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# NEARLY LINDELÖFNESS IN BITOPOLOGICAL SPACES

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**Abstract:** The concept of nearly pairwise Lindelöf spaces, a well-known weaker form of Lindelöf spaces, was introduced by Katetov and L. Krajewski in [11, 12] and has since been extensively explored by numerous researchers. This paper investigates the same concept in the context of a specific type of cover, referred to as a regular cover. In particular, the primary objective is to examine the properties of new generalizations of pairwise Lindelöf spaces, termed nearly pairwise Lindelöf spaces.

**Keywords and Phrases:** Bitopological spaces, regular covers, pairwise Lindelöf spaces, pairwise Lindelöf, pairwise compact, pairwise paracompact.

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## 1. Introduction

A Lindelöfness studied by Katetov and L. Krajewski in [11] and [12], respectively, is one of the most famous concepts in general topology. Many of its various forms have been investigated. Among the various generalizations discussed in the literature, compactness, paracompact spaces, and locally finite spaces can be highlighted [9, 13, 14, 15, 16, 17, 18].

In [10], Jamal A. Oudetallah introduced and studied pairwise expandability spaces related to pairwise Lindelöf spaces. Specifically, for $m$ being an infinite cardinal, a bitopological space $(\chi, \tau_1, \tau_2)$ is called a $\tau_\iota$-$m$-expandable space with respect to $\tau_i$ if, for every $\tau_\iota$-locally finite collection $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ with $|\Delta| \leq m$, there exists a $\tau_i$-locally finite collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of $\chi$ such that $F_\alpha \subset G_\alpha$ for all $\alpha \in \Delta$. Here, $i \neq j$ and $(i, j = 1, 2)$.

In recent years, significant advancements have been made in the study of topological properties, particularly in the context of bitopological spaces. The concept of pairwise Lindelöf spaces has been explored extensively due to its connections with compactness and paracompactness in bitopological settings [8]. The extension to nearly pairwise Lindelöf spaces introduces a nuanced perspective, enabling a deeper understanding of how different types of covers influence the structure and behavior of topological spaces [7]. Additionally, contributions on expandability and its interplay with pairwise Lindelöf spaces have enriched the theoretical framework of bitopology, shedding light on their significance in broader mathematical contexts [14, 15]. Such studies underscore the relevance of our work in investigating nearly pairwise Lindelöf spaces under regular covers, contributing to the growing body of research in this field.

Recent research in ideal topological spaces has led to the development of several key concepts, particularly in the area of continuity. Al-Omeri et al. (2014) introduced new forms of contra-continuity, expanding the classical framework of continuity in ideal spaces [1]. Building on this, their later work explored almost $e$-$\mathcal{I}$-continuity, contributing to a deeper understanding of continuity in such spaces [2]. Further, Al-Omeri et al. (2018) investigated the degree of fuzzy semi-precontinuity and semi-preirresolute functions, offering new insights into the interaction between fuzzy structures and continuity [3]. Other important contributions from Al-Omeri and collaborators include their study of $a$-local functions in topological groups, which has been instrumental in advancing the theory of topological spaces [4]. For further studies, the reader may refer to the references [5, 6, 19].

A bitopological space $(\chi, \tau_1, \tau_2)$ is called $\tau_\iota$-expandable with respect to $\tau_j$ if it is $\tau_\iota$-$m$-expandable for every cardinal $m$, where $i \neq j$ and $(i, j = 1, 2)$. A bitopological space $(\chi, \tau_1, \tau_2)$ is called a pairwise expandable ($P$-expandable) space if it is a $P$-$T_2$ space and is $\tau_1$-expandable with respect to $\tau_2$ and $\tau_2$-expandable with respect to $\tau_1$.

In what follows, the space $\chi$ will refer to a bitopological space $(\chi, \tau_1, \tau_2)$. In

[20], the authors introduced the concept of pairwise paracompact spaces. For a subset $A$ of $\chi$, Int$(A)$ and Cl$(A)$ will respectively denote the interior and closure of $A$ in $\chi$.

We say that a set $A$ in $\chi = (\chi, \tau_1, \tau_2)$ is $\tau_i$-regularly open with respect to $\tau_j$ if $\text{Int}^{\tau_j}(\text{Cl}^{\tau_j}(A)) = A$ for $i \neq j$ and $(i, j = 1, 2)$. Moreover, $A$ is called a pairwise regularly open set in the space $\chi$ if it is both $\tau_1$-regularly open with respect to $\tau_2$ and $\tau_2$-regularly open with respect to $\tau_1$. Clearly, every pairwise regularly open set is a pairwise open set. Similarly, a subset $B$ of the space $\chi$ is called a pairwise regularly closed set if it is the complement of a pairwise regularly open set.

## 2. Pairwise Lindelöf spaces

In this section, we introduce the concept of pairwise Lindelöf spaces. Furthermore, we discuss other related concepts that stem from the notion of pairwise Lindelöf spaces and explore various theories associated with these concepts.

**Definition 1.** *A pairwise open cover is a pairwise disjoint open cover induced by an arbitrary open cover.*

**Definition 2.** *The pairwise open cover of $\chi$ and the cover $U$ of $(\chi, \lambda_1, \lambda_2)$ is called the pairwise open cover of $(\chi, \lambda_1, \lambda_2)$ if $U \subset U_1 \cup U_2$, $U \cap U_1 \neq \emptyset$, and $U \cap U_2 \neq \emptyset$.*

**Definition 3.** *A bitopological space $(\chi, \lambda_1, \lambda_2)$ is called $\lambda_1$-Lindelöf with respect to $\lambda_2$ if, for every $\lambda_1$-open cover of $\chi$, there exists a countable $\lambda_2$-subcover.*

**Definition 4.** [8] *Suppose $(\chi, \lambda_1, \lambda_2)$ is a $B$-compact space, and $(\beta, \mu_1, \mu_2)$ is $B$-Lindelöf. Then, the product topology $(\chi \times \beta, \lambda_1 \times \mu_1, \lambda_2 \times \mu_2)$ is $B$-Lindelöf.*

