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RESOLVING NONLINEAR PHYSICS PROBLEMS WITH AN EFFICIENT SEVENTH ORDER ITERATIVE APPROACH

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Abstract: Our research introduces a novel seventh-order iterative method specifically designed to address nonlinear equations having multiple roots. Inspired by the pioneering work of Sharma et al. (2019), our approach represents a significant advancement in computational techniques for solving complex mathematical problems. Through rigorous convergence analysis, we establish that our proposed method achieves seventh-order convergence. To evaluate its efficacy, we conduct extensive numerical experiments utilizing a range of nonlinear equations encountered in applied physics domains, including Planck's Law, electron trajectory problems, and Newton's beam designing problem. Our findings reveal that the suggested method consistently outperforms other existing techniques of similar nature available in the literature. Notably, our method demonstrates exceptional convergence behavior even in challenging scenarios involving multiple roots, indicating its suitability for solving complex problems encountered in applied physics and related fields. This superiority is evidenced by its ability to efficiently converge to solutions even in scenarios involving multiple roots. The practical implications of our research extend to various fields reliant on nonlinear equation.

Keywords and Phrases: Multiple roots, Iterative methods, Nonlinear equations, Beam designing problem, Electron trajectory problem. Planck's Law.

2020 Mathematics Subject Classification: 49M15, 41A25, 65H05.

1. Introduction

Problems in science and technology are typically of mathematical nature, leading to the linear and nonlinear equations. The considered scientific problems are solved by finding solutions to these equations. It is difficult to find solutions for such nonlinear equations like $f(x)$=0. If the function's initial $m-1$ derivative disappears at γ and $f^{(m)}(\gamma) \neq 0$, and the root γ is a multiple root with multiplicity m, the situation is more severe.

Since solving such problems using direct methods is not dependable. Iterative processes are therefore necessary. For the purpose of computing multiple roots with known multiplicity m, the modified Newton technique [12] is widely utilized and is given as:

$$x_{t+1} = x_t - m\frac{f(x_t)}{f'(x_t)}. \tag{1}$$

This is a one-point method with second order of convergence.

Intense research is being done to create new iterative techniques with better order of convergence. In his outstanding book, J.F. Traub explains why multi-point iterative methods are preferable to one-point iterative methods (see [16]). He underlines that using these techniques to approximate the roots of nonlinear equations is more effective. In order to improve the convergence of iterative methods for multiple roots, some researchers, such as Dong [2, 3], Neta et al. [9-11, 18], and Li et al. [7, 8] have developed iterative methods with higher order of convergence. Some of these methods are of order three [2, 3, 10, 18], while others are of order four [7-9, 11].

Furthermore, in recent years researchers are working to develop sixth and seventh order methods, for instance, Geum et. al. [4] developed sixth order methods, Sharma et. al. [14] developed seventh order methods. Recently, Kumar et. al. [6] has given a three step seventh order scheme with two weight functions utilizing $2f$ and $2f'$ evaluations. Inspired by the ongoing work in this direction, we here introduce yet another three step iterative scheme composed of three weighted Newton steps having three univariate weight functions based on $3f$ and $1f'$ evaluations. The scheme is presenting more generalized form and it has been proved that the presented scheme outperforms the existing schemes in terms of accuracy and computing efficiency.

This manuscript is structured as follows: The Taylor series expansion was used to analyze the scheme's convergence in Section 2. In Section 3, approaches found in the literature are cited for comparison and the numerical findings are examined. Basins of attractions are presented for visual comparison in Section 4. The paper ends with the conclusion in Section 5.

2. Convergence Analysis of Scheme

We create the iterative scheme and go over the requirements to derive seventh-order procedure from it using the computer programming system Mathematica [20]. We consider the iterative scheme:

$$\begin{aligned} y_t &= x_t - m\frac{f(x_t)}{f'(x_t)}, \\ z_t &= y_t - mu_t H(u_t)\frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} &= z_t - mu_t P(u_t)G(v_t)\frac{f(x_t)}{f'(x_t)}, \end{aligned} \tag{2}$$

where the weight functions $H, P, G : \mathbb{C} \to \mathbb{C}$ are univariate analytic functions given by

