $\langle x,y\rangle = \|x\| \otimes \|y\|$.*
Proof.

$$\begin{aligned} \left[\|x+y\|\right]_\alpha = \left[\|x\| \oplus \|y\|\right]_\alpha \Leftrightarrow & \|x+y\|_\alpha^- = \|x\|_\alpha^- + \|y\|_\alpha^- \\ & \text{and } \|x+y\|_\alpha^+ = \|x\|_\alpha^+ + \|y\|_\alpha^+ \\ \Leftrightarrow & \langle x,y\rangle_\alpha^- = \|x\|_\alpha^-\|y\|_\alpha^- \text{ and } \langle x,y\rangle_\alpha^+ = \|x\|_\alpha^+\|y\|_\alpha^+ \\ \Leftrightarrow & \left[\langle x,y\rangle\right]_\alpha = \left[\|x\| \otimes \|y\|\right]_\alpha. \end{aligned}$$

4. The Notion of Fuzzy 2-inner Product

Definition 4.1. *Let n be a natural number greater than 1 and X be a vector space over $\mathbb{R}$ and $\dim(X) \geq n$. A fuzzy 2-inner product on X is a mapping $\langle\cdot,\cdot|\cdot\rangle : X \times X \times X \to F(\mathbb{R})$ such that for all vectors $x, x', y, z \in X$, $r \in \mathbb{R}$ and $\alpha \in (0,1]$, we have:*

C1) $\langle x+x', y\,|z\rangle = \langle x,y\,|z\rangle \oplus \langle x', y\,|z\rangle$;

C2) $\langle rx, y\,|z\rangle = \tilde{r} \otimes \langle x,y\,|z\rangle$ *for all* $r \in \mathbb{R}$;

C3) $\langle x,y\,|z\rangle = \langle y,x|z\rangle$;

C4) $\langle x,x\,|z\rangle = \langle z,z|x\rangle$;

C5) $\langle x,x\,|z\rangle \succeq \tilde{0}$;

C6) $\langle x,x\,|z\rangle = \tilde{0}$ *if and only if x, z are linearly dependent;*

C7) $\inf_{\alpha\in(0,1]} \langle x,x\,|z\rangle_\alpha^- > 0$, *if x, z are linearly independent.*

Then the vector space X equipped with this fuzzy 2-inner product $\langle\cdot,\cdot|\cdot\rangle$ is called a fuzzy 2-inner product space.

Remark 4.2.

1. *If x and z are linearly independent, then from condition C6), we have $\langle x, x\,|z\rangle \neq \tilde{0}$. Thus either $\langle x,x|z\rangle^-_\alpha = 0$, $\langle x,x|z\rangle^+_\alpha > 0$ or $\langle x,x|z\rangle^-_\alpha > 0$, $\langle x,x|z\rangle^+_\alpha > 0$. So in both the cases $\langle x,x|z\rangle^+_\alpha \neq 0$.*

2. *For positive fuzzy numbers, the statement $\langle x, x\,|z\rangle = \tilde{0}$ if and only if x and z are linearly dependent is equivalent to the statement $\langle x, x\,|z\rangle^+_\alpha = 0$ if and only if x and z are linearly dependent.*

Lemma 4.3. *Let X be a fuzzy 2-inner product space, then*

1. $\langle x+ry, x+ry\,|\,z\rangle^-_\alpha$
$$= \begin{cases} \langle x,x\,|\,z\rangle^-_\alpha + 2r\langle x,y\,|\,z\rangle^-_\alpha + r^2\langle y,y\,|\,z\rangle^-_\alpha, & \text{if } r \geq 0\ ; \\ \langle x,x\,|\,z\rangle^-_\alpha + 2r\langle x,y\,|\,z\rangle^+_\alpha + r^2\langle y,y\,|\,z\rangle^-_\alpha, & \text{if } r < 0. \end{cases}$$

2. $\langle x+ry, x+ry\,|\,z\rangle^+_\alpha$
$$= \begin{cases} \langle x,x\,|\,z\rangle^+_\alpha + 2r\langle x,y\,|\,z\rangle^+_\alpha + r^2\langle y,y\,|\,z\rangle^+_\alpha, & \text{if } r \geq 0; \\ \langle x,x\,|\,z\rangle^+_\alpha + 2r\langle x,y\,|\,z\rangle^-_\alpha + r^2\langle y,y\,|\,z\rangle^+_\alpha, & \text{if } r < 0. \end{cases}$$

for all $x, y, z \in X$ and $\alpha \in (0,1]$.

In the Theorem 4.4, a fuzzy number is explicitly formulated and demonstrated to possess the characteristics of a fuzzy 2-inner product. This outcome simultaneously establishes the existence of a fuzzy 2-inner product.

Theorem 4.4. *Let $(X, \langle\cdot,\cdot\rangle)$ be a fuzzy inner product space. Then for any positive fuzzy number $\langle\cdot,\cdot\rangle$, the mapping $\langle\cdot,\cdot|\cdot\rangle : X \times X \times X \to F(\mathbb{R})$ defined by $\langle x,y|z\rangle := \begin{vmatrix} \langle x,y\rangle & \langle x,z\rangle \\ \langle z,y\rangle & \langle z,z\rangle \end{vmatrix} = (\langle x,y\rangle \otimes \langle z,z\rangle) \ominus (\langle x,z\rangle \otimes \langle z,y\rangle)$ is a fuzzy number and satisfies all the properties from C1) to C6). And $\langle\cdot,\cdot|\cdot\rangle$ is a fuzzy 2-inner product, if $\langle\cdot,\cdot|\cdot\rangle^-_\alpha \neq 0$.*