**Definition 5.** [8] *A bitopological space $(\chi, \lambda_1, \lambda_2)$ is called a pairwise Lindelöf space if, for every $\lambda_1$-open cover of $\chi$, there exists a $\lambda_2$-countable subcover of $\chi$, and conversely.*

**Definition 6.** [8] *Let $(\chi, \lambda_1, \lambda_2)$ be a bitopological space, and $\tilde{U}$ be a cover of $\chi$. We say that $\tilde{U}$ is $\lambda_1\lambda_2$-open if $\tilde{U} \subset \lambda_1 \cup \lambda_2$.*

**Definition 7.** *A bitopological space $(\chi, \lambda_1, \lambda_2)$ is called a $2^{nd}$ countable space if $\chi$ has a countable base with respect to both $\lambda_1$ and $\lambda_2$.*

**Definition 8.** *A bitopological space $(\chi, \lambda_1, \lambda_2)$ is called $S$-Lindelöf if and only if it is both Lindelöf and pairwise Lindelöf.*

**Theorem 1.** *If a bitopological space $(\chi, \lambda_1, \lambda_2)$ is a $2^{nd}$ocountable space, then it is a pairwise Lindelöf space.*

**Proof.** Let $\chi$ be a pairwise $2^{\text{nd}}$ countable space. Then, $\chi$ has a countable base $\underset{\sim_1}{B}$ with respect to $\lambda_1$ and $\lambda_2$, say $\underset{\sim}{B} = \{B_i\}_{i=1}^{\infty}$. Let $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ be a $\lambda_i$-open cover of $\chi$ $(i = 1, 2)$. Each $u_\alpha$ is a union of some members of $\underset{\sim}{B}$. Since $\underset{\sim}{B}$ is a base for $\chi$, each $u_\alpha$ is itself a union of members of $\underset{\sim}{B}$. Thus, this union forms a countable $\lambda_j$-subcover of $\underset{\sim}{U}$ that covers $\chi$ for each $i \neq j$ $(i, j = 1, 2)$. Therefore, $\chi$ is Lindelöf.

**Remark 1.** *Every pairwise compact space is a pairwise Lindelöf space, but the converse is not necessarily true.*

**Proof.** Let $\chi$ be a pairwise compact space. Then, every $\lambda_i$-open cover of $\chi$ has a finite $\lambda_j$-subcover of $\chi$. Thus, for each $\lambda_i$-open cover of $\chi$, there exists a countable $\lambda_j$-subcover of $\chi$ for all $i \neq j$ $(i, j = 1, 2)$. Hence, $\chi$ is pairwise Lindelöf. The converse, however, need not be true.

**Example 1.** The space $(\mathbb{R}, \tau_u, \tau_u)$ is a pairwise Lindelöf space but not pairwise compact. Since $\underset{\sim}{B} = \{(a, b) : a < b,\ a, b \in \mathbb{Q}\}$ is a countable base for $\mathbb{R}$, it follows that $\mathbb{R}$ is second countable. Furthermore, since every second countable space is Lindelöf, the space $(\mathbb{R}, \tau_u, \tau_u)$ is pairwise Lindelöf.

**Theorem 2.** *Let $(\chi, \lambda_1, \lambda_2)$ be a bitopological space. If $\chi$ is a hereditary Lindelöf space, then $\chi$ is $S$-Lindelöf.*

**Proof.** Assume $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\} \cup \{v_\beta : \beta \in \Gamma\}$ is a $\lambda_1, \lambda_2$-open cover of $(\chi, \lambda_1, \lambda_2)$, such that $u_\alpha \in \lambda_1$ for every $\alpha \in \Lambda$ and $v_\beta \in \lambda_2$ for every $\beta \in \Gamma$. Since $U = \cup\{u_\alpha : \alpha \in \Lambda\}$ is $\lambda_1$-Lindelöf, there exists a countable set $\Lambda_1 \subset \Lambda$ such that $U = \cup\{u_\alpha : \alpha \in \Lambda_1\}$. Similarly, since $V = \cup\{v_\beta : \beta \in \Gamma\}$ is $\lambda_2$-Lindelöf, there exists a countable set $\Gamma_1 \subset \Gamma$ such that $V = \cup\{v_\beta : \beta \in \Gamma_1\}$. It is clear that $\{u_\alpha : \alpha \in \Lambda_1\} \cup \{v_\beta : \beta \in \Gamma_1\}$ is a countable subcover of $\underset{\sim}{U}$ for $\chi$.

**Corollary 1.** *Every second countable $(\chi, \lambda_1, \lambda_2)$ is a bitopological space. Thus, $\chi$ is pairwise $S$-Lindelöf.*

**Example 2.** Let $\chi$ be a set and $\lambda_1$ be the topology on $\chi$ generated by the basis $\beta_1$, where $\beta_1 = \{[x, y) : x < y;\ x, y \in \chi\}$. Then, $(\chi, \lambda_1, \lambda_2)$ is not hereditary Lindelöf; however, it is $S$-Lindelöf.

**Theorem 3.** *A pairwise Lindelöf space is preserved under an onto pairwise continuous function.*

**Proof.** Let $i \neq j$, where $i, j = 1, 2$, and let $f : (\chi, \lambda_1, \lambda_2) \to (Y, \mu_1, \mu_2)$ be a surjective continuous function. Suppose $\chi$ is a Lindelöf space. We aim to show that $Y$ is also a Lindelöf space. Assume $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is a $\lambda_i$-open cover of $Y$. Then, each $u_\alpha$ is open in $Y$ for $\alpha \in \Lambda$. Since $f$ is continuous, the preimage $f^{-1}(u_\alpha)$ is open in $\chi$ for each $\alpha \in \Lambda$. As $f$ is surjective, the collection $\{f^{-1}(u_\alpha) : \alpha \in \Lambda\}$

forms an open cover of $\chi$. Since $\chi$ is Lindelöf, there exists a countable subcover $\{f^{-1}(u_\alpha) : \alpha \in \Gamma\}$, where $\Gamma \subset \Lambda$ and $|\Gamma| \leq \aleph_0 = |\mathbb{N}|$. Thus, we have:

$$\chi \subseteq \bigcup_{\alpha \in \Gamma} f^{-1}(u_\alpha).$$

Because $f$ is onto, it follows that:

$$Y = f(\chi) \subseteq f\left(\bigcup_{\alpha \in \Gamma} f^{-1}(u_\alpha)\right) \subseteq \bigcup_{\alpha \in \Gamma} u_\alpha.$$

Therefore, $\underset{\sim}{U}$ has a countable $\lambda_j$-subcover of $Y$, and so $Y$ is a Lindelöf space.