$$H(u_t) \approx H_0 + H_1 u_t + \frac{H_2}{2}u_t^2 + \frac{H_3}{6}u_t^3 + O(u_t^4), \tag{3}$$

$$P(u_t) \approx P_0 + P_1 u_t + \frac{P_2}{2}u_t^2 + \frac{P_3}{6}u_t^3 + O(u_t^4) \quad \text{and} \tag{4}$$

$$G(v_t) \approx G_0 + G_1 v_t + \frac{G_2}{2}v_t^2 + \frac{G_3}{6}v_t^3 + O(v_t^4), \tag{5}$$

where $u_t = \left(\frac{f(y_t)}{f(x_t)}\right)^{\frac{1}{m}}$, $v_t = \left(\frac{f(z_t)}{f(y_t)}\right)^{\frac{1}{m}}$.

Theorem 2.1. *Let us consider an analytic function $f : \mathbb{C} \to \mathbb{C}$ with $t = \gamma$ (say) as a multiple root having multiplicity $m \geq 1$. The presented scheme (2) converges to order seven, if the considered weight functions satisfy the following conditions: $G_0 = 0$, $G_1 = \frac{1}{P_0}$, $G_2 = \frac{2}{P_0}$, $H_2 = 0$, $P_1 = 2P_0$, $P_2 = 2P_0$.*

Proof. We will investigate specific conditions to ensure that the suggested plan reaches the required order of convergence by using the idea of Taylor series expansion. To complete the extensive computations, the Mathematica software [20] is used.

Assume that for t^{th} iteration, the error is $e_t = x_t - \gamma$. Following the Taylor's series expansion about γ, we may expand $f(x_t)$ and $f'(x_t)$ to obtain

$$f(x_t) = \frac{f^{(m)}(\gamma)}{m!}e_t^m(1 + c_1 e_t + c_2 e_t^2 + c_3 e_t^3 + c_4 e_t^4 + c_5 e_t^5 + c_6 e_t^6 + \dots), \quad \text{and} \tag{6}$$

$$f'(x_t) = \frac{f^{(m)}(\gamma)}{m!} e_t^{m-1}(m+(m+1)c_1e_t+(m+2)c_2e_t^2+(m+3)c_3e_t^3+(m+4)c_4e_t^4+\dots), \tag{7}$$

where $c_k = \frac{m!}{(m+k)!}\frac{f^{(m+k)}(\gamma)}{f^{(m)}(\gamma)}$, for $k \in \mathbb{N}$.

By inserting the above expression from Eq. (6) and Eq. (7) in the first sub-step of scheme (2) we obtain:

$$e_{y_t} = y_t - \gamma = \frac{c_1}{m}e_t^2 + \left(\frac{-(m+1)c_1^2 + 2mc_2}{m^2}\right)e_t^3 + \sum_{i=0}^{4}\phi_i e_t^{i+3} + O(e_t^8), \tag{8}$$

where $\phi_i = \phi_i(m, c_1, c_2, \dots, c_8)$, $i = 0, 1, 2, \dots$, are given in terms of $m, c_1, c_2, \dots, c_8$.

Expressions of ϕ_i that are exceedingly long are not written directly for the sake of brevity. Long expressions that we receive from the upcoming computations will likewise not be written down.

With the help of Taylor's series expansion and Eq. (8). we have

$$f(y_t) = \frac{f^{(m)}(\gamma)}{m!}e_{y_t}^m(1 + c_1e_{y_t} + c_2e_{y_t}^2 + c_3e_{y_t}^3 + c_4e_{y_t}^4 + \dots). \tag{9}$$

By using Eq. (6) and Eq. (9), we get following equation:

$$u_t = \left(\frac{f(y_t)}{f(x_t)}\right)^{\frac{1}{m}} = \frac{c_1e_t}{m} + \frac{(2mc_2 - (m+2)c_1^2)e_t^2}{m^2} + \sum_{i=0}^{5}\eta_i e_t^{i+2} + O(e_t^8), \tag{10}$$

where $\eta_i = \eta_i(m, c_1, c_2, \dots, c_8)$, $i = 0, 1, 2, \dots,$.

