South East Asian J. of Mathematics and Mathematical Sciences
Vol. 20, No. 2 (2024), pp. 39-52

DOI: 10.56827/SEAJMMS.2024.2002.4 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

A STUDY ON THE REVERSE EULER SOMBOR INDEX OF VARIOUS GRAPHS

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(**Received:** *Jun. 15, 2024* **Accepted:** *Aug. 27, 2024* **Published:** *Aug. 30, 2024*)

Abstract: In this paper an attempt to define vertex degree-based topological index, reverse Euler Sombor index is made and its mathematical properties are established. Reverse Euler Sombor index
$REU(G) = \sum_{uv\in E(G)} \sqrt{(\Delta - d_u + 1)^2 + (\Delta - d_v + 1)^2 + (\Delta - d_u + 1)(\Delta - d_v + 1)}$,
where d_u is the degree of the vertex $u \in V(G)$ and Δ is the maximum vertex degree of the graph G. REU index is computed for standard graphs like path, cycle, complete, crown, star, wheel, friendship, ladder, butterfly, complete bipartite, helm and regular. The bounds of reverse Euler Sombor index are found using famous Cauchy-Schwarz inequality and Jensen inequality. This study is extended for computing reverse Euler Sombor index for the family of thorn graphs.

Keywords and Phrases: Vertex degree-based topological indices, reverse Euler Sombor index, simple graphs, thorn graphs.

2020 Mathematics Subject Classification: 05C07, 05C09, 05C35, 05C38.

1. Introduction

Topological index is a numerical value that signifies the graph and its molecular structure [11, 19, 24]. There are numerous topological indices designed by various researchers throughout the world and are significant for a particular application in QSAR/QSPR based on the algorithm used [12, 13]. These indices play a vital role in environmental chemistry, drug design and material science. Apart from this, there are various fields in which the researchers are studying to connect the indices. The compound is modelled as a graph to predict the behaviour of molecules without using expensive apparatus needed in performing the experiment and labour time [5, 14, 15, 21, 32].

The oldest index was first proposed by Harold Weiner named after him as the Weiner index [22, 31]. This is a distance-based index, and it has been in the limelight since its inception in predicting the boiling point, melting point, and other thermodynamic properties of alkanes. It is considered as the most familiar index and still the researchers use this index in comparison to the novel indices to study the analysis of the work. After proposing and using the first index, experts throughout the world designed many indices which are used in various applications of drug design and development, materials science, pharmacy and many other fields [5, 7, 16, 17, 23, 33].

Let $G = (V, E)$ be a finite, simple connected graph. The degree d_u be the number of vertices adjacent to u. Let $\Delta(G)$ be the maximum degree among the vertices. The reverse vertex degree of vertex u is defined as $c_u = \Delta(G) - d_u + 1$. Let u and v be the end vertices of reverse edge uv. Recently, Gutman proposed Sombor index, Elliptic Sombor index, Euler Sombor index and are defined as follows

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2},$$

$$ESO(G) = \sum_{uv \in E(G)} (d_u + d_v)(\sqrt{d_u^2 + d_v^2}),$$

$$EU(G) = \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2 + d_u d_v}.$$

In 2022 Narahari et al., [30] introduced Reverse Sombor index

$$RSO(G) = \sum_{uv \in E(G)} \sqrt{c_u^2 + c_v^2}.$$

Reverse Elliptic Sombor index [20] is defined as

$$RESO(G) = \sum_{uv \in E(G)} (c_u + c_v)(\sqrt{c_u^2 + c_v^2}).$$

The following vertex degree based topological indices are used in this paper.