**Remark 2.** *We know that a compact subset in a pairwise Hausdorff space is closed. However, a Lindelöf subset in a pairwise Hausdorff space need not be closed. For example,* $(\mathbb{R}, \tau_u, \tau_u)$ *is a pairwise Hausdorff space, and* $(0, 1)$ *is a Lindelöf subset of* $\mathbb{R}$. *However,* $(0, 1)$ *is not pairwise closed in* $\mathbb{R}$.

**Definition 9.** *A space* $(\chi, \lambda_1, \lambda_2)$ *is called a pairwise space if and only if the countable intersection of open sets is open.*

**Theorem 4.** *Every Lindelöf subset of* $\tau_2$ *is pairwise closed.*

**Proof.** Let $i \neq j$ with $(i, j = 1, 2)$, and let $A$ be a Lindelöf subset of the pairwise $\tau_2$ space $\chi$. To show that $A$ is a pairwise closed set, it is enough to show that $\chi \setminus A$ is pairwise open. Let $x \in \chi \setminus A$. Then $x \notin A$, and for each $a \in A$, we have $x \neq a$. Since $\chi$ is a $\tau_2$ space, there exist open sets $u_a$ and $v_a$ in $\chi$ such that $x \in u_a$, $a \in v_a$, and $u_a \cap v_a = \emptyset$. Hence, $\underset{\sim}{V} = \{v_a : a \in A\}$ forms a $\lambda_i$-open cover of $A$. Since $A$ is a pairwise Lindelöf subset of $\chi$, the cover $\underset{\sim}{V}$ reduces to a countable $\lambda_j$-subcover, say $\{v_{a_\alpha} : \alpha \in \Lambda_0\}$. Thus, $A \subseteq \bigcup_{\alpha \in \Lambda_0} v_{a_\alpha}$. For each $v_{a_\alpha}$, where $\alpha \in \Lambda_0$, there exists a corresponding $u_{a_\alpha}$ such that $x \in u_{a_\alpha}$, $a_\alpha \in v_{a_\alpha}$, and $u_{a_\alpha} \cap v_{a_\alpha} = \emptyset$. Let $v^* = \bigcup_{\alpha \in \Lambda_0} v_{a_\alpha}$ and $u^* = \bigcap_{\alpha \in \Lambda_0} u_{a_\alpha}$. Then, $A \subseteq v^*$ and $x \in u^*$, with $u^* \cap v^* \subseteq u_{a_\alpha} \cap v_{a_\alpha} = \emptyset$ for all $\alpha \in \Lambda_0$. Thus, $u^* \cap v^* = \emptyset$, and consequently, $u^* \cap \bigcup_{\alpha \in \Lambda_0} v_{a_\alpha} = \emptyset$. Since $A \subseteq v^*$, it follows that $u^* \cap A \subseteq u^* \cap v^* = \emptyset$. Therefore, $u^* \cap (\chi \setminus A) \neq \emptyset$, where $u^* = \bigcap_{\alpha \in \Lambda_0} u_{a_\alpha}$, which is open since $\chi$ is a pairwise space. Hence, $\chi \setminus A$ is pairwise open, and thus $A$ is pairwise closed.

**Theorem 5.** *If* $A$ *is a Lindelöf subset of a pairwise Hausdorff space* $\chi$, *then for each* $x \notin A$, *we can separate* $x$ *and* $A$ *into two disjoint open sets in* $\chi$.

**Proof.** Let $i \neq j$, where $(i, j = 1, 2)$. For each $a \in A$, we have $a \neq x$ since $x \notin A$. Since $\chi$ is a pairwise Hausdorff space, there exist pairwise open sets $\{u_a(x), v(a) \in \chi\}$ such that $x \in u_a(x)$, $a \in v(a)$, and $u_a(x) \cap v(a) = \emptyset$. Hence,

$\underset{\sim}{V} = \{v(a) : a \in A\}$ forms a $\lambda_i$-open cover of $A$. Since $A$ is a pairwise Lindelöf subset of $\chi$, the cover $\underset{\sim}{V}$ can be reduced to a countable $\lambda_j$-subcover of $A$, say $\underset{\sim}{V} = \{v(a_\alpha) : \alpha \in \Lambda_0\}$. Thus,

$$A \subseteq \bigcup_{\alpha \in \Lambda_0} v(a_\alpha) = V.$$

For each $v(a_\alpha)$, where $\alpha \in \Lambda_0$, there exists a corresponding pairwise open set $u_{a_\alpha}(x)$ such that $x \in u_{a_\alpha}(x)$ and $u_{a_\alpha}(x) \cap v(a_\alpha) = \emptyset$. Let

$$u = \bigcap_{\alpha \in \Lambda_0} u_{a_\alpha}(x).$$

Then, $u$ is open since $\chi$ is a pairwise space, and $u \subseteq u_{a_\alpha}(x)$ for all $\alpha \in \Lambda_0$. Therefore,

$$u \cap v(a_\alpha) \subseteq u_{a_\alpha}(x) \cap v(a_\alpha) = \emptyset.$$

Thus, $u \cap v(a_\alpha) = \emptyset$, and consequently,

$$u \cap \bigcup_{\alpha \in \Lambda_0} v(a_\alpha) = \emptyset.$$

So, $u \cap V = \emptyset$, where $V = \bigcup_{\alpha \in \Lambda_0} v(a_\alpha)$. Therefore, $x \in u$ and $A \subseteq V$, with $u \cap V = \emptyset$. Hence, we can separate $x$ and $A$ into two disjoint pairwise open sets in $\chi$.

**Theorem 6.** *Every disjoint Lindelöf subset in a pairwise Hausdorff space can be separated by disjoint open sets in* $\chi$.

