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INCLUSION PROPERTIES INVOLVING GAUSSIAN HYPERGEOMETRIC FUNCTIONS FOR UNIVALENT FUNCTIONS WITH BOUNDED BOUNDARY ROTATION

S. Kavitha and M. Raja Rajeswari*

Department of Mathematics,
SDNB Vaishnav College for Women,
Chennai - 600044, Tamil Nadu, INDIA

E-mail : kavi080716@gmail.com

*Department of Mathematics,
Ethiraj College for Women,
Chennai - 600008, Tamil Nadu, INDIA

E-mail : rajarajeswari_m@ethirajcollege.edu.in

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Abstract: In this current article, the authors obtain certain inclusion relation between few subclasses of univalent functions with bounded boundary rotation and the familiar Gauss hypergeometric functions. The results obtained in this article includes earlier results available in the literature.

Keywords and Phrases: Analytic, Univalent, Gaussian Hypergeometric, Bounded boundary rotation.

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1. Introduction

Let $\mathcal{A}$ be the class of functions f normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be starlike of order γ $(0 \leq \gamma < 1)$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma \quad (z \in \mathbb{U}).$$

This function class is denoted by $\mathcal{S}^*(\gamma)$. We also write $\mathcal{S}^*(0) =: \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that are starlike in $\mathbb{U}$ with respect to the origin.

A function $f \in \mathcal{A}$ is said to be convex of order γ $(0 \leq \gamma < 1)$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma \quad (z \in \mathbb{U}).$$

This class is denoted by $\mathcal{C}(\gamma)$. Further, $\mathcal{C} = \mathcal{C}(0)$, the well-known standard class of convex functions. It is an established fact that

$$f \in \mathcal{C}(\gamma) \Longleftrightarrow zf' \in \mathcal{S}^*(\gamma).$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UCD}(\beta)$, $\beta \in \mathbb{R}$, if

$$\Re(f'(z)) \geq \beta|zf''(z)| \quad (z \in \mathbb{U}).$$

The class $\mathcal{UCD}(\beta)$ is introduced by Breaz [2].

Let $m \geq 2$ and $0 \leq \gamma < 1$. Let $\mathcal{P}_m(\gamma)$ denote the class of functions p, that are analytic and normalized with $p(0) = 1$, satisfying the condition

$$\int_0^{2\pi} \left|\frac{\Re(p(z)) - \gamma}{1 - \gamma}\right| d\theta \leq m\pi,$$

where $z = re^{i\theta} \in \mathbb{U}$. The class $\mathcal{P}_m(\gamma)$ was introduced by Padmanabhan and Parvatham [12] If $\gamma = 0$, we denote $\mathcal{P}_m(0)$ as $\mathcal{P}_m$. Hence the class $\mathcal{P}_m$ (defined by Pinchuk [13]) represents the class of analytic functions $p(z)$, with $p(0) = 1$ and the function $p \in \mathcal{P}_m$ will be having a representation

$$p(z) = \int_0^{2\pi} \left|\frac{1 - ze^{it}}{1 + ze^{it}}\right| d\mu(t),$$

where μ is a real-valued function with bounded variation satisfying

$$\int_0^{2\pi} d\mu(t) = 2 \text{ and } \int_0^{2\pi} |d\mu(t)| \leq m,\, m \geq 2.$$

Remark 1. *$\mathcal{P} \equiv \mathcal{P}_2$ is the class of analytic functions with positive real part in $\mathbb{U}$, familiarly called as the class of Carathéodory functions.*

For the class $\mathcal{P}_m$ the following lemma was proved.

Lemma 1. [13] *For $p \in \mathcal{P}_m$, there exists $p_1, p_2 \in \mathcal{P}$ such that*

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z).$$

Let $\mathcal{R}_m(\gamma)$ represent the class of analytic functions $h(z)$ in $\mathbb{U}$ with $h(0) = 0$, $h'(0) = 1$ and satisfying

$$\frac{zh'(z)}{h(z)} \in \mathcal{P}_m(\gamma).$$

This class generalizes the class $\mathcal{S}^*(\gamma)$ of starlike functions of the order γ, investigated by Robertson [16]. For $\gamma = 0$, we get the class $\mathcal{R}_m(0) \equiv \mathcal{R}_m$, the class of all functions of bounded boundary rotation. Therefore, the functions $h \in \mathcal{R}_m$ will be having a representation

$$h(z) = z \exp\left\{ \int_0^{2\pi} -\log\left(1 - ze^{it}\right) d\mu(t) \right\},$$

where μ is a real-valued function with bounded variation satisfying

$$\int_0^{2\pi} d\mu(t) = 2 \text{ and } \int_0^{2\pi} |d\mu(t)| \leq m,\, m \geq 2.$$