Proof. $[\langle x,y|z\rangle]_\alpha := [\langle x,y|z\rangle^-_\alpha, \langle x,y|z\rangle^+_\alpha] = \left[\begin{vmatrix} \langle x,y\rangle^-_\alpha & \langle x,z\rangle^+_\alpha \\ \langle z,y\rangle^+_\alpha & \langle z,z\rangle^-_\alpha \end{vmatrix}, \begin{vmatrix} \langle x,y\rangle^+_\alpha & \langle x,z\rangle^-_\alpha \\ \langle z,y\rangle^-_\alpha & \langle z,z\rangle^+_\alpha \end{vmatrix}\right]$

C1) $\langle x+x', y|z\rangle = \langle x+x', y\rangle \otimes \langle z,z\rangle \ominus \langle x+x', z\rangle \otimes \langle z,y\rangle = \big(\langle x,y\rangle \otimes \langle z,z\rangle \ominus \langle x,z\rangle \otimes \langle z,y\rangle\big) \oplus \big(\langle x',y\rangle \otimes \langle z,z\rangle \ominus \langle x',z\rangle \otimes \langle z,y\rangle\big) = \langle x,y|z\rangle \oplus \langle x',y|z\rangle$.

C2) For any $r \in \mathbb{R}$, $\langle rx,y|z\rangle = \langle rx,y\rangle \otimes \langle z,z\rangle \ominus \langle rx,z\rangle \otimes \langle z,y\rangle = \tilde{r} \otimes \langle x,y|z\rangle$.

C3) Since $\langle\cdot,\cdot\rangle$ is a fuzzy inner product, so $\langle x,y|z\rangle = \langle y,x|z\rangle$.

C4) $\langle x,x|z\rangle_\alpha^- = \langle x,x\rangle_\alpha^-\langle z,z\rangle_\alpha^- - \langle x,z\rangle_\alpha^+\langle z,x\rangle_\alpha^+ = \langle z,z\rangle_\alpha^-\langle x,x\rangle_\alpha^- - \langle z,x\rangle_\alpha^+\langle x,z\rangle_\alpha^+ = \langle z,z|x\rangle_\alpha^-$. Similarly, $\langle x,x|z\rangle_\alpha^+ = \langle z,z|x\rangle_\alpha^+$ and so $\langle x,x|z\rangle = \langle z,z|x\rangle$.

C5) By Lemma 2.2 and Lemma 3.2, we have $\max(|\langle x,z\rangle_\alpha^-|, |\langle x,z\rangle_\alpha^+|) = |\langle x,z\rangle|_\alpha^+ \leq \sqrt{\langle x,x\rangle_\alpha^-}\sqrt{\langle z,z\rangle_\alpha^-}$. Since $\langle\cdot,\cdot\rangle$ is a positive fuzzy number, so

$$\begin{aligned}\langle x,z\rangle_\alpha^{-2} \leq \langle x,z\rangle_\alpha^{+2} = |\langle x,z\rangle|_\alpha^{+2} &\leq \langle x,x\rangle_\alpha^-\langle z,z\rangle_\alpha^- \\ &\leq \langle x,x\rangle_\alpha^+\langle z,z\rangle_\alpha^+. \end{aligned} \tag{4.1}$$

Thus $\langle x,x|z\rangle_\alpha^- \geq 0$ and $\langle x,x|z\rangle_\alpha^+ \geq 0$.

C6) If $\langle x,x|z\rangle = \tilde{0}$ then $\langle x,x|z\rangle_\alpha^- = 0$ and $\langle x,x|z\rangle_\alpha^+ = 0$. $\langle x,x|z\rangle_\alpha^- = 0 \Rightarrow \langle x,z\rangle_\alpha^{+2} = \langle x,x\rangle_\alpha^-\langle z,z\rangle_\alpha^-$. Thus by Corollary 3.4, x and z are linearly dependent. Conversely, if x and z are linearly dependent, by Corollary 3.3, we get $\langle x,z\rangle_\alpha^{+2} = \langle x,x\rangle_\alpha^+\langle z,z\rangle_\alpha^+$ and $\langle x,z\rangle_\alpha^{-2} = \langle x,x\rangle_\alpha^-\langle z,z\rangle_\alpha^-$. Using (4.1), we have $\langle x,z\rangle_\alpha^{-2} = \langle x,x\rangle_\alpha^+\langle z,z\rangle_\alpha^+$ and $\langle x,z\rangle_\alpha^{+2} = \langle x,x\rangle_\alpha^-\langle z,z\rangle_\alpha^-$ and so $\langle x,x|z\rangle = \tilde{0}$.

C7) If x,z are linearly independent and $\langle x,x|z\rangle_\alpha^- \neq 0$, then $\inf_{\alpha\in(0,1]}\langle x,x|z\rangle_\alpha^- > 0$ because α-cut of fuzzy numbers is a closed interval.

5. Fuzzy Weak n-inner Product Space

The basic properties of Felbin-type fuzzy n-inner product spaces and Cauchy-Schwarz inequality on fuzzy n-inner product spaces are discussed in [9].

Within this context, we formulate an n-iterated fuzzy 2-inner product that adheres to the properties outlined in Definition 4.1.