Now, inserting Eqs.(3), (6) - (10) in the second sub-step of scheme (2)

$$e_{z_t} = z_t - \gamma = -\frac{c_1(H_0-1)e_t^2}{m} + \frac{(-2mc_2(H_0-1) + c_1^2(-m-1+(m+3)H_0 - H_1))e_t^3}{m^2} + \dots + O(e_t^8). \tag{11}$$

On substituting $H_0 = 1$ and $H_1 = 2$ in above expression, we obtain the optimal fourth order convergence.

$$e_{z_t} = z_t - \gamma = \frac{(c_1^3(m+9-H_2) - 2mc_1c_2)\,e_t^4}{2m^3} + \sum_{i=0}^{4}\psi_i e_t^{i+3} + O(e_t^8), \tag{12}$$

where $\psi_i = \psi_i(m, c_1, c_2, \dots, c_8)$, $i = 0, 1, 2, \dots$

Again, using Taylor's series expansion for $f(z_t)$, we get

$$f(z_t) = \frac{f^{(m)}(\gamma)}{m!}e_{z_t}^m(1 + c_1e_{z_t} + c_2e_{z_t}^2 + c_3e_{z_t}^3 + c_4e_{z_t}^4 + \dots). \tag{13}$$

Employing Eq. (9) and Eq. (13), we get

$$v_t = \left(\frac{f(z_t)}{f(y_t)}\right)^{\frac{1}{m}} = \frac{(c_1^2(m+9-H_2)-2mc_2)\,e_t^2}{2m^2} + \sum_{i=0}^{5}\varphi_i e_t^{i+2} + O(e_t^8), \tag{14}$$

where $\varphi_i = \varphi_i(m, c_1, c_2, \ldots, c_8)$.

Using Eqs.(4)-(14) in proposed scheme (2) and simplifying, we obtain:

$$e_{t+1} = -\frac{c_1 G_0 P_0}{m} e_t^2 + \sum_{i=0}^{4} D_i e_t^{i+3} + O(e_t^8), \tag{15}$$

where $D_i = D_i(m, c_1, c_2, \ldots, c_8)$.

In order to achieve seventh order convergence, substitute $G_0 = 0$, the resultant equation is given by:

$$e_{t+1} = -\frac{(-2mc_1c_2(G_1P_0-1)+(c_1^3(H_2(G_1P_0-1)+(m+9)((G_1P_0-1)))}{2m^3}e_t^4+\cdots+O(e_t^8), \tag{16}$$

Proceeding in same way, on substituting $G_1 = \frac{1}{P_0}$, $G_2 = \frac{2}{P_0}$, $H_2 = 0$, $P_1 = 2P_0$, $P_2 = 2P_0$, in the given order, we end up with desired error equation.

Hence, the equation reduces to

$$e_{t+1} = \frac{c_1^2\big((9+m)c_1^2-2mc_2\big)\Big(-12mc_2P_0 + c_1^2\big((30+6m+H_3)P_0 - P_3\big)\Big)e_t^7}{12m^6P_0} + O(e_t^8). \tag{17}$$

The expression in Eq.(17) shows that the presented scheme (2) obtains the desired seventh order of convergence. The scheme utilizes only four functional evaluations i.e., $f(x_t), f'(x_t), f(y_t)$ and $f(z_t)$ per iteration.

On substituting $P_0 = 1$ and $H_3 = 0$, the proposed scheme in its final form, is given as:

$$\begin{aligned}
y_t &= x_t - m\frac{f(x_t)}{f'(x_t)},\\
z_t &= y_t - mu_t(1+2u_t)\frac{f(x_t)}{f'(x_t)},\\
x_{t+1} &= z_t - mu_t(1+2u_t+u_t^2)(v_t+v_t^2)\frac{f(x_t)}{f'(x_t)}.
\end{aligned} \tag{18}$$

This proposed method is considered as PM in the rest of the paper.

3. Numerical Experimentation

For the purpose of comparison, we consider the following sixth and seventh order methods present in the literature to compare with the proposed method. The programming package Mathematica [20] is used for performing all the computations. For numerical testing, the stopping criterion is considered as $|x_{t+1} - x_t| < 10^{-350}$. All computations are done on Intel(R) Core(TM) i3-7020U CPU @ 2.30 GHz with 4.00 GB RAM. Here, n denotes the number of iterations, Columns 3, 4 and 5 represent error estimations, penultimate column shows the Computational Order of Convergence (COC) [19]

$$COC = \frac{\log|(x_{t+2} - \gamma)/(x_{t+1} - \gamma)|}{\log|(x_{t+1} - \gamma)/(x_t - \gamma)|},$$

and the last column deals with CPU time. The CPU time is expressed in seconds.