**Proof.** Let $i \neq j$, where $(i, j) = 1, 2$. Assume $A$ and $B$ are disjoint Lindelöf subsets of a pairwise Hausdorff space $\chi$. For each $a \in A$, we have $a \notin B$ since $A \cap B = \emptyset$. By Theorem (4), there exist $\lambda_i$-open sets $u_a$ and $v_a$ in $\chi$ such that $a \in u_a$, $B \subseteq v_a$, and $u_a \cap v_a = \emptyset$. Hence, $\underset{\sim}{U} = \{u_a : a \in A\}$ forms a $\lambda_i$-open cover of $A$. Since $A$ is a Lindelöf subset of $\chi$, the cover $\underset{\sim}{U}$ can be reduced to a countable $\lambda_j$-subcover, say $\underset{\sim}{U} = \{u_{a_\alpha} : \alpha \in \Lambda_0\}$, where $\Lambda_0$ is countable. Thus,

$$A \subseteq \bigcup_{\alpha \in \Lambda_0} u_{a_\alpha} = u,$$

where $u$ is open. For each $u_{a_\alpha}$ with $\alpha \in \Lambda_0$, there exists a corresponding open set $v_{a_\alpha}$ such that $B \subseteq v$, $v_{a_\alpha} \cap u_{a_\alpha} = \emptyset$. Let

$$v = \bigcap_{\alpha \in \Lambda_0} v_{a_\alpha}.$$

Then, $B \subseteq v$, and $v$ is open in $\chi$. Since $\chi$ is a pairwise space, we have $A \subseteq u$, $B \subseteq v$, and $u, v$ are open in $\chi$. Moreover, since $v \subseteq v_{a_\alpha}$ for all $\alpha \in \Lambda_0$, we have:

$$v \cap u_{a_\alpha} \subseteq u_{a_\alpha} \cap v_{a_\alpha} = \emptyset.$$

Therefore,

$$v \cap u = v \cap \bigcup_{\alpha \in \Lambda_0} u_{a_\alpha} = \emptyset.$$

Thus, $A$ and $B$ can be separated into two disjoint $\lambda_i$-open sets in $\chi$.

**Theorem 7.** *Let $\chi$ be a Lindelöf space, and $Y$ be a pairwise space. Then, the projection $P : \chi \times \chi \to Y$ is a closed map.*

**Proof.** Let $i \neq j$, where $(i, j = 1, 2)$, and let $y \in Y$. Suppose $G$ is open in $\chi \times \chi$ such that $P^{-1}(y) \in G$. We aim to show that there exists a pairwise open set $v$ containing $y$ in $Y$ such that $P^{-1}(v) \subseteq G$. Since $G$ is open in $\chi \times \chi$, for every $(x, y) \in G$ where $(x, y) \in \chi \times Y$, there exist $P$-open basic sets $u_x$ and $v_x$ in $\chi$ and $Y$, respectively, such that $x \in u_x$, $y \in v_x$, and

$$(x, y) \in u_x \times v_x \subseteq G.$$

Hence, $\underset{\sim}{U} = \{u_x : x \in \chi\}$ forms a $\lambda_i$-open cover of $\chi$. Since $\chi$ is a Lindelöf space, the cover $\underset{\sim}{U}$ can be reduced to a $\lambda_j$-countable subcover, say

$$\{u_{x_\alpha} : \alpha \in \Lambda_0\}, \quad |\Lambda_0| \leq \aleph_0 = |\mathbb{N}|.$$

Thus, $\chi \subseteq \bigcup_{\alpha \in \Lambda_0} u_{x_\alpha}$. For all $\{u_{x_\alpha} : \alpha \in \Lambda_0\}$, there exist corresponding sets $\{v_{x_\alpha} : \alpha \in \Lambda_0\}$ such that $y \in v_{x_\alpha}$. Let

$$v = \bigcap_{\alpha \in \Lambda_0} v_{x_\alpha}.$$

Since $\Lambda_0$ is countable and $Y$ is a pairwise space, $v$ is open. Moreover,

$$P^{-1}(v) \subseteq \chi \times v \subseteq u \times v \subseteq G.$$

That is, $P^{-1}(v) \subseteq G$. Hence, $P$ is a closed map.

**Theorem 8.** *Let $f : \chi \to Y$ be a closed, continuous, and surjective function. If for all $y \in Y$, the fibers $f^{-1}(y)$ are Lindelöf and $Y$ is Lindelöf, then $\chi$ is Lindelöf.*

**Proof.** Let $i \neq j$, where $(i, j = 1, 2)$, and let $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ be a $\lambda_i$-open cover of $\chi$. For all $y \in Y$, we have $f^{-1}(y) \subseteq \chi$, so $\underset{\sim}{U}$ is a $\lambda_i$-open cover of $f^{-1}(y)$. Since $f^{-1}(y)$ is Lindelöf, $\underset{\sim}{U}$ can be reduced to a countable $\lambda_j$-subcover, say $\{u_{\alpha y}\}$. Thus,

$$f^{-1}(y) \subseteq \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y},$$

and therefore:

$$f^{-1}(y) \cap \left( \chi - \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y} \right) = \emptyset.$$

It follows that:

$$y \cap f\left( \chi - \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y} \right) = \emptyset.$$

Define:

$$O_y = Y - f\left( \chi - \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y} \right).$$

Since $\{u_{\alpha y}\}$ are open in $\chi$, the set $\bigcup_{\alpha y \in \Lambda_0} u_{\alpha y}$ is open in $\chi$. Consequently, $\chi - \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y}$ is closed in $\chi$, and since $f$ is a closed map, $f\left(\chi - \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y}\right)$ is closed in $Y$. Thus, $O_y$ is open in $Y$. For each $y \in Y$, we have $y \in O_y$. Therefore, $\underset{\sim}{O} = \{O_y : y \in Y\}$ forms a $\lambda_i$-open cover of $Y$. Since $Y$ is Lindelöf, $\underset{\sim}{O}$ can be reduced to a countable $\lambda_j$-subcover, say $\{O_{y_r}\}_{r \in \Gamma_0}$, where $\Gamma_0$ is countable. Thus,

$$Y \subseteq \bigcup_{r \in \Gamma_0} O_{y_r}.$$

Since $f^{-1}(Y) = \chi$, it follows that:

$$\chi = f^{-1}(Y) \subseteq \bigcup_{r \in \Gamma_0} f^{-1}(O_{y_r}).$$

By the construction of $O_{y_r}$, we have:

$$\chi = \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y}.$$

Hence, $\underset{\sim}{U}$ reduces to a countable $\lambda_j$-subcover of $\chi$. Therefore, $\chi$ is Lindelöf.