First, second Zagreb indices and forgotten index [6, 25, 26] are defined as

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v),$$
$$M_2(G) = \sum_{uv \in E(G)} (d_u \times d_v),$$
$$F(G) = \sum_{uv \in E(G)} ((d_u)^2 + (d_v)^2).$$

The reverse Zagreb indices introduced by Ediz [4]

$$RM_1^{\alpha}(G) = \sum_{u \in V(G)} (c(u)^2),$$
$$RM_1^{\beta}(G) = \sum_{uv \in E(G)} (c(u) + c(v)),$$
$$RM_2(G) = \sum_{uv \in E(G)} (c(u) \times c(v)).$$

In the year 2020, Gutman introduced Sombor index [8] which has become very familiar in most of the recent studies [2, 3, 27, 28, 29, 34]. Subsequently, various versions of this index are designed, and a lot of researchers are working on these indices such as elliptic Sombor [9], Euler Sombor index [10]. Motivated by the above studies, an attempt is made to define the reverse Euler Sombor index [18] for the various types of graphs in this paper.

The reverse Euler Sombor index is defined as

$$REU(G) = \sum_{uv \in E(G)} \sqrt{c_u^2 + c_v^2 + c_u c_v}$$
$$= \sum_{uv \in E(G)} \sqrt{(\Delta - d_u + 1)^2 + (\Delta - d_v + 1)^2 + (\Delta - d_u + 1)(\Delta - d_v + 1)}$$

where $c_u = \Delta - d_u + 1$ for any vertex $u \in V(G)$ and Δ is the maximum vertex degree of the graph G.

The flow of this study is initiated by defining a novel vertex degree-based topological index called reverse Euler Sombor index for which mathematical properties are derived using various types of graphs. The bounds of this novel index are determined using two familiar inequalities. The study is further carried out by computing the REU index for a family of Thorn graphs.

Basic mathematical properties of reverse Euler Sombor index (REU) are established in the subsequent sections.

2. Mathematical properties of reverse Euler Sombor index

Let G be a connected graph with $|V(G)| > 2$. Then maximum degree of graph G is $\Delta \geq 2$. The reverse Euler Sombor index of some standard graphs are computed and its bounds are established using graph parameters and topological indices.

Proposition 2.1. *Let P_n be path with $n \geq 3$, $REU(P_n) = 2\sqrt{7} + (n-3)\sqrt{3}$.*

Proposition 2.2. *Let C_n be cycle with $n \geq 3$, $REU(P_n) = n\sqrt{3}$.*

Proposition 2.3. *Let K_n be complete graph then, $REU(K_n) = \frac{n(n-1)}{2}\sqrt{3}$.*

Proposition 2.4. *Let G be crown graph then, $REU(G) = n(n-1)\sqrt{3}$.*

Proposition 2.5. *Let G be a star graph with n vertices and $n-1$ edges then $REU(G) = (n-1)\sqrt{n^2 - n + 1}$.*

Proposition 2.6. *Let $W_{1,n}$ be a wheel graph with $2n$ vertices and $2(n-1)$ edges, then $REU(W_{1,n}) = (n-1)(n-3)\sqrt{3} + (n-1)\sqrt{n^2 - 5n + 7}$.*

Proposition 2.7. *Let F_n be the friendship graph, then $REU(F_n) = n(2n-1)\sqrt{3} + 2n\sqrt{4n^2 - 2n + 1}$.*

Proposition 2.8. *For ladder graph L_n, $REU(L_n) = (3n-4)\sqrt{3} + 4\sqrt{7}$.*

Proposition 2.9. *Let G be a butterfly graph, then $REU(G) = 4\sqrt{12n^2 - 18n + 7} + 4\sqrt{4n^2 - 2n + 1} + (2n-4)\sqrt{4n^2 - 6n + 3} + \sqrt{3}(4n^2 - 16n + 12)$.*

Proposition 2.10. *Let $K_{m,n}$ be complete bipartite graph with $m+n$ vertices and mn edges then*

$$REU(K_{m,n}) = mn\sqrt{(m-n+1)^2 + (m-n) + 2}, \quad if \ \ \Delta = m,$$
$$REU(K_{m,n}) = mn\sqrt{(n-m+1)^2 + (n-m) + 2}, \quad if \ \ \Delta = n.$$