Let $\mathcal{V}_m(\gamma)$ denote the class of all analytic functions $h(z)$ in $\mathbb{U}$ normalized by $h(0) = 0$ and $h'(0) = 1$, satisfying

$$1 + \frac{zh''(z)}{h'(z)} \in \mathcal{P}_m(\gamma),\, 0 \leq \gamma < 1.$$

For $\gamma = 0$, we get the class $\mathcal{V}_m(0) \equiv \mathcal{V}_m$, the class of all analytic functions of bounded boundary rotation studied by Paatero [11]. Therefore, the functions $h \in \mathcal{V}_m$ will be having a representation

$$h'(z) = \exp\left\{\int_0^{2\pi} -\log\left(1 - ze^{it}\right) d\mu(t)\right\},$$

where μ is a real-valued function with bounded variation satisfying

$$\int_0^{2\pi} d\mu(t) = 2 \text{ and } \int_0^{2\pi} |d\mu(t)| \leq m,\, m \geq 2.$$

This class $\mathcal{V}_m(\gamma)$ generalize the class of all convex functions $\mathcal{C}(\gamma)$ of order γ introduced by Robertson [16]. Interesting connection for the classes $\mathcal{V}_m(\gamma)$ and $\mathcal{R}_m(\gamma)$ with $\mathcal{P}_m(\gamma)$ was established by Pinchuk [13] and are given by

$$h(z) \in \mathcal{V}_m(\gamma) \Longleftrightarrow 1 + \frac{zh''(z)}{h'(z)} \in \mathcal{P}_m(\gamma),$$

$$h(z) \in \mathcal{R}_m(\gamma) \Longleftrightarrow \frac{zh'(z)}{h(z)} \in \mathcal{P}_m(\gamma)$$

and

$$h(z) \in \mathcal{V}_m(\gamma) \Longleftrightarrow zh'(z) \in \mathcal{R}_m(\gamma).$$

Let $\mathcal{S}_m$ be the subclass of $\mathcal{V}_m$ whose members are univalent in $\mathbb{U}$. It was pointed out by Paatero [11] that $\mathcal{V}_m$ coincides with $\mathcal{S}_m$ whenever $2 \leq m \leq 4$. Pinchuk [13] also proved that functions in $\mathcal{V}_m$ are close-to-convex in $\mathbb{U}$ if $2 \leq m \leq 4$ and hence are univalent. Brannan [1] showed that $\mathcal{V}_m$ is a subclass of the class $\mathcal{K}(\gamma)$ of close-to-convex of order $\gamma = \frac{m}{2} - 1$. If $f \in \mathcal{V}_m(\gamma)$, and $n = 2, 3$, the sharp results $|a_2| \leq \frac{m}{2}$, and $|a_3| \leq \frac{m^2+2}{6}$ was proved by Lehto [8]. Coefficient bounds for few subclasses of bi-univalent functions involving bounded boundary rotation has been obtained by Sharmal et al. very recently [18].

The Gaussian hypergeometric function $\mathcal{F}(a, b; c; z)$ given by

$$\mathcal{F}(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n \quad (z \in \mathbb{U}) \tag{1.2}$$

is the solution of the homogenous hypergeometric differential equation

$$z(1-z)w''(z)+[c-(a+b+1)z]w'(z)-abw(z)=0$$

and has affluent application in diverse field such as conformal mappings, quasi conformal theory, continued fractions and so on.