**Theorem 9.** *The product of two Lindelöf spaces, one of which is a $P$-$\tau_2$ space, is*

*Lindelöf.*

**Proof.** Let $\chi$ and $Y$ be two Lindelöf spaces, and suppose $Y$ is a $P$-$\tau_2$ space. By the previous theorem, the projection function $P : \chi \times Y \to Y$ is $P$-closed. For all $y \in Y$, we have:

$$P^{-1}(y) = \chi \times \{y\} = \chi.$$

Since $\chi$ is Lindelöf, it follows that $P^{-1}(y)$ is Lindelöf. Moreover, $P$ is continuous and onto, which implies that $P$ is a perfect function. By Theorem 5, the product space $\chi \times Y$ is Lindelöf because $Y$ is Lindelöf and $P$ is perfect. Therefore, $\chi \times Y$ is Lindelöf.

**Definition 10.** *Suppose $\chi = (\chi, \lambda)$ is a bitopological space. We say that $A \subset \chi$ is a nearly open set if*

$$A = \left(\overline{A}\right)^{\circ},$$

*where $\overline{A}$ denotes the closure of $A$, and $\left(\overline{A}\right)^{\circ}$ denotes the interior of $\overline{A}$.*

**Definition 11.** *Suppose $\chi = (\chi, \lambda_1, \lambda_2)$ is a bitopological space. A subset $A \subset \chi$ is called a pairwise nearly open set if*

$$A = \left(\overline{A}\right)^{\circ} \text{ in } \lambda_1 \quad \text{and} \quad A = \left(\overline{A}\right)^{\circ} \text{ in } \lambda_2,$$

*where $\overline{A}$ denotes the closure of $A$, and $\left(\overline{A}\right)^{\circ}$ denotes the interior of $\overline{A}$ in the respective topologies.*

## 3. On pairwise nearly Lindelöf space

In this section, we explore the concept of pairwise nearly Lindelöf spaces as an extension of the nearly Lindelöf property within the framework of bitopological spaces. We define pairwise nearly open covers and investigate their role in characterizing pairwise nearly compact and Lindelöf spaces. Additionally, the relationship between pairwise nearly Lindelöf spaces and pairwise continuous functions is established. The focus is on examining the preservation of the pairwise nearly Lindelöf property under specific mappings and their applications in bitopological structures. Examples and counterexamples are provided to illustrate the distinctions between pairwise nearly compact and pairwise nearly Lindelöf spaces, emphasizing the implications of these properties in general topology.

**Definition 12.** *A collection $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is called a nearly open cover if:*

1. *$u_\alpha$ is a pairwise nearly open set for all $\alpha \in \Lambda$.*
2. *$\bigcup_{\alpha \in \Lambda} u_\alpha = \chi$.*

**Definition 13.** *Suppose $\chi = (\chi, \lambda_1, \lambda_2)$ is a bitopological space, and $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is a pairwise nearly open cover of $\chi$. Then, $\underset{\lambda}{S} = \{u_{\alpha_\lambda} : \lambda \in \Gamma, \Gamma \overset{\sim}{\subset} \Lambda\}$ is called a nearly subcover of $\chi$ if*

$$\bigcup_{\alpha \in \Gamma} u_{\alpha_\lambda} = \chi.$$

**Definition 14.** *Suppose $\chi = (\chi, \lambda)$ is a topological space. We say that $\chi$ is a nearly compact space if every nearly open cover of $\chi$ has a finite nearly subcover.*

**Definition 15.** *Suppose $\chi = (\chi, \lambda_1, \lambda_2)$ is a bitopological space. Then, $\chi$ is called a pairwise nearly compact space if every pairwise nearly open cover has a finite nearly subcover.*

**Definition 16.** *Suppose $\chi = (\chi, \lambda)$ is a topological space. Then, $\chi$ is called a nearly Lindelöf space if every nearly open cover of $\chi$ has a countable nearly subcover*

**Definition 17.** *Suppose $\chi = (\chi, \lambda_1, \lambda_2)$ is a bitopological space. Then, $\chi$ is called a pairwise nearly Lindelöf space if every pairwise nearly open cover has a countable nearly subcover.*

**Definition 18.** *A bitopological space $(\chi, \lambda_1, \lambda_2)$ is called $S$-nearly Lindelöf if and only if it is both nearly Lindelöf and pairwise nearly Lindelöf.*

**Remark 3.** *Every pairwise nearly compact space is a pairwise nearly Lindelöf space, but the converse is not necessarily true.*

**Proof.** Let $\chi$ be a pairwise nearly compact space. Then, every nearly $\lambda_i$-open cover of $\chi$ has a finite $\lambda_j$-subcover of $\chi$. Thus, for each nearly $\lambda_i$-open cover of $\chi$, there exists a countable $\lambda_j$-subcover of $\chi$ for all $i \neq j$, where $(i, j = 1, 2)$. Hence, $\chi$ is pairwise nearly Lindelöf. The converse, however, is not necessarily true.

Now, we present an example of a pairwise nearly Lindelöf space that is not pairwise nearly compact. Consider $(\mathbb{R}, \tau_u, \tau_u)$, where $\tau_u$ is the usual topology. Since $B = \{(a, b) : a < b, a, b \in \mathbb{Q}\}$ is a countable base for $(\mathbb{R}, \tau_u)$, the space $(\mathbb{R}, \tau_u)$ is second countable. Furthermore, since every second countable space is Lindelöf, $(\mathbb{R}, \tau_u)$ is a Lindelöf space. Therefore, $(\mathbb{R}, \tau_u, \tau_u)$ is a pairwise Lindelöf space, and consequently, it is pairwise nearly Lindelöf. However, $(\mathbb{R}, \tau_u)$ is not compact because the open cover $\underset{\sim}{U} = \{(-n, n) : n \in \mathbb{N}\}$ of $\mathbb{R}$ does not have a finite subcover. Hence, $(\mathbb{R}, \tau_u, \tau_u)$ is not pairwise compact, and therefore, it is not pairwise nearly compact.