Method proposed by Geum et al. [4], named as GM

$$\begin{aligned} y_t &= x_t - m\frac{f(x_t)}{f'(x_t)}, \\ z_t &= x_t - m(1 + u_t + 2u_t^2)\frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} &= x_t - (1 + u_t + 2u_t^2 + (1 + 2u_t)v_t)\frac{f(x_t)}{f'(x_t)}, \end{aligned} \tag{19}$$

with $u_t = \left(\frac{f(y_t)}{f(x_t)}\right)^{\frac{1}{m}}$, $v_t = \left(\frac{f(z_t)}{f(x_t)}\right)^{\frac{1}{m}}$.

Method given by Sharma et al. [15], denoted as SM_1

$$\begin{aligned} y_t &= x_t - m\frac{f(x_t)}{f'(x_t)}, \\ z_t &= y_t - mu_t(1 + u_t - u_t^2)\frac{f(x_t)}{f'(x_t)}, \\ x_{t+1} &= z_t - mv_t(1 + 2u_t + w_t)\frac{f(x_t)}{f'(x_t)}, \end{aligned} \tag{20}$$

with $u_t = \left(\frac{f(y_t)}{f(x_t)}\right)^{\frac{1}{m}}$, $v_t = \left(\frac{f(z_t)}{f(x_t)}\right)^{\frac{1}{m}}$ and $w_t = \left(\frac{f(z_t)}{f(y_t)}\right)^{\frac{1}{m}}$.

Method discussed by Sharma et al. [15], designated as SM_2

$$\begin{aligned} y_t &= x_t - m\frac{f(x_t)}{f'(x_t)},\\ z_t &= y_t - mu_t(1+u_t-u_t^2)\frac{f(x_t)}{f'(x_t)},\\ x_{t+1} &= z_t - mv_t\left(2u_t+\frac{1}{1-w_t}\right)\frac{f(x_t)}{f'(x_t)}, \end{aligned} \tag{21}$$

with $u_t = \left(\frac{f(y_t)}{f(x_t)}\right)^{\frac{1}{m}}$, $v_t = \left(\frac{f(z_t)}{f(x_t)}\right)^{\frac{1}{m}}$ and $w_t = \left(\frac{f(z_t)}{f(y_t)}\right)^{\frac{1}{m}}$
Method by Kumar et al. [6], named as KM_1

$$\begin{aligned} y_t &= x_t - m\frac{f(x_t)}{f'(x_t)},\\ z_t &= y_t - mu_t\left(\frac{1+u_t}{1+\frac{1+m}{1-m}u_t+\frac{2m(m+1)}{(m-1)^2}u_t^2}\right)\frac{f(x_t)}{f'(x_t)},\\ x_{t+1} &= z_t - mv_t\left(1+\frac{m-1}{m}\frac{v_t}{u_t}\right)\left(1+2u_t+\frac{m^2-2m-1}{m(m-1)}u_t^2\right)\frac{f(x_t)}{f'(x_t)}. \end{aligned} \tag{22}$$

Another method by Kumar et al. [6], designated as KM_2

$$\begin{aligned} y_t &= x_t - m\frac{f(x_t)}{f'(x_t)},\\ z_t &= y_t - mu_t\left(1-\frac{mu_t}{m-1}+\frac{3m^2u_t^2}{2(m-1)^2}\right)^{-2}\frac{f(x_t)}{f'(x_t)},\\ x_{t+1} &= z_t - mv_t\left(1+\frac{m-1}{m}\frac{v_t}{u_t}\right)\left(1+2u_t+\frac{m^2-2m-1}{m(m-1)}u_t^2\right)\frac{f(x_t)}{f'(x_t)}, \end{aligned} \tag{23}$$

with $u_t = \left(\frac{f'(y_t)}{f'(x_t)}\right)^{\frac{1}{m-1}}$ and $v_k = \left(\frac{f(z_t)}{f(x_t)}\right)^{\frac{1}{m}}$.