At this juncture, a, b, c are complex numbers such that $c \neq 0, -1, -2, -3, \ldots$, $(a)_0 = 1$ for $a \neq 0$, and for each positive integer n, $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$ is the Pochhammer symbol. In the case of $c = -k$, $k = 0, 1, 2, \ldots$, $\mathcal{F}(a, b; c; z)$ is defined if $a = -j$ or $b = -j$ where $j \leq k$. In this circumstances, $\mathcal{F}(a, b; c; z)$ becomes a polynomial of degree j with respect to z. Results concerning $\mathcal{F}(a, b; c; z)$ when $\Re\ (c-a-b)$ is positive, zero or negative are abundant in the literature. In exacting when $\Re\ (c-a-b) > 0$, the function is bounded. This and the zero balanced case $\Re\ (c-a-b) = 0$ are elaborately discussed by lots of authors (see [14]). The hypergeometric function $\mathcal{F}(a, b; c; z)$ has been considered widely by different authors and play an imperative role in Geometric Function Theory. It is handy in unifying various functions by giving appropriate values to the parameters a, b and c. We pass on to [3, 7, 15, 17, 19, 20, 21, 22] and references in that for some significant consequences.

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}. \tag{1.3}$$

For $f \in \mathcal{A}$, we recall the operator $\mathcal{I}_{a,b,c}(f)$ of Hohlov [6] which maps $\mathcal{A}$ into itself defined by means of Hadamard product as

$$\mathcal{I}_{a,b,c}(f)(z) = z\mathcal{F}(a, b; c; z) * f(z). \tag{1.4}$$

Therefore, for a function f defined by (1.1), we have

$$\mathcal{I}_{a,b,c}(f)(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n. \tag{1.5}$$

Using the integral representation,

$$\mathcal{F}(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{\mathrm{d}t}{(1-tz)^a}, \quad \Re(c) > \Re(b) > 0,$$

we can write

$$[\mathcal{I}_{a,b,c}(f)](z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \frac{f(tz)}{t} \mathrm{d}t * \frac{z}{(1-tz)^a}.$$

When $f(z)$ equals the convex function $\dfrac{z}{1-z}$, then the operator $\mathcal{I}_{a,b,c}(f)$ in this case becomes $z\mathcal{F}(a,b;c;z)$. If $a = 1, b = 1+\delta$, $c = 2+\delta$ with $\Re(\delta) > -1$ then the convolution operator $\mathcal{I}_{a,b,c}(f)$ turns into Bernardi operator

$$B_f(z) = [\mathcal{I}_{a,b,c}(f)](z) = \frac{1+\delta}{z^\delta} \int_0^1 t^{\delta-1} f(t)\mathrm{d}t.$$

Indeed, $I_{1,1,2}(f)$ and $I_{1,2,3}(f)$ are known as Alexander and Libera operators, respectively.

Next, let us consider the integral operator $G(a,b;c;z)$ as follows.

$$\mathcal{G}(a,b;c;z) = \int_0^z \mathcal{G}(a,b;c;t)dt = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} z^n.$$

. For $f \in \mathcal{A}$, let us consider the operator

$$\begin{aligned} \mathcal{J}(a,b;c;z) &= \mathcal{G}(a,b;c;z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} a_n z^n \end{aligned}$$

To prove the main results, we need the following Lemmas.

Lemma 2. [2] *A function $f \in \mathcal{A}$ is in the class $\mathcal{UCD}(\beta)$ if*

$$\sum_{n=2}^{\infty} n(1+\beta(n-1))|a_n| \leq 1. \tag{1.6}$$

Lemma 3. [9] *If $f \in \mathcal{R}_m(\gamma)$ is of form (1.1), then*

$$|a_n| \leq \frac{(m(1-\gamma))_{n-1}}{(n-1)!}, \quad \textit{for all } n \geq 2. \tag{1.7}$$

The result is sharp.

Lemma 4. [9] *If $f \in \mathcal{V}_m(\gamma)$ is of form (1.1), then*

$$|a_n| \leq \frac{(m(1-\gamma))_{n-1}}{n!}, \quad \textit{for all } n \geq 2. \tag{1.8}$$

The result is sharp.

In this current article, the authors obtain certain inclusion relation between certain inclusion relations involving the classes $\mathcal{V}_m(\gamma)$, $\mathcal{R}_m(\gamma)$ of functions with bounded boundary rotation and $\mathcal{UCD}(\beta)$. The results obtained in this article includes earlier results available in the literature.