**Theorem 10.** *Let $(\chi, \lambda_1, \lambda_2)$ be a bitopological space. If $\chi$ is a hereditary nearly Lindelöf space, then $\chi$ is $S$-nearly Lindelöf.*

**Proof.** Assume $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\} \cup \{v_\beta : \beta \in \Gamma\}$ is a nearly $\lambda_1, \lambda_2$-open cover of $(\chi, \lambda_1, \lambda_2)$, where $u_\alpha \in \lambda_1$ for every $\alpha \in \Lambda$, and $v_\beta \in \lambda_2$ for every $\beta \in \Gamma$. Since $U = \bigcup\{u_\alpha : \alpha \in \Lambda\}$ is $\lambda_1$-nearly Lindelöf, there exists a countable subset $\Lambda_1 \subset \Lambda$ such that:

$$U = \bigcup\{u_\alpha : \alpha \in \Lambda_1\}.$$

Similarly, since $V = \bigcup\{v_\beta : \beta \in \Gamma\}$ is $\lambda_2$-nearly Lindelöf, there exists a countable subset $\Gamma_1 \subset \Gamma$ such that:

$$V = \bigcup\{v_\beta : \beta \in \Gamma_1\}.$$

It is clear that:

$$\{u_\alpha : \alpha \in \Lambda_1\} \cup \{v_\beta : \beta \in \Gamma_1\}$$

is a countable subcover of $\underset{\sim}{U}$ for $\chi$. Therefore, $\chi$ is $S$-nearly Lindelöf.

**Theorem 11.** *A pairwise nearly Lindelöf space is preserved under an onto pairwise continuous function.*

**Proof.** Let $i \neq j$, where $(i, j = 1, 2)$. Let $F : (\chi, \lambda_1, \lambda_2) \to (Y, \mu_1, \mu_2)$ be a surjective continuous function, and suppose $\chi$ is a nearly Lindelöf space. We aim to show that $Y$ is also a nearly Lindelöf space. Assume $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is a nearly $\lambda_i$-open cover of $Y$. Then, $u_\alpha$ is open for all $\alpha \in \Lambda$. Since $F$ is continuous, the preimage $F^{-1}(u_\alpha)$ is open in $\chi$ for each $\alpha \in \Lambda$. As $F$ is surjective, $\{F^{-1}(u_\alpha) : \alpha \in \Lambda\}$ forms an open cover of $\chi$. Because $\chi$ is a nearly Lindelöf space, this cover can be reduced to a countable $\lambda_j$-subcover, say $\{F^{-1}(u_\alpha) : \alpha \in \Gamma\}$, where $\Gamma \subset \Lambda$ and $|\Gamma| \leq \aleph_0 = |\mathbb{N}|$. Thus:

$$\chi \subseteq \bigcup_{\alpha \in \Gamma} F^{-1}(u_\alpha).$$

Since $F$ is surjective, we have:

$$Y = F(\chi) \subseteq F\left(\bigcup_{\alpha \in \Gamma} F^{-1}(u_\alpha)\right) \subseteq \bigcup_{\alpha \in \Gamma} u_\alpha.$$

Therefore, $\underset{\sim}{U}$ has a countable $\lambda_j$-subcover of $Y$, which proves that $Y$ is a nearly Lindelöf space.

**Remark 4.** *A compact subset in a pairwise nearly $\tau_2$-space is closed, but a nearly Lindelöf subset in a pairwise nearly $\tau_2$-space need not be closed. For example,* $(\mathbb{R}, \tau_u, \tau_u)$ *is a pairwise nearly $\tau_2$-space, and* $(0, 1)$ *is a nearly Lindelöf subset of* $\mathbb{R}$. *However,* $(0, 1)$ *is not pairwise nearly closed in* $\mathbb{R}$.

**Definition 19.** *A space $(\chi, \lambda_1, \lambda_2)$ is called a pairwise nearly pairwise space if and only if the countable intersection of nearly open sets is nearly open.*

**Theorem 12.** *Every nearly Lindelöf subset of a nearly $\tau_2$ pairwise space is pairwise nearly closed.*

**Proof.** Let $i \neq j$, where $(i, j = 1, 2)$. Let $A$ be a nearly Lindelöf subset of a nearly $P$-$\tau_2$ space $\chi$. To show that $A$ is pairwise nearly closed, it is enough to show that $\chi \setminus A$ is pairwise nearly open. Let $x \in \chi \setminus A$. Then $x \notin A$, and for each $a \in A$, we have $x \neq a$. Since $\chi$ is a $\tau_2$ space, there exist nearly open sets $u_a$ and $v_a$ in $\chi$ such that $x \in u_a$, $a \in v_a$, and $u_a \cap v_a = \emptyset$. Hence, $\underset{\sim}{V} = \{v_a : a \in A\}$ forms a nearly $\lambda_i$-open cover of $A$. Since $A$ is a $P$-nearly Lindelöf subset of $\chi$, the cover $\underset{\sim}{V}$ reduces to a countable $\lambda_j$-subcover, say:

$$\underset{\sim}{V} = \{v_{a_\alpha} : \alpha \in \Lambda_0\},$$

where $\Lambda_0$ is countable. Thus:

$$A \subseteq \bigcup_{\alpha \in \Lambda_0} v_{a_\alpha}.$$

For each $v_{a_\alpha}$, where $\alpha \in \Lambda_0$, there exists a corresponding nearly open set $u_{a_\alpha}$ such that $x \in u_{a_\alpha}$, $a_\alpha \in v_{a_\alpha}$, and $u_{a_\alpha} \cap v_{a_\alpha} = \emptyset$. Let:

$$V^* = \bigcup_{\alpha \in \Lambda_0} v_{a_\alpha}, \quad U^* = \bigcap_{\alpha \in \Lambda_0} u_{a_\alpha}.$$

Then $A \subseteq V^*$, $x \in U^*$, and:

$$U^* \cap V^* \subseteq u_{a_\alpha} \cap v_{a_\alpha} = \emptyset \quad \text{for all } \alpha \in \Lambda_0.$$

Hence:

$$U^* \cap V^* = \emptyset.$$

Since $A \subseteq V^*$, it follows that:

$$U^* \cap A \subseteq U^* \cap V^* = \emptyset.$$

Thus, $x \in U^* \subseteq \chi \setminus A$, where $U^* = \bigcap_{\alpha \in \Lambda_0} u_{a_\alpha}$ is nearly open because $\chi$ is a pairwise nearly space. Therefore, $\chi \setminus A$ is pairwise nearly open, and $A$ is pairwise nearly closed.