Consider the following real-world physics' problems and academic problems for the sake of numerical experiments. Each Table after example demonstrates the superiority of proposed method PM in contrast with other methods existing in literature.

Example 1. We first consider the Newton's Beam designing problem (see [21]). Consider figure 1 the beam DB is of length a unit lying on one of the edge of cubical box of side 2 units which lies on the floor adjacent to wall. The one end say D of beam touches the wall (y-axis) and another end B touches the floor (x-axis). The

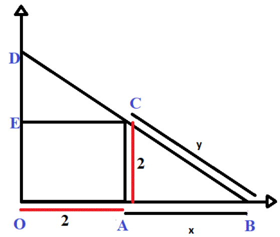


Figure 1: Newton's Beam problem.

problem here is to calculate the distance between foot of wall and the point B i.e., distance OB, where O is the origin. Here, we assign BC = y and AB = x, now there is need to calculate the distance $x+2$ units (see Fig. 1). In triangles △CAB and △DEC are similar. Therefore

$$\frac{x}{y} = \frac{2}{a-y} \tag{24}$$

Since △ CAB is a right angle,

$$x^4 + 4x^3 + (8-a^2)x^2 + 16x + 16 = 0 \tag{25}$$

For $a^2 = 32$, we get the nonlinear equations

$$f_1(x) = x^4 + 4x^3 - 24x^2 + 16x + 16,$$

with $x = 2$ as the root having multiplicity 2. Taking initial guess $x_0 = 3$ to find the root. Comparison of different methods with respect to $f_1(x)$ is given in Table 1.

Table 1: Comparison for $f_1(x) = x^4 + 4x^3 - 24x^2 + 16x + 16$

Method	n	$\lvert x_2 - x_1\rvert$	$\lvert x_3 - x_2\rvert$	$\lvert x_4 - x_3\rvert$	COC	CPU time
$x_0 = 3$						
GM	4	2.39×10^{-3}	5.78×10^{-18}	1.15×10^{-105}	6.00	0.047
SM_1	4	1.13×10^{-3}	6.52×10^{-23}	1.41×10^{-157}	7.00	0.062
SM_2	4	9.26×10^{-4}	1.63×10^{-23}	8.75×10^{-162}	7.00	0.062
KM_1	4	6.90×10^{-4}	1.00×10^{-23}	6.73×10^{-93}	6.00	0.047
KM_2	4	2.49×10^{-3}	1.29×10^{-17}	2.57×10^{-103}	6.00	0.047
PM	4	6.05×10^{-4}	2.36×10^{-25}	3.28×10^{-175}	7.00	0.035

Example 2. Next, we consider the Electron Trajectory problem [5]

$$f_2(x) = \left(x - \frac{1}{2}\cos x + \frac{\pi}{4}\right)^5, \tag{26}$$

here, the root is -0.30909327 , m=5 and $x_0 = 1$.

Table 2: Comparison for $f_2(x) = \left(x - \frac{1}{2}\cos x + \frac{\pi}{4}\right)^5$

Method	n	$\lvert x_2 - x_1\rvert$	$\lvert x_3 - x_2\rvert$	$\lvert x_4 - x_3\rvert$	COC	CPU time
$x_0 = 6$						
GM	4	9.44×10^{-3}	3.14×10^{-14}	4.36×10^{-83}	6.00	0.453
SM_1	4	5.12×10^{-3}	4.14×10^{-18}	9.42×10^{-124}	7.00	0.391
SM_2	4	4.02×10^{-3}	7.66×10^{-19}	7.07×10^{-129}	7.00	0.437
KM_1	4	4.02×10^{-3}	1.46×10^{-19}	3.72×10^{-38}	6.07	0.875
KM_2	4	4.97×10^{-3}	2.66×10^{-16}	6.36×10^{-96}	6.00	0.516
PM	4	3.23×10^{-3}	4.42×10^{-20}	3.92×10^{-138}	7.00	0.359

Example 3. The spectral density of electromagnetic radiations released by a black body at a specific temperature and at thermal equilibrium can be found using the Planck's radiation equation [1],

$$G(y) = \frac{8\pi c h y^{-5}}{e^{\frac{ch}{ykT}} - 1},$$

where c, h, y, k denote the speed of light in the vacuum medium, Planck's constant, the radiation wavelength, the Boltzmann constant and T stand for the