2. Main results

Theorem 1. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number. If $f \in \mathcal{V}_m(\gamma)$, and the inequality*

$$\begin{aligned}&{}_3\mathcal{F}_2(|a|, |b|, m(1-\gamma); c, 1; 1)\\ &\qquad + \frac{m(1-\gamma)\beta|ab|}{c}\ {}_3\mathcal{F}_2(|a|+1, |b|+1, m(1-\gamma)+1; c+1, 2; 1) \leq 2 \end{aligned} \tag{2.1}$$

is satisfied, then $\mathcal{I}_{a,b,c}(f) \in \mathcal{UCD}(\beta)$.

Proof. Let f be given by (1.1). By (1.6), to show $\mathcal{I}_{a,b,c}(f) \in \mathcal{UCD}(\beta)$ it is sufficient to prove that

$$\sum_{n=2}^{\infty} n\left[1+\beta(n-1)\right] \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n\right| \leq 1. \tag{2.2}$$

Applying the estimates for the coefficients given by (1.8), and making use of the relation $|(a)_n| \leq (|a|)_n$, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\left[1+\beta(n-1)\right] \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n\right| \\
\leq\ & \sum_{n=2}^{\infty} n[1+\beta(n-1)] \frac{(|a|)_{n-1}(|b|)_{n-1}(m(1-\gamma))_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_n} \\
=\ & \sum_{n=2}^{\infty} \frac{(|a|)_{n-1}(|b|)_{n-1}(m(1-\gamma))_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}} \\
&\qquad + \beta\sum_{n=2}^{\infty}(n-1)\frac{(|a|)_{n-1}(|b|)_{n-1}(m(1-\gamma))_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}} \\
=\ & \left[{}_3\mathcal{F}_2(|a|, |b|, P_1; c, 1; 1) - 1\right] \\
&\qquad + \frac{m(1-\gamma)\beta|ab|}{c}\ {}_3\mathcal{F}_2(|a|+1, |b|+1, m(1-\gamma)+1; c+1, 2; 1) \\
\leq\ & 1
\end{aligned}$$

provided the condition (2.1) is satisfied.

If $|b| = |a|$ we can rewrite the Theorem 1 as follows.

Corollary 1. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Suppose that $|b| = |a|$. Also, let c be a real number. If $f \in \mathcal{V}_m(\gamma)$ and the inequality*

$$
\begin{aligned}
&{}_3\mathcal{F}_2(|a|, |a|, m(1-\gamma); c, 1; 1) \\
&\quad + \frac{m(1-\gamma)\beta|a|^2}{c}\ {}_3\mathcal{F}_2(|a|+1, |a|+1, m(1-\gamma)+1; c+1, 2; 1) \leq 2 \qquad (2.3)
\end{aligned}
$$

is satisfied, then $\mathcal{I}_{a,b,c}(f) \in \mathcal{UCD}(\beta)$.

In the special case when $b = 1$, Theorem 1 immediately yields a result concerning the Carlson-Shaffer operator $\mathcal{L}(a, c)(f) = I_{a,1,c}(f)$.

Corollary 2. *Let $a \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number. If $f \in \mathcal{V}_m(\gamma)$ and the inequality*

$$
\begin{aligned}
&\mathcal{F}(|a|, m(1-\gamma); c; 1) \\
&\quad + \frac{m(1-\gamma)\beta|a|}{c}\ \mathcal{F}(|a|+1, m(1-\gamma)+1; c+1; 1) \leq 2 \qquad (2.4)
\end{aligned}
$$

is satisfied, then $\mathcal{L}(a, c)(f) \in \mathcal{UCD}(\beta)$.

For the choice of $\gamma = 0$ and $m = 2$, Theorem 1 immediately gives the following corollary as in Corollary 3.

Corollary 3. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number such that $c > |a|+|b|+2$. Let $f \in \mathcal{A}$ and be of the form (1.1). If the hypergeometric inequality*

$$
\begin{aligned}
&\frac{\Gamma(c)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}[(c-|a|-|b|-2)(c-|a|-|b|-1) \\
&\quad + \beta|ab|(1+|a|)(1+|b|) + (1+2\beta)|ab|(c-|a|-|b|-2)] \leq 2
\end{aligned}
$$

is satisfied, then

$$\mathcal{I}_{a,b,c}(f) \in \mathcal{UCD}(\beta).$$

Theorem 2. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number. If, $f \in \mathcal{R}_m(\gamma)$ and the inequality*

$$
\begin{aligned}
&{}_3\mathcal{F}_2(|a|, |b|, m(1-\gamma); c, 1; 1) \\
&\quad + (1+\beta)\frac{m(1-\gamma)|ab|}{c}\ {}_3\mathcal{F}_2(|a|+1, |b|+1, m(1-\gamma)+1; c+1, 2; 1) \\
&\quad + \frac{m(1-\gamma)\beta|ab|}{c}\ {}_3\mathcal{F}_2(|a|+1, |b|+1, m(1-\gamma)+1; c+1, 1; 1) \\
&\quad \leq 2 \qquad (2.5)
\end{aligned}
$$

is satisfied, then $\mathcal{I}_{a,b,c}(f) \in \mathcal{UCD}(\beta)$.