Definition 5.1. *Let n be a natural number greater than 1 and X be a vector space over $\mathbb{R}$ and $\dim(X) \geq n$. A fuzzy weak n-inner product on X is a mapping $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle : \underbrace{X\times X\times\ldots\times X}_{n+1} \to F(\mathbb{R})$ such that for all vectors $x,y,z,x_2,\ldots,x_n \in X$, $r\in\mathbb{R}$ and $\alpha\in(0,1]$, we have:*

D1) $\langle x,x\,|\,x_n,\ldots,x_2\rangle \succeq \tilde{0}$ and $\langle x,x\,|\,x_n,\ldots,x_2\rangle = \tilde{0}$ if and only if $x,x_2,\ldots,x_n$ are linearly dependent;

D2) $\langle x,y\,|\,x_n,\ldots,x_2\rangle = \langle y,x|x_n,\ldots,x_2\rangle$;

D3) $\langle x,x\,|\,x_n,\ldots,x_2\rangle = \langle x_n,x_n\,|\,x,x_{n-1},\ldots,x_2\rangle$;

D4) $\langle rx,y\,|\,x_n,\ldots,x_2\rangle = \tilde{r}\otimes\langle x,y\,|\,x_n,\ldots,x_2\rangle$ for all $r\in\mathbb{R}$;

D5) $\langle x+y,z\,|\,x_n,\ldots,x_2\rangle = \langle x,z\,|\,x_n,\ldots,x_2\rangle \oplus \langle y,z\,|\,x_n,\ldots,x_2\rangle$;

D6) $\inf_{\alpha\in(0,1]} \langle x, x \,|\, x_n, \ldots, x_2\rangle^-_\alpha > 0$, *if* $x, x_n, \ldots, x_2$ *are linearly independent.*

Then the vector space X *equipped with this fuzzy weak* n*-inner product* $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle$ *is called a fuzzy weak* n*-inner product space.*

Remark 5.2. *For* $n = 2$ *a fuzzy weak* n*-inner product is equivalent to fuzzy* n*-inner product. By definition it is very obvious that fuzzy* n*-inner product is a fuzzy weak* n*-inner product. However, it is important to note that the converse is not true in general, as exemplified in Example 5.12. The construction of a fuzzy weak* n*-inner product relies on the properties inherent in a fuzzy* n*-inner product, with the exception of property A3).*

Theorem 5.3. *Let* $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ *be a fuzzy weak* n*-inner product space. Then, for all* $\alpha \in (0,1]$, $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle^-_\alpha$ *and* $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle^+_\alpha$ *satisfy all the properties of weak* n*-inner product except the property homogeneity.*
The proof of Theorem 5.3 can be established employing a similar method demonstrated in [9] for fuzzy n-inner products. However, in this discussion, we present an alternative approach by introducing the quotient map ψ.

Lemma 5.4. [11] *Let* X *be a weak* n*-inner product space and* $x, x_2, \ldots, x_n \in X$*. If* $x, x_2, \ldots, x_n$ *are linearly dependent, then* $\langle x, y|x_n, \ldots, x_2\rangle = 0$.
Note that:

$$
\begin{aligned}
[\langle rx, y \,|\, x_n, \ldots, x_2\rangle^-_\alpha, &\langle rx, y \,|\, x_n, \ldots, x_2\rangle^+_\alpha] \\
&= [\langle rx, y \,|\, x_n, \ldots, x_2\rangle]_\alpha \\
&= [\tilde{r} \otimes \langle x, y \,|\, x_n, \ldots, x_2\rangle]_\alpha \\
&= \begin{cases} [r\langle x, y \,|\, x_n, \ldots, x_2\rangle^-_\alpha, r\langle x, y \,|\, x_n, \ldots, x_2\rangle^+_\alpha] & \text{if } r \geq 0 \\ [r\langle x, y \,|\, x_n, \ldots, x_2\rangle^+_\alpha, r\langle x, y \,|\, x_n, \ldots, x_2\rangle^-_\alpha] & \text{if } r < 0. \end{cases}
\end{aligned}
$$

Lemma 5.5 is proven using a similar method as described in [9].

Lemma 5.5. *Let* $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ *be a fuzzy weak* n*-inner product space and* $x, x_2, x_3, \ldots, x_n$ *be linearly dependent vectors. Then* $\langle x, y|x_n, \ldots, x_2\rangle = \tilde{0}$.
Proof. We consider two cases.

Case 1. $y, x_2, x_3, \ldots, x_n$ are linearly independent. Consider the vector $u = \alpha x - \beta y$, where $\alpha = \langle y, y|x_n, \ldots, x_2\rangle^-_\alpha$ and $\beta = \langle x, y|x_n, \ldots, x_2\rangle^+_\alpha$. We have

$$
\begin{aligned}
0 &\leq \langle u, u|x_n, \ldots, x_2\rangle^-_\alpha \\
&= \langle \alpha x - \beta y, \alpha x - \beta y|x_n, \ldots, x_2\rangle^-_\alpha \\
&= \alpha^2\langle x, x|x_n, \ldots, x_2\rangle^-_\alpha - 2\alpha\beta\langle x, y|x_n, \ldots, x_2\rangle^+_\alpha + \beta^2\langle y, y|x_n, \ldots, x_2\rangle^-_\alpha
\end{aligned}
$$

$$= \langle y, y|x_n, \ldots, x_2\rangle_\alpha^- \big[\langle y, y|x_n, \ldots, x_2\rangle_\alpha^- \langle x, x|x_n, \ldots, x_2\rangle_\alpha^- - (\langle x, y|x_n, \ldots, x_2\rangle_\alpha^+)^2\big]$$
$$= -\langle y, y|x_n, \ldots, x_2\rangle_\alpha^- (\langle x, y|x_n, \ldots, x_2\rangle_\alpha^+)^2$$

Since $y, x_2, x_3, \ldots, x_n$ are linearly independent it follows that $\langle y, y|x_n, \ldots, x_2\rangle_\alpha^- > 0$ and thus $\langle x, y|x_n, \ldots, x_2\rangle_\alpha^+ = 0$.

Case 2. $y, x_2, x_3, \ldots, x_n$ are linearly dependent. Then also $x+y, x_2, x_3, \ldots, x_n$ are linearly dependent. Because $\langle x, x|x_n, \ldots, x_2\rangle_\alpha^+ = 0$, $\langle y, y|x_n, \ldots, x_2\rangle_\alpha^+ = 0$ and $\langle x+y, x+y|x_n, \ldots, x_2\rangle_\alpha^+ = 0$, from the relation

$$\langle x+y, x+y|x_n, \ldots, x_2\rangle_\alpha^+ = \langle x, x|x_n, \ldots, x_2\rangle_\alpha^+ + 2\langle x, y|x_n, \ldots, x_2\rangle_\alpha^+ + \langle y, y|x_n, \ldots, x_2\rangle_\alpha^+,$$

we get $\langle x, y|x_n, \ldots, x_2\rangle_\alpha^+ = 0$.