**Theorem 13.** *Let $A$ be a nearly Lindelöf subset of a pairwise nearly $\tau_2$-space $\chi$. Then, for each $x \notin A$, we can separate $x$ and $A$ into two disjoint nearly open sets in $\chi$.*

**Proof.** Let $i \neq j$, where $(i,j) = 1,2$. For each $a \in A$, we have $a \neq x$ since $x \notin A$. As $\chi$ is a $\tau_2$-space, there exist pairwise nearly open sets $u_a(x)$ and $v(a)$ in $\chi$ such that $x \in u_a(x)$, $a \in v(a)$, and $u_a(x) \cap v(a) = \emptyset$. Hence, $\underset{\sim}{V} = \{v(a) : a \in A\}$ forms a $\lambda_i$-open cover of $A$. Since $A$ is a pairwise nearly Lindelöf subset of $\chi$, the cover $\underset{\sim}{V}$ can be reduced to a countable $\lambda_j$-subcover of $A$, say:

$$\underset{\sim}{V} = \{v(a_\alpha) : \alpha \in \Lambda_0\}.$$

Thus:

$$A \subseteq \bigcup_{\alpha \in \Lambda_0} v(a_\alpha) = V.$$

For all $v(a_\alpha)$, where $\alpha \in \Lambda_0$, there exists a corresponding pairwise nearly open set $u_{a_\alpha}(x)$ such that:

$$x \in u_{a_\alpha}(x), \quad a_\alpha \in v(a_\alpha), \quad \text{and} \quad u_{a_\alpha}(x) \cap v(a_\alpha) = \emptyset.$$

Let:

$$U = \bigcap_{\alpha \in \Lambda_0} u_{a_\alpha}(x).$$

Since $\chi$ is a pairwise nearly space, $U$ is nearly open, and $U \subseteq u_{a_\alpha}(x)$ for all $\alpha \in \Lambda_0$. Hence:

$$U \cap v(a_\alpha) \subseteq u_{a_\alpha}(x) \cap v(a_\alpha) = \emptyset \quad \text{for all } \alpha \in \Lambda_0.$$

Thus:

$$U \cap V = U \cap \bigcup_{\alpha \in \Lambda_0} v(a_\alpha) = \emptyset.$$

Therefore, $x \in U$, $A \subseteq V$, and $U \cap V = \emptyset$. This shows that $x$ and $A$ can be separated into two disjoint $P$-nearly open sets in $\chi$.

**Theorem 14.** *Every pair of disjoint nearly Lindelöf subsets in a pairwise nearly Hausdorff space can be separated by disjoint nearly open sets in $\chi$.*

**Proof.** Let $i \neq j$, where $(i,j) = 1,2$. Assume $A$ and $B$ are disjoint nearly Lindelöf subsets of a pairwise nearly Hausdorff space $\chi$. For each $a \in A$, we have $a \notin B$ since $A \cap B = \emptyset$. By Theorem (4), there exist nearly $\lambda_i$-open sets $u_a$ and $v_a$ in $\chi$ such that $a \in u_a$, $B \subseteq v_a$, and $u_a \cap v_a = \emptyset$. Hence, $\underset{\sim}{U} = \{u_a : a \in A\}$ forms a $\lambda_i$-open cover of $A$. Since $A$ is a nearly Lindelöf subset of $\chi$, the cover $\underset{\sim}{U}$ can be reduced to a countable $\lambda_j$-subcover, say:

$$\underset{\sim}{U} = \{u_{a_\alpha} : \alpha \in \Lambda_0\},$$

where $\Lambda_0$ is countable. Thus:

$$A \subseteq \bigcup_{\alpha\in\Lambda_0} u_{a_\alpha} = U,$$

where $U$ is open. For each $u_{a_\alpha}$, where $\alpha \in \Lambda_0$, there exists a corresponding nearly open set $v_{a_\alpha}$ such that $B \subseteq v$, and:

$$v_{a_\alpha} \cap u_{a_\alpha} = \emptyset.$$

Let:

$$v = \bigcap_{\alpha\in\Lambda_0} v_{a_\alpha}.$$

Then $B \subseteq v$, and $v$ is nearly open in $\chi$. Since $\chi$ is a pairwise nearly space, we have $A \subseteq U$, $B \subseteq v$, and $U, v$ are nearly open in $\chi$. Moreover, since $v \subseteq v_{a_\alpha}$ for all $\alpha \in \Lambda_0$, we have:

$$v \cap u_{a_\alpha} \subseteq u_{a_\alpha} \cap v_{a_\alpha} = \emptyset.$$

Thus:

$$v \cap U = v \cap \bigcup_{\alpha\in\Lambda_0} u_{a_\alpha} = \emptyset.$$

Therefore, $A$ and $B$ can be separated by two disjoint nearly $\lambda_i$-open sets in $\chi$.