Proof. Let f be given by (1.1). By (1.6), to show $\mathcal{I}_{a,b,c}(f) \in \mathcal{UCD}(\beta)$ it is sufficient to prove that

$$\sum_{n=2}^{\infty} n\,[1+\beta(n-1)]\ \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \leq 1. \tag{2.6}$$

$$\begin{aligned}\sum_{n=2}^{\infty} n\,[1+\beta(n-1)]\ &\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \\ &= \sum_{n=2}^{\infty} (n-1+1)\,[1+\beta(n-1)]\ \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \end{aligned}\tag{2.7}$$

Applying the estimates for the coefficients given by (1.7), and making use of the relation $|(a)_n| \leq (|a|)_n$, we get

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\,[1+\beta(n-1)]\ \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right| \\
\leq\ & \sum_{n=2}^{\infty}[1+\beta(n-1)]\frac{(|a|)_{n-1}(|b|)_{n-1}(m(1-\gamma))_{n-1}}{(c)_{n-1}(1)_{n-2}(1)_{n-1}} \\
+\ & \sum_{n=2}^{\infty}[1+\beta(n-1)]\frac{(|a|)_{n-1}(|b|)_{n-1}(m(1-\gamma))_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}} \\
=\ & \sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(m(1-\gamma))_{n-1}}{(c)_{n-1}(1)_{n-2}(1)_{n-1}} \\
+\ & \beta\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(m(1-\gamma))_{n-1}}{(c)_{n-1}(1)_{n-2}(1)_{n-2}} \\
+\ & \sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(m(1-\gamma))_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}} \\
+\ & \beta\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(m(1-\gamma))_{n-1}}{(c)_{n-1}(1)_{n-2}(1)_{n-1}}
\end{aligned}$$

$$\begin{aligned}
=\ & \frac{|ab|m(1-\gamma)}{c}\ {}_3\mathcal{F}_2(1+|a|,1+|b|,1+m(1-\gamma);1+c,2;1) \\
+\ & \beta\frac{|ab|m(1-\gamma)}{c}\ {}_3\mathcal{F}_2(1+|a|,1+|b|,1+m(1-\gamma);1+c,1;1) \\
+\ & {}_3\mathcal{F}_2(|a|,|b|,m(1-\gamma);c,1;1)-1 \\
+\ & \beta\frac{|ab|m(1-\gamma)}{c}\ {}_3\mathcal{F}_2(1+|a|,1+|b|,1+m(1-\gamma);1+c,2;1).
\end{aligned}$$

The last inequality is bounded above by 1 provided the condition (2.5) is satisfied.

If $|b| = |a|$ we can rewrite the Theorem 2 as follows.

Corollary 4. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Suppose that $|b| = |a|$. Also, let c be a real number. If $f \in \mathcal{V}_m(\gamma)$ and the inequality*

$$
\begin{aligned}
&{}_3\mathcal{F}_2(|a|, |a|, m(1-\gamma); c, 1; 1) \\
&+ \frac{m(1-\gamma)(1+\beta)|a|^2}{c} \, {}_3\mathcal{F}_2(|a|+1, |a|+1, m(1-\gamma)+1; c+1, 2; 1) \\
&\quad + \frac{m(1-\gamma)\beta|a|^2}{c} \, {}_3\mathcal{F}_2(|a|+1, |a|+1, m(1-\gamma)+1; c+1, 1; 1) \leq 2
\end{aligned}
$$

is satisfied, then $\mathcal{I}_{a,b,c}(f) \in \mathcal{UCD}(\beta)$.

In the special case when $b = 1$, Theorem 2 immediately yields a result concerning the Carlson-Shaffer operator $\mathcal{L}(a, c)(f) = I_{a,1,c}(f)$.