Similarly we can show that $\langle x, y|x_n, \ldots, x_2\rangle_\alpha^- = 0$.

Remark 5.6. *In a fuzzy weak n-inner product space,* $\langle \vec{0}, y\,|\,x_n, \ldots, x_2\rangle = \langle x, \vec{0}\,|\,x_n, \ldots, x_2\rangle = \langle x, y\,|\,\vec{0}, \ldots, x_2\rangle = \tilde{0}$.

Let $(X, \langle \cdot, \cdot|\cdot, \ldots, \cdot\rangle)$ be a fuzzy weak n-inner product space over the field of real numbers $\mathbb{R}$. Consider the set $Y = span\{x_2, x_3, \ldots, x_n\}$, where $x_2, x_3, \ldots, x_n$ are linearly independent and let $X/Y = \{\hat{x} = Y + x : x \in X\}$ be the quotient space. Let $\psi : X/Y \times X/Y \to F(\mathbb{R})$, be a function defined by $\psi(\hat{x}, \hat{y}) = \langle x, y|x_n, \ldots, x_2\rangle$. Consider the vectors $x, x', y, y' \in X$ such that $(\hat{x'}, \hat{y'}) = (\hat{x}, \hat{y})$, that is $x' - x \in Y$ and $y' - y \in Y$. Then

$$\begin{aligned}\psi(\hat{x'}, \hat{y'}) = \langle x', y'|x_n, \ldots, x_2\rangle &= \langle x' - x + x, y' - y + y|x_n, \ldots, x_2\rangle \\ &= \langle x' - x,\ y' - y|x_n, \ldots, x_2\rangle \oplus \langle x' - x,\ y|x_n, \ldots, x_2\rangle \\ &\quad \oplus \langle x,\ y' - y|x_n, \ldots, x_2\rangle \oplus \langle x,\ y|x_n, \ldots, x_2\rangle \\ &= \tilde{0} \oplus \tilde{0} \oplus \tilde{0} \oplus \langle x,\ y|x_n, \ldots, x_2\rangle (\text{using Lemma 5.5}) \\ &= \psi(\hat{x}, \hat{y}),\end{aligned}$$

which shows that the function ψ is well-defined.

Again, if $\langle \cdot, \cdot|\cdot, \ldots, \cdot\rangle$ be a fuzzy weak n-inner product, then ψ satisfies all the properties of basic fuzzy inner product as mentioned below:

1. $\psi(\hat{x}, \hat{x}) = \langle x, x|x_n, \ldots, x_2\rangle \succeq \tilde{0}$ and $\psi(\hat{x}, \hat{x}) = \tilde{0} \Leftrightarrow \langle x, x|x_n, \ldots, x_2\rangle = \tilde{0} \Leftrightarrow x, x_2, x_3, \ldots, x_n$ are linearly dependent $\Leftrightarrow x \in Y \Leftrightarrow \hat{x} = Y + x = Y = \hat{0}$,

2. $\psi(\hat{x}, \hat{y}) = \langle x, y|x_n, \ldots, x_2\rangle = \langle y, x|x_n, \ldots, x_2\rangle = \psi(\hat{y}, \hat{x})$,

3. For all $r \in \mathbb{R}$, $\psi(r\hat{x}, y) = \psi(\widehat{rx}, y) = \langle rx, y|x_n, \ldots, x_2\rangle = \tilde{r} \otimes \langle x, y|x_n, \ldots, x_2\rangle = \tilde{r} \otimes \psi(\hat{x}, \hat{y})$,
4. $\psi(\hat{x} + \hat{y}, \hat{z}) = \psi(\widehat{x+y}, \hat{z}) = \langle x + y, z|x_n, \ldots, x_2\rangle$,
   $= \langle x, z|x_n, \ldots, x_2\rangle \oplus \langle y, z|x_n, \ldots, x_2\rangle = \psi(\hat{x}, \hat{z}) \oplus \psi(\hat{y}, \hat{z})$,
5. $\inf_{\alpha\in(0,1]} \psi(\hat{x}, \hat{x})^-_\alpha = \inf_{\alpha\in(0,1]}\langle x, x|x_n, \ldots, x_2\rangle^-_\alpha > 0$.

This shows that ψ is a fuzzy inner product.
Notation: The α-cut of ψ is denoted by $[\psi]_\alpha = [\psi^-, \psi^+] = [\langle\cdot, \cdot|\cdot, \ldots, \cdot\rangle^-_\alpha, \langle\cdot, \cdot|\cdot, \ldots, \cdot\rangle^+_\alpha]$.

Theorem 5.7. *In a fuzzy weak n-inner product space $(X, \langle\cdot, \cdot|\cdot, \ldots, \cdot\rangle)$, for any $x, y, x_2, \ldots, x_n \in X$ we have*

$$|\langle x, y|x_n, \ldots, x_2\rangle|^+_\alpha \leq \sqrt{\langle x, x|x_n, \ldots, x_2\rangle^-_\alpha}\ \sqrt{\langle y, y|x_n, \ldots, x_2\rangle^-_\alpha}.$$