Proposition 2.11. *For Helm graph with $(2n+1)$ vertices and $3n$ edges, provided $n \geq 5$ is given by $REU(H_n) = n\sqrt{3n^2 - 9n + 9} + n\sqrt{3}(n-3) + n\sqrt{n^2 - 5n + 7}$.*

Theorem 2.12. *Let G be regular graph with m edges, then $REU(G) = m\sqrt{3}$*

Proof. If G is r-regular then, $\Delta = d_u = r$, $c_u = r - r + 1 = 1$ Thus

$$REU(G) = \sum_{uv \in E(G)} \sqrt{1^2 + 1^2 + 1 \times 1} = m\sqrt{3}.$$

Theorem 2.13. *If G is not regular with m edges, then $2m\sqrt{3} \leq REU(G) < m\sqrt{3}\Delta$.*

Proof. Suppose G is not regular and $\Delta \geq 2$. For each vertex u in V, $c_u = \Delta - d_u + 1 \geq 2$ and $d_u \geq 1$, so that $c_u = \Delta - d_u + 1 \leq \Delta$

$$REU(G) = \sum_{uv \in E(G)} \sqrt{c_u^2 + c_v^2 + c_u c_v} \geq m\sqrt{2^2 + 2^2 + 2 \times 2} \geq 2m\sqrt{3},$$

$$REU(G) = \sum_{uv \in E(G)} \sqrt{c_u^2 + c_v^2 + c_u c_v} < \sqrt{\Delta^2 + \Delta^2 + \Delta \times \Delta} < m\Delta\sqrt{3}.$$

Theorem 2.14. *For any graph G of size $m \geq 1$,*

$$REU(G) \leq \sqrt{m\,[3(\Delta+1)^2 + F(G) - 3(\Delta+1)M_1(G) + M_2(G)]}.$$

Proof. Using Cauchy-Schwarz inequality, we have

$$\left[\sum_{uv \in E(G)} \sqrt{c_u^2 + c_v^2 + c_u c_v}\right]^2 \leq \sum_{uv \in E(G)} 1 \sum_{uv \in E(G)} \left[c_u^2 + c_v^2 + c_u c_v\right]$$
$$= m \sum_{uv \in E(G)} \left[c_u^2 + c_v^2 + c_u c_v\right]$$
$$= m \sum_{uv \in E(G)} \left[(\Delta - d_u + 1)^2 + (\Delta - d_v + 1)^2 + (\Delta - d_u + 1)(\Delta - d_v + 1)\right]$$
$$= m \sum_{uv \in E(G)} [(\Delta+1)^2 + d_u^2 - 2(\Delta+1)d_u + (\Delta+1)^2 + d_v^2 - 2(\Delta+1)d_v$$
$$+ (\Delta+1)^2 - (\Delta+1)d_u - (\Delta+1)d_v + d_u d_v]$$
$$= m \sum_{uv \in E(G)} \left[3(\Delta+1)^2 + d_u^2 + d_v^2 - 3(\Delta+1)(d_u + d_v) + d_u d_v\right]$$
$$REU^2 \leq m\left[3(\Delta+1)^2 + F(G) - 3(\Delta+1)M_1(G) + M_2(G)\right].$$

Theorem 2.15. *For any graph G, $REU(G) \geq \frac{1}{\sqrt{2}}\left(RM_1^{\beta}(G) + RM_2(G)\right)$.*

Proof. For a concave function g(x), by Jensen inequality

$$g\left(\frac{1}{n}\sum x_i\right) \geq \frac{1}{n}\sum g(x_i)$$

with equality for a strict concave function if $x_1 = x_2 = x_3 .... = x_n$. Choosing $g(x) = \sqrt{x}$ we get

$$\sqrt{\frac{c_u^2 + c_v^2 + c_u c_v}{2}} \geq \frac{c_u + c_v + c_u c_v}{2}$$

$$\sum_{uv \in E(G)} \sqrt{c_u^2 + c_v^2 + c_u c_v} \geq \frac{1}{\sqrt{2}} \sum_{uv \in E(G)} [c_u + c_v + c_u c_v]$$

$$REU(G) \geq \frac{1}{\sqrt{2}} \left[RM_1^{\beta}(G) + RM_2(G) \right].$$

2.1. Thorn ring

The m-thorn ring, denoted by $C_{n,m}$ has a cycle C_n as the parent for which every vertex of the cycle contains $m-2$ thorns provided $m > 2$ [1]. By adding pendant vertices to each vertex of the cycle gives rise to thorn ring.