Corollary 5. *Let $a \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number. If $f \in \mathcal{R}_m(\gamma)$ and the inequality*

$$
\begin{aligned}
&\mathcal{F}(|a|, m(1-\gamma); c; 1) \\
&\quad + \frac{m(1-\gamma)\beta|a|}{c} \, \mathcal{F}(|a|+1, m(1-\gamma)+1; c+1; 1) \leq 2
\end{aligned}
\tag{2.8}
$$

is satisfied, then $\mathcal{L}(a, c)(f) \in \mathcal{UCD}(\beta)$.

Now we shall prove few inclusion results for the integral operator $\mathcal{J}(a, b; c; z)$ to be in the class $\mathcal{UCD}(\beta)$ when f belongs to different classes as discussed earlier for the Hohlov operator.

Theorem 3. *Let $a, b \in \mathbb{C} \setminus \{0\}$. Also, let c be a real number. If $f \in \mathcal{V}_m(\gamma)$, and the inequality*

$$
(1-\beta) {}_3\mathcal{F}_2(|a|, |b|, m(1-\gamma); c, 2; 1) + \beta {}_3\mathcal{F}_2(|a|, |b|, m(1-\gamma); c, 1; 1) \leq 2 \tag{2.9}
$$

is satisfied, then $\mathcal{J}_{a,b,c}(f) \in \mathcal{UCD}(\beta)$.

Proof. Let f be given by (1.1). By (1.6), to show $\mathcal{J}_{a,b,c}(f) \in \mathcal{UCD}(\beta)$ it is sufficient to prove that

$$
\sum_{n=2}^{\infty} n \left[1 + \beta(n-1)\right] \left| \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n} a_n \right| \leq 1. \tag{2.10}
$$

Applying the estimates for the coefficients given by (1.8), and making use of the relation $|(a)_n| \leq (|a|)_n$, we get

$$\sum_{n=2}^{\infty} n\,[1+\beta(n-1)]\ \left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}a_n\right|$$

$$\begin{aligned}
&\leq \sum_{n=2}^{\infty}(1+\beta(n-1))\frac{(|a|)_{n-1}(|b|)_{n-1}(m(1-\gamma))_{n-1}}{(c)_{n-1}(1)_n(1)_{n-1}}\\
&= (1-\beta)\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}(m(1-\gamma))_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_n}\\
&\qquad +\beta\sum_{n=2}^{\infty}(n-1)\frac{(|a|)_{n-1}(|b|)_{n-1}(m(1-\gamma))_{n-1}}{(c)_{n-1}(1)_{n-1}(1)_{n-1}}\\
&= (1-\beta)\left[{}_3\mathcal{F}_2(|a|,|b|,m(1-\gamma);c,2;1)-1\right]+\beta\left[{}_3\mathcal{F}_2(|a|,|b|,m(1-\gamma);c,1;1)-1\right]\\
&\leq 1
\end{aligned}$$

provided the condition (2.9) is satisfied.

Theorem 4. *Let $a, b \in \mathbb{C}\setminus\{0\}$. Also, let c be a real number. If $f \in \mathcal{R}_m(\gamma)$, and the inequality*

$$ {}_3\mathcal{F}_2(|a|,|b|,m(1-\gamma);c,1;1)+\frac{|a||b|m(1-\gamma)\beta}{c}\,{}_3\mathcal{F}_2(|a|,|b|,m(1-\gamma);c,2;1)\leq 2 \tag{2.11}$$

is satisfied, then $\mathcal{J}_{a,b,c}(f)\in\mathcal{UCD}(\beta)$.

The proof of theorem can be followed by the same lines as done in the proof of Theorem 4 and hence we omit the details of proof.

It is to be remarked at this moment, one can deduce interesting corollaries for Theorem 3 and Theorem 4 for special choices of a and b as stated for Theorem 1 and Theorem 2. However, those details are left for interested readers.

Concluding remarks and observations

In this current article, the authors obtained certain inclusion relation between few subclasses of univalent functions with bounded boundary rotation and the familiar Gauss hypergeometric functions. Interesting corollaries on the main results including for special are also given. Apart from these remarks which are given in the present article, more corollaries and remarks can be stated for the choice of $\gamma = 0$ and $m = 2$ and those details are omitted for the readers to explore.

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