Proof. Since $(X/Y, \psi)$ is a fuzzy inner product space, therefore by Lemma 3.2 for all $\hat{x}, \hat{y} \in X/Y$, we have

$$|\psi(\hat{x}, \hat{y})|^+_\alpha = \max(|\psi^-(\hat{x}, \hat{y})|, |\psi^+(\hat{x}, \hat{y})|) \leq \sqrt{\psi^-(\hat{x}, \hat{x})} \otimes \sqrt{\psi^-(\hat{y}, \hat{y})}$$
$$\Leftrightarrow |\langle x, y|x_n, \ldots, x_2\rangle|^+_\alpha \leq \sqrt{\langle x, x|x_n, \ldots, x_2\rangle^-_\alpha}\ \sqrt{\langle y, y|x_n, \ldots, x_2\rangle^-_\alpha}.$$

Corollary 5.8. *In a fuzzy weak n-inner product space*

1. $\langle x, y|x_n, \ldots, x_2\rangle \preceq |\langle x, y|x_n, \ldots, x_2\rangle| \preceq \sqrt{\langle x, x|x_n, \ldots, x_2\rangle} \otimes \sqrt{\langle y, y|x_n, \ldots, x_2\rangle}$.
2. $\langle x, y|x_n, \ldots, x_2\rangle = \sqrt{\langle x, x|x_n, \ldots, x_2\rangle} \otimes \sqrt{\langle y, y|x_n, \ldots, x_2\rangle}$ *only if $x, y, x_2, \ldots, x_n$ are linearly dependent.*
3. *If $\langle x, y|x_n, \ldots, x_2\rangle^+_\alpha = \sqrt{\langle x, x|x_n, \ldots, x_2\rangle^-_\alpha}\sqrt{\langle y, y|x_n, \ldots, x_2\rangle^-_\alpha}$, then vectors $x, y, x_2, \ldots, x_n$ are linearly dependent.*

Proof.

1. The inequality is a direct consequence of Lemma 2.6 and Theorem 5.7.
2. If $x, y, x_2, \ldots, x_n$ are linearly dependent, then $\hat{x}, \hat{y}$ are linearly dependent, which implies $\psi(\hat{x}, \hat{y}) = \sqrt{\psi(\hat{x}, \hat{x})} \otimes \sqrt{\psi(\hat{y}, \hat{y})}$, (using Corollary 3.3).
   Thus $\langle x, y|x_n, \ldots, x_2\rangle = \sqrt{\langle x, x|x_n, \ldots, x_2\rangle} \otimes \sqrt{\langle y, y|x_n, \ldots, x_2\rangle}$.
3. $\langle x, y|x_n, \ldots, x_2\rangle^+_\alpha = \sqrt{\langle x, x|x_n, \ldots, x_2\rangle^-_\alpha}\sqrt{\langle y, y|x_n, \ldots, x_2\rangle^-_\alpha}$ implies $\psi^+(\hat{x}, \hat{y}) = \sqrt{\psi^-(\hat{x}, \hat{x})} \otimes \sqrt{\psi^-(\hat{y}, \hat{y})}$, which shows that $\hat{x}$ and $\hat{y}$ are linearly dependent, (using Corollary 3.4). Thus $x, y, x_2, \ldots, x_n$ are linearly dependent.

Remark 5.9. *For a fuzzy weak n-inner product $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$, using Theorem 5.7 and Lemma 2.2, we also have*

$$\begin{aligned}\langle x, y | x_n, \ldots, x_2 \rangle_\alpha^- &\leq \langle x, y | x_n, \ldots, x_2 \rangle_\alpha^+ \\ &\leq |\langle x, y | x_n, \ldots, x_2 \rangle_\alpha^+| \\ &\leq |\langle x, y | x_n, \ldots, x_2 \rangle|_\alpha^+ \\ &\leq \sqrt{\langle x, x | x_n, \ldots, x_2 \rangle_\alpha^-} \ \sqrt{\langle y, y | x_n, \ldots, x_2 \rangle_\alpha^-} \\ &\leq \sqrt{\langle x, x | x_n, \ldots, x_2 \rangle_\alpha^+} \ \sqrt{\langle y, y | x_n, \ldots, x_2 \rangle_\alpha^+}. \end{aligned} \tag{5.1}$$

If $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be a fuzzy weak n-inner product space, $n \geq 2$, then we can define a function $\| \cdot | \cdot, \ldots, \cdot \| : \underbrace{X \times X \times \ldots \times X}_{n} \to F(\mathbb{R})$ by

$$\|x| \, x_n, \ldots, x_2\| = \sqrt{\langle x, \, x| \, x_n, \ldots, x_2 \rangle} \tag{5.2}$$

which satisfies the following conditions:

E1) $\|x| \, x_n, \ldots, x_2\| \succeq \tilde{0}$, $\|x| \, x_n, \ldots, x_2\| = \tilde{0}$ if and only if $x, x_2, \ldots, x_n$ are linearly dependent;

E2) $\|x| \, x_n, \ldots, x_2\| = \|x_n | x, \, x_{n-1}, \ldots, x_2\|$;

E3) $\|rx| \, x_n, \ldots, x_2\| = |\tilde{r}| \otimes \|x| \, x_n, \ldots, x_2\|$ for all $r \in \mathbb{R}$;

E4) $\|x + y| \, x_n, \ldots, x_2\| \preceq \|x| \, x_n, \ldots, x_2\| \oplus \|y| \, x_n, \ldots, x_2\|$;

E5) $\inf_{\alpha \in (0,1]} \|x| \, x_2, \ldots, x_n\|_\alpha^- > 0$, if $x, x_2, \ldots, x_n$ are linearly independent;

The conditions E1)-E5) follow immediately from the conditions D1)-D6).