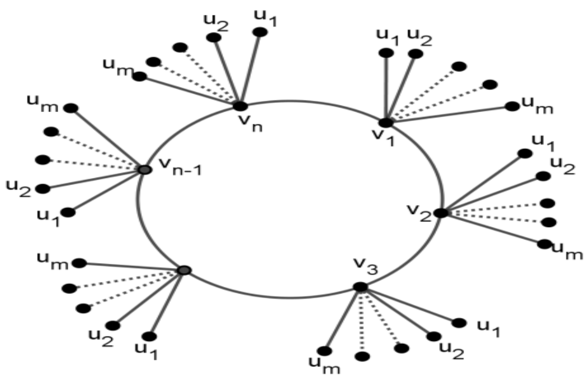

Figure 1: Thorn ring $C_{n,m}$

Theorem 2.16. *Let $C_{n,m}$ be the thorn ring with n vertices and m thorns then,* $REU(C_{n,m}) = mn\sqrt{m^2 + 5m + 7} + n\sqrt{3}$.

Proof. Let C_n denote the cycle with vertices $v_1, v_2, v_3, ..., v_n$. Let $u_1, u_2, u_3, ..., u_m$ be the pendant vertices attached to each vertex of cycle C_n. Here m and n denotes number of vertices of cycle and number of pendant vertices of cycle respectively. Here degrees of vertices $d(v_i) = m + 2$, $d(u_j) = 1$, $d(v_{n+1}) = v_1$. Here maximum degree $\Delta = m + 2$, then reverse degrees $c_u = (m + 2) - 1 + 1 = m + 2$ and $c_v = (m+2) - (m+2) + 1 = 1$ for edge partition $(m+2, 1)$. Similarly $c_u = 1, c_v = 1$ for edge partition $(m + 2, m + 2)$.

The reverse Euler Sombor index is

$$REU(C_{n,m}) = \sum_{uv \in E(G)} \sqrt{c_u^2 + c_v^2 + c_u c_v}$$
$$REU(C_{n,m}) = mn\sqrt{(m+2)^2 + 1^2 + (m+2) \times 1} + n\sqrt{1^2 + 1^2 + 1 \times 1}$$
$$REU(C_{n,m}) = mn\sqrt{m^2 + 5m + 7} + n\sqrt{3}.$$

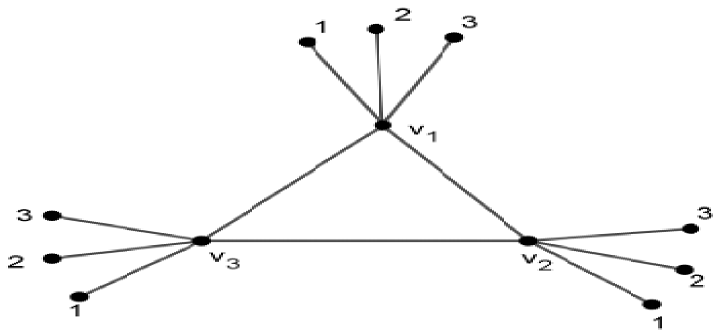

Figure 2: Thorn cycle with 5-thorn ring $C_{3,5}$

Corollary 2.17. *Reverse Euler Sombor index of 5-thorn ring* $C_{3,5}(m-2=3)$ *is* $9\sqrt{31} + 3\sqrt{3}$.

2.2. Thorn path

By adding q number of pendant vertices to the end vertices of the path P_n while r pendant vertices to the intermediate vertices of the end vertices of the path P_n, collectively gives rise to thorn path denoted by $P_{n,r,s}$.