Table 3: Comparison for $f_3(x) = \left(e^{-x} - 1 + \frac{x}{5}\right)^3$

Method	n	$\lvert x_2 - x_1\rvert$	$\lvert x_3 - x_2\rvert$	$\lvert x_4 - x_3\rvert$	COC	CPU time
$x_0 = 3$						
GM	4	1.93×10^{-7}	5.81×10^{-47}	4.23×10^{-284}	6.00	0.312
SM_1	4	8.34×10^{-9}	2.19×10^{-64}	1.88×10^{-453}	7.00	0.406
SM_2	4	7.49×10^{-9}	1.04×10^{-64}	1.01×10^{-455}	7.00	0.359
KM_1	4	5.99×10^{-9}	8.78×10^{-66}	1.28×10^{-463}	7.00	0.359
KM_2	4	2.95×10^{-8}	1.86×10^{-52}	1.17×10^{-317}	6.00	0.359
PM	4	4.69×10^{-9}	1.74×10^{-66}	1.68×10^{-468}	7.00	0.359

black body's absolute temperature respectively. We arrived at the following equation to assess the wavelength y, which yields the maximum energy density $G(y)$. Additionally, the nonlinear equation is created by putting $x = \frac{ch}{ykT}$, we get

$$f_3(x) = \left(e^{-x} - 1 + \frac{x}{5}\right)^3.$$

The precise root of multiplicity $m = 3$ is $\gamma = 4.965114231744428$.

Example 4. Let us consider the following academic problem for further comparison.

$$f_4(x) = (x^4 - 1)^5.$$

The desired root of given function is 1 with multiplicity 5. We assume the initial root as 2.

Table 4: Comparison for $f_4(x) = (x^4 - 1)^5$

Method	n	$\lvert x_2 - x_1\rvert$	$\lvert x_3 - x_2\rvert$	$\lvert x_4 - x_3\rvert$	COC	CPU time
$x_0 = 3$						
GM	5	1.85×10^{-3}	7.59×10^{-15}	3.68×10^{-83}	5.999	0.110
SM_1	5	6.54×10^{-4}	5.22×10^{-20}	1.08×10^{-132}	6.999	0.031
SM_2	5	2.78×10^{-4}	1.33×10^{-22}	7.42×10^{-151}	6.999	0.078
KM_1	5	3.38×10^{-4}	1.07×10^{-22}	1.08×10^{-43}	1.14	0.171
KM_2	5	4.19×10^{-4}	4.06×10^{-19}	3.36×10^{-109}	6.00	0.093
PM	5	1.56×10^{-4}	6.11×10^{-25}	8.46×10^{-168}	7.000	0.030

Example 5. We further take up the following test function whose multiple root is 2 with multiplicity 50:

$$f_5(x) = ((x - 1)^3 - 1)^{50}.$$

To begin with, we consider 3 as an initial root.

Example 6. Next we have

$$f_6(x) = (x^4 - 2x^4 + 1)^3.$$

The test function has 1 as a multiple root with multiplicity 6. The chosen initial approximation is 3.

Table 5: Comparison for $f_5(x) = ((x-1)^3 - 1)^{50}$

Method	n	$\|x_2 - x_1\|$	$\|x_3 - x_2\|$	$\|x_4 - x_3\|$	COC	CPU time
$x_0 = 3$						
GM	5	1.08×10^{-5}	4.17×10^{-29}	1.40×10^{-169}	6.00	0.093
SM_1	5	9.47×10^{-7}	6.44×10^{-41}	4.32×10^{-280}	7.00	0.093
SM_2	5	2.84×10^{-7}	1.43×10^{-44}	1.14×10^{-305}	7.00	0.11
KM_1	5	4.27×10^{-8}	1.42×10^{-51}	6.18×10^{-356}	7.00	0.093
KM_2	5	1.70×10^{-7}	1.29×10^{-40}	2.50×10^{-239}	6.00	0.078
PM	5	7.36×10^{-8}	2.92×10^{-49}	4.50×10^{-339}	7.00	0.078