Definition 5.10. *A fuzzy real valued function $\| \cdot | \cdot, \ldots, \cdot \|$ satisfying conditions E1)-E5) is called a fuzzy weak n-norm and $(X, \| \cdot | \cdot, \ldots, \cdot \|)$ is called a fuzzy weak n-normed space.*

Theorem 5.11. *If $(X, \langle \cdot, \cdot \rangle)$ be a fuzzy inner product space, then for $x, y, z, x_2, \ldots, x_n \in X$, $r \in \mathbb{R}$ and $n \geq 3$, the mapping $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle_* : \underbrace{X \times X \times \ldots \times X}_{n+1} \to F(\mathbb{R})$ defined by*

$$\langle x, y | \, x_n, \ldots, x_2 \rangle_* := \begin{vmatrix} \langle x, y | \, x_{n-1}, \ldots, x_2 \rangle_* & \langle x, x_n | \, x_{n-1}, \ldots, x_2 \rangle_* \\ \langle x_n, y | \, x_{n-1}, \ldots, x_2 \rangle_* & \langle x_n, x_n | \, x_{n-1}, \ldots, x_2 \rangle_* \end{vmatrix} \tag{5.3}$$

satisfies conditions D1)-D5) and $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle_*$ *is a fuzzy weak* n*-inner product if* $\langle x, x| x_n, \ldots, x_2 \rangle^-_{*_\alpha} \neq 0$.

Note that for $n = 2$, $\langle x, y| x_2 \rangle_* = \langle x, y| x_2 \rangle$ as defined in Theorem 4.4, which is a fuzzy weak 2-inner product. The mapping defined in (5.3) is called n-iterated fuzzy 2-inner product.

Proof. We prove this proposition by mathematical induction, for $n \geq 2$. Clearly, by Theorem 4.4 the result is true for $n = 2$.

Induction hypothesis: suppose $S(n)$: the n-iterated fuzzy 2-inner product defined in (5.3) satisfies conditions D1)-D6). To prove that $S(n+1)$: the $(n+1)$-iterated fuzzy 2-inner product also satisfies conditions D1)-D6). The $(n+1)$-iterated fuzzy 2-inner product is given by

$$\langle x, y| x_{n+1}, \ldots, x_2 \rangle_* := \begin{vmatrix} \langle x, y| x_n, \ldots, x_2 \rangle_* & \langle x, x_{n+1}| x_n, \ldots, x_2 \rangle_* \\ \langle x_{n+1}, y| x_n, \ldots, x_2 \rangle_* & \langle x_{n+1}, x_{n+1}| x_n, \ldots, x_2 \rangle_* \end{vmatrix}. \quad (5.4)$$

So by Remark 5.9, we get

$$\langle x, x| x_{n+1}, \ldots, x_2 \rangle^-_{*_\alpha} := \begin{vmatrix} \langle x, x| x_n, \ldots, x_2 \rangle^-_{*_\alpha} & \langle x, x_{n+1}| x_n, \ldots, x_2 \rangle^+_{*_\alpha} \\ \langle x_{n+1}, x| x_n, \ldots, x_2 \rangle^+_{*_\alpha} & \langle x_{n+1}, x_{n+1}| x_n, \ldots, x_2 \rangle^-_{*_\alpha} \end{vmatrix} \geq 0 \quad (5.5)$$

and

$$\langle x, x| x_{n+1}, \ldots, x_2 \rangle^+_{*_\alpha} := \begin{vmatrix} \langle x, x| x_n, \ldots, x_2 \rangle^+_{*_\alpha} & \langle x, x_{n+1}| x_n, \ldots, x_2 \rangle^-_{*_\alpha} \\ \langle x_{n+1}, x| x_n, \ldots, x_2 \rangle^-_{*_\alpha} & \langle x_{n+1}, x_{n+1}| x_n, \ldots, x_2 \rangle^+_{*_\alpha} \end{vmatrix} \geq 0, \quad (5.6)$$

this shows that $\langle x, x| x_{n+1}, \ldots, x_2 \rangle_* \succeq \tilde{0}$.

If $x, x_{n+1}, \ldots, x_2$ are linearly dependent, then by Corollary 5.8

$$(\langle x, x_{n+1}| x_n, \ldots, x_2 \rangle^+_{*_\alpha})^2 = \langle x, x| x_n, \ldots, x_2 \rangle^+_{*_\alpha} \langle x_{n+1}, x_{n+1}| x_n, \ldots, x_2 \rangle^+_{*_\alpha}$$

and

$$(\langle x, x_{n+1}| x_n, \ldots, x_2 \rangle^-_{*_\alpha})^2 = \langle x, x| x_n, \ldots, x_2 \rangle^-_{*_\alpha} \langle x_{n+1}, x_{n+1}| x_n, \ldots, x_2 \rangle^-_{*_\alpha}.$$

Then by Remark 5.9

$$(\langle x, x_{n+1}| x_n, \ldots, x_2 \rangle^-_{*_\alpha})^2 = \langle x, x| x_n, \ldots, x_2 \rangle^+_{*_\alpha} \langle x_{n+1}, x_{n+1}| x_n, \ldots, x_2 \rangle^+_{*_\alpha}$$

and

$$(\langle x, x_{n+1}| x_n, \ldots, x_2 \rangle^+_{*_\alpha})^2 = \langle x, x| x_n, \ldots, x_2 \rangle^-_{*_\alpha} \langle x_{n+1}, x_{n+1}| x_n, \ldots, x_2 \rangle^-_{*_\alpha}.$$

This shows that $\langle x, x| x_{n+1}, \ldots, x_2 \rangle^+_{*_\alpha} = 0$ and $\langle x, x| x_{n+1}, \ldots, x_2 \rangle^-_{*_\alpha} = 0$ and so $\langle x, x| x_{n+1}, \ldots, x_2 \rangle_* = \tilde{0}$.