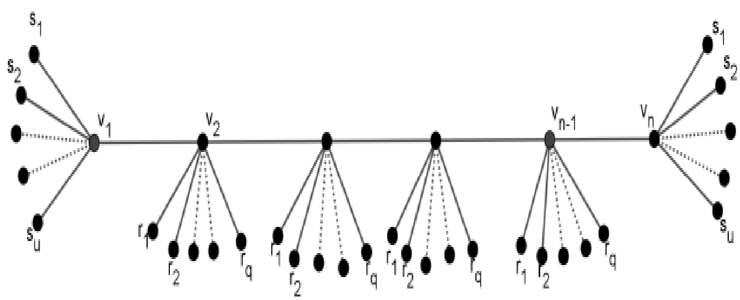

Figure 3: Thorn path $P_{n,r,s}$

Theorem 2.18. *Let $P_{n,r,s}$ be the thorn graph with n-tuple obtained from the connected graph G. Then, the reverse Euler Sombor index of a thorn path $P_{n,r,s}$, with $n \geq 3$ vertices is given by*

$$REU(P_{n,r,s}) = 2u\sqrt{u^2+3u+3} + (n-3)(u-q)\sqrt{3} + (n-2)q\sqrt{3u^2+q^2-3uq+3u-q+1} + 2\sqrt{u^2+q^2-2uq+u-q+1}.$$

Proof. Let $P_{n,r,s}, n \geq 3$ denote the thorn path obtained from the parent P_n, for each non-terminal vertex, r_q neighbors are added, while for each terminal vertex, s_u neighbors are added. Let $v_1, v_2, ..., v_n$ be the vertices of path, $s_1, s_2, ..., s_u$ be the pendant vertices of terminal vertex and $r_1, r_2, ..., r_n$ be pendant vertices attached to intermediate vertices of path. Let $d(v_1) = d(v_n) = u+1$, $d(v_i) = q+2$, for all $i = 2, 3, .., n-1$ and $d(s_j) = d(r_k) = 1$, where $j = 1, 2, 3, ..., u$,$k = 1, 2, 3, ..., q$. Then

$$REU(P_{n,r,s}) = \sum_{uv \in E(G)} \sqrt{c_u^2 + c_v^2 + c_u c_v}$$

$$REU(P_{n,r,s}) = 2u\sqrt{(u+1)^2+1^2+(u+1)} + (n-3)\sqrt{(u-q)^2+(u-q)^2+(u-q)(u-q)} + (n-2)q\sqrt{(u+1)^2+(u-q)^2+(u+1)(u-q)} + 2\sqrt{1^2+(u-q)^2+(u-q)}$$

$$REU(P_{n,r,s}) = 2u\sqrt{u^2+3u+3} + (n-3)(u-q)\sqrt{3} + (n-2)q\sqrt{3u^2+q^2-3uq+3u-q+1} + 2\sqrt{u^2+q^2-2uq+u-q+1}.$$

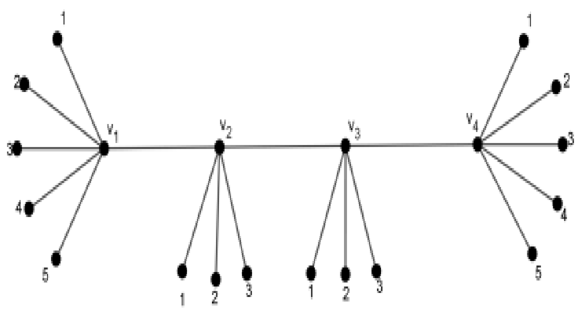

Figure 4: Thorn path $P_{4,3,5}$

Corollary 2.19. *Reverse Euler Sombor index of thorn path $P_{4,3,5}$ is* $10\sqrt{43}+2\sqrt{3}+6\sqrt{52}+2\sqrt{7}$

2.3. Thorn star

When u pendant vertices are joined to the central vertex V_0 of the star S_n and by joining q pendant vertices to its end vertices, gives rise to thorn star $S_{n,q,u}$.