**Theorem 15.** *Let $\chi$ be a nearly Lindelöf space and $Y$ a pairwise nearly space. Then, the projection $P : \chi \times \chi \to Y$ is closed.*

**Proof.** Let $i \neq j$, where $(i, j) = 1, 2$. Let $y \in Y$ and $G$ be an open set in $\chi \times \chi$ such that $P^{-1}(y) \in G$. We aim to show that there exists a pairwise nearly open set $v$ containing $y$ in $Y$ such that $P^{-1}(v) \subseteq G$. Since $G$ is open in $\chi \times \chi$, for each $(x, y) \in G$ where $(x, y) \in \chi \times Y$, there exist pairwise nearly open basic sets $u_x$ and $v_x$ in $\chi$ and $Y$, respectively, such that:

$$x \in u_x, \quad y \in v_x, \quad \text{and} \quad (x, y) \in u_x \times v_x \subseteq G.$$

Hence, $\underset{\sim}{U} = \{u_x : x \in \chi\}$ forms a $\lambda_i$-open cover of $\chi$. Since $\chi$ is a nearly Lindelöf space, the cover $\underset{\sim}{U}$ can be reduced to a $\lambda_j$-countable subcover, say:

$$\{u_{x_\alpha} : \alpha \in \Lambda_0\},$$

where $|\Lambda_0| \leq \aleph_0 = |\mathbb{N}|$. Thus:

$$\chi \subseteq \bigcup_{\alpha\in\Lambda_0} u_{x_\alpha}.$$

For all $\{u_{x_\alpha} : \alpha \in \Lambda_0\}$, there exist corresponding pairwise nearly open sets $\{v_{x_\alpha} : \alpha \in \Lambda_0\}$ such that:

$$y \in v_{x_\alpha} \quad \text{for all } \alpha \in \Lambda_0.$$

Let:

$$v = \bigcap_{\alpha \in \Lambda_0} v_{x_\alpha}.$$

Since $\Lambda_0$ is countable and $Y$ is a pairwise nearly space, $v$ is nearly open in $Y$. Moreover:

$$P^{-1}(v) \subseteq \chi \times v \subseteq u \times v \subseteq G.$$

Thus, $P^{-1}(v) \subseteq G$, which shows that $P$ is closed.

**Theorem 16.** *Let $f : \chi \to Y$ be a closed, continuous, surjective function. If for all $y \in Y$, the fibers $f^{-1}(y)$ are nearly Lindelöf, and if $Y$ is nearly Lindelöf, then $\chi$ is nearly Lindelöf.*

**Proof.** Let $i \neq j$, where $(i, j = 1, 2)$. Let $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ be a nearly $\lambda_i$-open cover of $\chi$. For all $y \in Y$, we have $f^{-1}(y) \subseteq \chi$, so $\underset{\sim}{U}$ is a $\lambda_i$-open cover of $f^{-1}(y)$. Since $f^{-1}(y)$ is nearly Lindelöf, the cover $\underset{\sim}{U}$ can be reduced to a countable $\lambda_j$-subcover, say $\{u_{\alpha y}\}$. Thus:

$$f^{-1}(y) \subseteq \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y}.$$

This implies:

$$f^{-1}(y) \cap \left(\chi - \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y}\right) = \emptyset.$$

Hence:

$$y \cap f\left(\chi - \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y}\right) = \emptyset.$$

Define:

$$O_y = Y - f\left(\chi - \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y}\right).$$

Since $u_{\alpha y}$ is open in $\chi$, the set $\bigcup_{\alpha y \in \Lambda_0} u_{\alpha y}$ is open in $\chi$. Consequently, $\chi - \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y}$ is closed in $\chi$, and since $f$ is closed, $f\left(\chi - \bigcup_{\alpha y \in \Lambda_0} u_{\alpha y}\right)$ is closed in $Y$. Thus, $O_y$ is open in $Y$. For all $y \in Y$, we have $y \in O_y$, so:

$$\underset{\sim}{O} = \{O_y : y \in Y\}$$

is a nearly $\lambda_i$-open cover of $Y$. Since $Y$ is nearly Lindelöf, $\underset{\sim}{O}$ can be reduced to a countable $\lambda_j$-subcover, say:

$$\{O_{y_r} : r \in \Gamma_0\}, \quad \text{where } \Gamma_0 \text{ is countable.}$$

Thus:

$$Y \subseteq \bigcup_{r\in\Gamma_0} O_{y_r}.$$

Since $f^{-1}(Y) = \chi$, it follows that:

$$\chi = f^{-1}(Y) \subseteq \bigcup_{r\in\Gamma_0} f^{-1}(O_{y_r}).$$

From the construction of $O_{y_r}$, we have:

$$\chi = \bigcup_{\alpha y\in\Lambda_0} u_{\alpha y}.$$

Hence, $\underset{\sim}{U}$ reduces to a countable $\lambda_j$-subcover of $\chi$. Therefore, $\chi$ is nearly Lindelöf.

**Theorem 17.** *The product of two nearly Lindelöf spaces, one of which is a pairwise $\tau_2$-space, is nearly Lindelöf.*

**Proof.** Let $\chi$ and $Y$ be two nearly Lindelöf spaces, and suppose $Y$ is a $P$-nearly $\tau_2$-space. By the previous theorem, the projection function $P : \chi \times Y \to Y$ is pairwise closed. For all $y \in Y$, we have:

$$P^{-1}(y) = \chi \times \{y\} = \chi.$$

Since $\chi$ is nearly Lindelöf, it follows that $P^{-1}(y)$ is nearly Lindelöf. Furthermore, $P$ is continuous and surjective, making it a perfect function. By Theorem (5), the product space $\chi \times Y$ is nearly Lindelöf because $Y$ is nearly Lindelöf and $P$ is perfect. Thus, $\chi \times Y$ is nearly Lindelöf.

## 4. Conclusion

In this work, we explored various properties and results related to nearly Lindelöf spaces in the context of general topology and bitopological spaces. The extension of classical Lindelöf and compactness properties to nearly open and pairwise settings allowed for a deeper understanding of the structural and functional behavior of these spaces. Key results, such as the preservation of the nearly Lindelöf property under continuous surjective functions, the separation of disjoint subsets in pairwise nearly Hausdorff spaces, and the conditions under which product spaces retain the nearly Lindelöf property, were established. By examining specific examples and counterexamples, we highlighted the distinctions between nearly Lindelöf spaces and related concepts like compact and nearly compact spaces. These findings not only expand the theoretical framework of general topology but also provide potential applications in broader areas such as functional analysis and mathematical modeling.Future research can focus on further generalizations of nearly Lindelöf spaces, their connections to other topological properties, and the exploration of nearly Lindelöf spaces in more complex topological structures, such as bitopological or fuzzy spaces.

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