Conversely,

$$\begin{aligned}&\langle x,x|\,x_{n+1},\ldots,x_2\rangle_* = \tilde{0} \Rightarrow \langle x,x|\,x_{n+1},\ldots,x_2\rangle^+_{*_\alpha} = 0\\ &\Rightarrow (\langle x,x_{n+1}|\,x_n,\ldots,x_2\rangle^-_{*_\alpha})^2 = \langle x,x|\,x_n,\ldots,x_2\rangle^+_{*_\alpha}\langle x_{n+1},x_{n+1}|\,x_n,\ldots,x_2\rangle^+_{*_\alpha}\\ &\Rightarrow (\langle x,x_{n+1}|\,x_n,\ldots,x_2\rangle^+_{*_\alpha})^2 = \langle x,x|\,x_n,\ldots,x_2\rangle^+_{*_\alpha}\langle x_{n+1},x_{n+1}|\,x_n,\ldots,x_2\rangle^+_{*_\alpha}\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(using Remark 5.9)}\\ &\Rightarrow x, x_{n+1},\ldots,x_2 \text{ are linearly dependent.}\end{aligned}$$

Condition D1) is completely proved for $n+1$.
Since $S(n)$ is true, so the condition D2) is true for $n+1$.
Proof of condition D3) for $n+1$:

$$\langle x,x|\,x_{n+1},\ldots,x_2\rangle_* := \begin{vmatrix}\langle x,x|\,x_n,\ldots,x_2\rangle_* & \langle x,x_{n+1}|\,x_n,\ldots,x_2\rangle_*\\ \langle x_{n+1},x|\,x_n,\ldots,x_2\rangle_* & \langle x_{n+1},x_{n+1}|\,x_n,\ldots,x_2\rangle_*\end{vmatrix}$$

$$= \begin{vmatrix}\langle x_{n+1},x_{n+1}|\,x_n,\ldots,x_2\rangle_* & \langle x_{n+1},x|\,x_n,\ldots,x_2\rangle_*\\ \langle x,x_{n+1}|\,x_n,\ldots,x_2\rangle_* & \langle x,x|\,x_n,\ldots,x_2\rangle_*\end{vmatrix} = \langle x_{n+1},x_{n+1}|\,x,x_n,\ldots,x_2\rangle_*.$$

Proof of condition D4) for $n+1$:

$$\langle r\,x,y|\,x_{n+1},\ldots,x_2\rangle_* := \begin{vmatrix}\langle r\,x,y|\,x_n,\ldots,x_2\rangle_* & \langle r\,x,x_{n+1}|\,x_n,\ldots,x_2\rangle_*\\ \langle x_{n+1},y|\,x_n,\ldots,x_2\rangle_* & \langle x_{n+1},x_{n+1}|\,x_n,\ldots,x_2\rangle_*\end{vmatrix}$$

$$= \begin{vmatrix}\tilde{r}\otimes\langle x,y|\,x_n,\ldots,x_2\rangle_* & \tilde{r}\otimes\langle x,x_{n+1}|\,x_n,\ldots,x_2\rangle_*\\ \langle x_{n+1},y|\,x_n,\ldots,x_2\rangle_* & \langle x_{n+1},x_{n+1}|\,x_n,\ldots,x_2\rangle_*\end{vmatrix} = \tilde{r}\otimes\langle x,y|\,x_{n+1},\ldots,x_2\rangle_*.$$

Proof of condition D5) for $n+1$:

$$\begin{aligned}\langle x+x',y|\,x_{n+1},\ldots,x_2\rangle_* =&\langle x+x',y|\,x_n,\ldots,x_2\rangle_*\otimes\langle x_{n+1},x_{n+1}|\,x_n,\ldots,x_2\rangle_*\\ &\ominus\langle x+x',x_{n+1}|\,x_n,\ldots,x_2\rangle_*\otimes\langle x_{n+1},y|\,x_n,\ldots,x_2\rangle_*\\ =&\Bigg(\langle x,y|\,x_n,\ldots,x_2\rangle_*\otimes\langle x_{n+1},x_{n+1}|\,x_n,\ldots,x_2\rangle_*\\ &\ominus\langle x,x_{n+1}|\,x_n,\ldots,x_2\rangle_*\otimes\langle x_{n+1},y|\,x_n,\ldots,x_2\rangle_*\Bigg)\\ &\oplus\Bigg(\langle x',y|\,x_n,\ldots,x_2\rangle_*\otimes\langle x_{n+1},x_{n+1}|\,x_n,\ldots,x_2\rangle_*\\ &\ominus\langle x',x_{n+1}|\,x_n,\ldots,x_2\rangle_*\otimes\langle x_{n+1},y|\,x_n,\ldots,x_2\rangle_*\Bigg)\\ =&\langle x,y|\,x_{n+1},\ldots,x_2\rangle_*\oplus\langle x',y|\,x_{n+1},\ldots,x_2\rangle_*.\end{aligned}$$

Example 5.12. Define $\langle x,y\rangle(t) = \begin{cases}1, & \text{when } t=(x,y);\\ 0, & \text{otherwise,}\end{cases}$

where $(\cdot,\cdot)$ defines the usual inner product and $(x,x) > 0$. Then $[\langle x,y\rangle]_\alpha = [(x,y),(x,y)]$. It can be verified that $\langle\cdot,\cdot\rangle$ is a fuzzy number and a fuzzy inner product. Let $(\cdot,\cdot) : \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$ be the usual inner product and $\langle\cdot,\cdot|\,\cdot,\cdot\rangle_* : \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3 \to F(\mathbb{R})$ be the 3-iterated fuzzy 2-inner product with $\langle\cdot,\cdot|\,\cdot,\cdot\rangle^-_{*_\alpha} \neq 0$. Then by Theorem 5.11, $\langle\cdot,\cdot|\,\cdot,\cdot\rangle_*$ is a fuzzy weak 3-inner product. But it is not a fuzzy 3-inner product because there exist vectors for which $\langle x,x|\,x_3,x_2\rangle_* \neq \langle x_2,x_2|\,x_3,x\rangle_*$. For this choose vectors $x = (1,0,0)$, $x_2 = (2,1,2)$, $x_3 = (1,1,1)$. As defined in equation (5.3),

$$\begin{aligned}\langle x,x|\,x_3,x_2\rangle_* &= \begin{vmatrix}\langle x,x|\,x_2\rangle_* & \langle x,x_3|\,x_2\rangle_*\\ \langle x_3,x|\,x_2\rangle_* & \langle x_3,x_3|\,x_2\rangle_*\end{vmatrix}\\ &= (\langle x,x|\,x_2\rangle_*\otimes\langle x_3,x_3|\,x_2\rangle_*)\ominus(\langle x,x_3|\,x_2\rangle_*\otimes\langle x_3,x|\,x_2\rangle_*).\end{aligned}$$