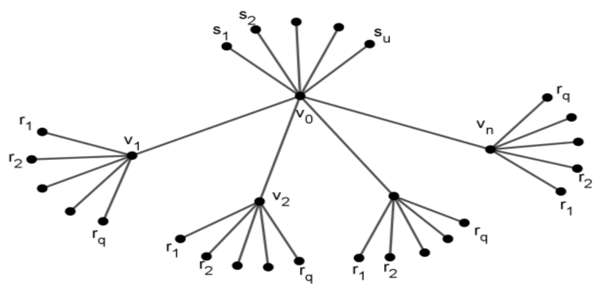

Figure 5: Thorn star $S_{n,q,u}$

Theorem 2.20. *Let $S_{n,q,u}$ be the thorn graph with n-tuple obtained from the connected graph G. Then, the reverse Euler Sombor index of thorn star $S_{n,q,u}$ is given by*

$$\begin{aligned}&u\sqrt{u^2+n^2+u+n+2un+1}\\&+n\sqrt{u^2+n^2+q^2+2un-2uq-2nq+u+n-q+1}\\&+nq\sqrt{3u^2+3n^2+q^2+6un-3qu-3nq}.\end{aligned}$$

Proof. Let S_n be star graph and $S_{n,q,u}$ be thorn star graph. Let v_0 be the central vertex with degree $u+n$. Let $s_1, s_2, ..., s_u$ be the pendant vertices attached to v_0 and $r_1, r_2, ..., r_q$ be the pendant vertices attached to end vertices of star $v_1, v_2, ..., v_n$ with degree $q+1$. Here P denotes n-tuple $P = (q, q, q...., u)$, Then reverse Euler Sombor index

$$\begin{aligned}REU(S_{n,q,u}) &= \sum_{uv\in E(G)} \sqrt{c_u^2+c_v^2+c_uc_v}\\REU(S_{n,q,u}) &= u\sqrt{(u+n)^2+1^2+u+n}+n\sqrt{1^2+(u+n-q)^2+u+n-q}\\&+nq\sqrt{(u+n)^2+(u+n-q)^2+(u+n)(u+n-q)}\\REU(S_{n,q,u}) &= u\sqrt{u^2+n^2+2un+u+n+1}\\&+n\sqrt{u^2+n^2+q^2+2un-2uq-2nq+u+n-q+1}\\&+nq\sqrt{3u^2+3n^2+q^2+6un-3qu-3nq}.\end{aligned}$$

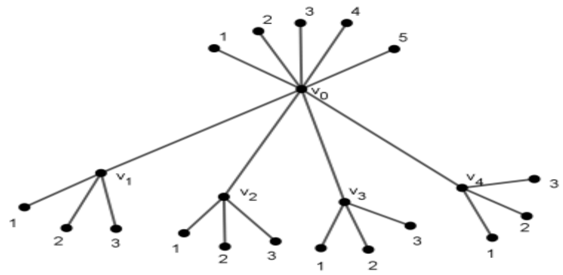

Figure 6: Thorn cycle with 5-thorn ring $C_{3,5}$

Corollary 2.21. *Consider a thorn star $S_{4,3,5}$, then reverse Euler Sombor index is $5\sqrt{91} + 4\sqrt{43} + 12\sqrt{171}$.*

3. Conclusion

This work pinpoints on the novel index namely reverse Euler Sombor index which is computed for simple and connected graphs such as path, cycle, complete, crown, star, wheel, friendship, ladder, butterfly, complete bipartite, helm, regular. Subsequently, mathematical properties and its bounds are established for newly introduced reverse Euler Sombor index with other indices. Followed by this, a family of thorn graphs are studied for which reverse Euler Sombor index is computed. The primary objective of this work is to introduce the reverse version of Euler Sombor index and establish its mathematical properties and bounds for the novel index using popular Cauchy-Schwarz and Jensen's inequalities.

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