Table 6: Comparison for $f_6(x) = (x^4 - 2x^4 + 1)^3$

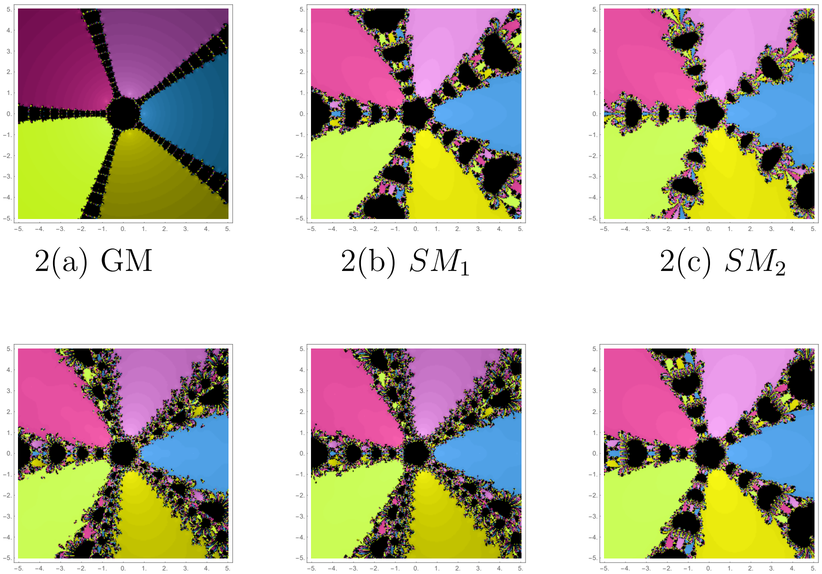
Method	n	$\|x_2 - x_1\|$	$\|x_3 - x_2\|$	$\|x_4 - x_3\|$	COC	CPU time
$x_0 = 3$						
GM	5	8.97×10^{-7}	4.90×10^{-37}	1.40×10^{-218}	6.00	0.062
SM_1	5	2.99×10^{-8}	3.67×10^{-53}	1.51×10^{-367}	7.00	0.062
SM_2	5	8.52×10^{-9}	5.52×10^{-57}	2.64×10^{-394}	7.00	0.062
KM_1	5	1.28×10^{-9}	5.33×10^{-64}	7.69×10^{-505}	7.00	0.141
KM_2	5	3.63×10^{-8}	7.59×10^{-46}	6.30×10^{-272}	6.00	0.046
PM	5	1.76×10^{-9}	2.50×10^{-62}	2.88×10^{-432}	7.00	0.046

Ultimately, the numerical experiments validate the theoretical findings, demonstrating the robustness and efficiency of the suggested seventh order approach. The suggested approach PM, in contrast to the other methods under discussion, achieves the theoretical order of convergence with more precision and less computational time, as shown by the Table 1-6. For high-precision computation, the proposed scheme is better since they can also produce high-precision solutions.

4. Basins of Attraction

The graphical tool, Basins af attraction, is used to compare the the methods under consideration visually. We a complex function $f(z) = (z^5 - 1)^3$, $z \in L$, which has five complex roots with multiplicity 3 each.

Here, we considered a rectangular mesh L = [-5, 5] × [-5, 5] ⊂ ℂ. Every root of $f(z) = 0$ is contained in this region. Let the initial point be $z_0 \in$ D. MATHEMATICA software [20] is used for generating high quality basins. Color shades are changing from light to dark as per number of iterations, darker tone suggests that the root is converging to desired root as shown in Figure [2]. If initial root fails to converge to the desired root, these points are shaded in black color.



2(a) GM 2(b) SM_1 2(c) SM_2

2(d) KM_1 2(e) KM_2 2(f) PM

Figure 2: The visual comparion of methods using Basins of attraction for $f(z) = (z^5 - 1)^3$, $z \in L$

[13, 17].

The pictures in Figure 2 present the basins of attraction for different methods. We observe the proposed method PM has an edge over other methods. On careful study, the behavior and suitability of any method can be judged and particular method can be used depending upon requirements.

5. Conclusion

The iterative procedure described in this manuscript uses four functional evaluations per iteration to achieve seventh order of convergence. Moreover, the presented method PM demonstrated excellent performance for well-known physics problems along with academic problems. In addition, Tables 1-6 illustrate that the method PM that is being provided yields exceptionally good results when compared to other approaches that are being studied in terms of COC, elapsed CPU time, and least estimated error.

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