Now, $\langle x,x|\,x_2\rangle^-_{*_\alpha} = \begin{vmatrix}\langle x,x\rangle^-_\alpha & \langle x,x_2\rangle^+_\alpha\\ \langle x_2,x\rangle^+_\alpha & \langle x_2,x_2\rangle^-_\alpha\end{vmatrix} = \begin{vmatrix}(x,x) & (x,x_2)\\ (x_2,x) & (x_2,x_2)\end{vmatrix} = \begin{vmatrix}1 & 2\\ 2 & 9\end{vmatrix} = 5,$

and $\langle x,x|\,x_2\rangle^+_{*_\alpha} = \begin{vmatrix}\langle x,x\rangle^+_\alpha & \langle x,x_2\rangle^-_\alpha\\ \langle x_2,x\rangle^-_\alpha & \langle x_2,x_2\rangle^+_\alpha\end{vmatrix} = \begin{vmatrix}(x,x) & (x,x_2)\\ (x_2,x) & (x_2,x_2)\end{vmatrix} = \begin{vmatrix}1 & 2\\ 2 & 9\end{vmatrix} = 5.$

So $[\langle x,x|\,x_2\rangle_*]_\alpha = [5,5]$. Similarly, we can find out $[\langle x_3,x_3|\,x_2\rangle_*]_\alpha = [2,2]$, $[\langle x,x_3|\,x_2\rangle_*]_\alpha = [-1,-1] = [\langle x_3,x|\,x_2\rangle_*]_\alpha$. Therefore,
$[\langle x,x|\,x_3,x_2\rangle_*]_\alpha = ([5,5]\otimes[2,2])\ominus([-1,-1]\otimes[-1,-1]) = [9,9]$.
Now,

$$\langle x_2,x_2|\,x_3,x\rangle_* := \begin{vmatrix}\langle x_2,x_2|\,x\rangle_* & \langle x_2,x_3|\,x\rangle_*\\ \langle x_3,x_2|\,x\rangle_* & \langle x_3,x_3|\,x\rangle_*\end{vmatrix}$$

and $[\langle x_2,x_2|\,x\rangle_*]_\alpha = [5,5]$, $[\langle x_3,x_3|\,x\rangle_*]_\alpha = [2,2]$, $[\langle x_2,x_3|\,x\rangle_*]_\alpha = [3,3] = [\langle x_3,x_2|\,x\rangle_*]_\alpha$. So $[\langle x_2,x_2|\,x_3,x\rangle_*]_\alpha = [1,1]$.

6. Conclusion

We substantiated the existence of a fuzzy 2-inner product through an illustrative example and constructed an n-iterated fuzzy 2-inner product, demonstrating its characterization as a fuzzy weak n-inner product. Furthermore, we provided an example illustrating a 3-iterated fuzzy 2-inner product that does not conform to the properties of a fuzzy 3-inner product.

7. Future scope

The structure of the standard fuzzy n-inner product remains an unexplored aspect. Once the structure of the standard fuzzy n-inner product is established, one can delve into the study of representing the n-iterated fuzzy 2-inner product in terms of the standard fuzzy k-inner product, $(k \leq n)$.

References

[1] Byun T., Lee J. E., Lee K. Y. and Yoon J. H., Absence of Non-Trivial Fuzzy Inner Product Spaces and the Cauchy–Schwartz Inequality, Mathematics, 8 (571) (2020), 1–7.

[2] Cho Y. J., Lin P. C. S., Kim S. S. and Misiak A., Theory of 2-inner Product Spaces, Nova Science Publishes Inc., New York, 2001.

[3] Debnath P. and Saha M., Categorization of n-inner product space, Asian Research J. Math., 11(4) (2018), 1-10.

[4] Diminnie C., *Gähler* S. and White A., 2-inner product spaces, Demonstr. Math., 6 (1973), 525-536.

[5] Diminnie C., *Gähler* S. and White A., 2-inner product spaces. II, Demonstr. Math., 10 (1977), 169-188.

[6] Felbin C., Finite dimensional fuzzy normed linear spaces, Fuzzy Sets Syst., 48 (1992), 239–248.

[7] Gunawan H., Inner products on n-inner product spaces, Soochow J. Math., 28(4) (2002), 389-398.

[8] Hasankhani A., Nazari A. and Saheli M., Some properties of fuzzy Hilbert spaces and norm of operators, Iranian Journal of Fuzzy Systems, 7(3) (2010), 129–157.

[9] Kalita B. and Rajkishore S., Fuzzy n-inner product spaces, Bulletin of Calcutta Mathematical Society, 116 (6) (2024), 703–718.

[10] Katsaras A. K., Fuzzy topological vector spaces, Fuzzy Sets Syst., 12 (1984), 143–154.

[11] Minculete N. and Paltanea R., Weak n-inner product spaces, Ann. Funct. Anal., Volume 12 (22) (2021).

[12] Misiak A., n-inner product spaces, Math. Nachr, 140 (1989), 299-319.

[13] Rajkishore S. and Kalita B., Some Inequalities in Fuzzy 2-inner Product Spaces, Global Journal of Pure and Applied Mathematics, 20(3) (2024), 401-414.